Learning and Labor.
LIBRARY
OF THE
University of Illinois.
CLASS. BOOK. VOLUME.
510.6 AMS 2
READING ROOM
MATHEMATICS LIBRARY
Accession No.

Return this book on or before the **Latest Date** stamped below. A charge is made on all overdue books.

University of Illinois Library

April 3, 1953

AUG 30 1963

SEP 9 1968

AUG 30 REC'D

JUN 28 1969

JUN 23 REC'D

JUL 30 1969

JUL 29 REC'D

MAR 14 1991

MAR 13 REC'D

FEB 28 1999

FEB 25 REC'D

L161—H41

Digitized by the Internet Archive
in 2021 with funding from
University of Illinois Urbana-Champaign

https://archive.org/details/transactionsofam2190unse

TRANSACTIONS

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

ELIAKIM HASTINGS MOORE

ERNEST WILLIAM BROWN THOMAS SCOTT FISKE

PUBLISHED QUARTERLY BY THE SOCIETY WITH
THE COÖPERATION OF

HARVARD UNIVERSITY
YALE UNIVERSITY
PRINCETON UNIVERSITY
COLUMBIA UNIVERSITY
HAVERFORD COLLEGE
NORTHWESTERN UNIVERSITY
CORNELL UNIVERSITY
THE UNIVERSITY OF CALIFORNIA
BRYN MAWR COLLEGE
THE UNIVERSITY OF CHICAGO

VOLUME 2
1901

LANCASTER, PA., AND NEW YORK
THE MACMILLAN COMPANY
AGENTS FOR THE SOCIETY
1901

PRESS OF
THE NEW ERA PRINTING COMPANY,
LANCASTER, PA.

v.w. 2 Ap 02

TABLE OF CONTENTS

VOLUME 2 1901

PAGES

BLICHFELDT, H. F., of Stanford University, Cal. Note on the functions of the form

$$f(x) \equiv \phi(x) + d_1 x^{n-1} + d_2 x^{n-2} + \cdots + d_n$$

which in a given interval differ the least possible from zero . . 100–102

———— A new determination of the primitive continuous groups in two variables 249–258

BÔCHER, MAXIME, of Cambridge, Mass. Certain cases in which the vanishing of the Wronskian is a sufficient condition for linear dependence 139–149

———— An elementary proof of a theorem of Sturm 150–151

———— On certain pairs of transcendental functions whose roots separate each other 428–436

BOLZA, O., of Chicago, Ill. New proof of a theorem of Osgood's in the calculus of variations 422–427

CAJORI, FLORIAN, of Colorado Springs, Col. Divergent and conditionally convergent series whose product is absolutely convergent 25– 36

DICKSON, LEONARD EUGENE, of Chicago, Ill. Canonical forms of quaternary abelian substitutions in an arbitrary Galois field . 103–138

———— Theory of linear groups in an arbitrary field 363–394

FIELDS, J. C., of Hamilton, Ontario, Canada. On the reduction of the general abelian integral 49– 86

HILBERT, DAVID, of Göttingen, Germany. Ueber Flächen von constanter Gaussscher Krümmung 87– 99

MCDONALD, J. H., of Lafayette, Ind. On the system of a binary cubic and quadratic and the reduction of hyperelliptic integrals of genus two to elliptic integrals by a transformation of the fourth order 437–458

METZLER, W. H., of Syracuse, N. Y. On certain aggregates of determinant minors 395–403

MERTENS, F., of Vienna, Austria. Zur linearen Transformation der ϑ-Reihen 331–342

MILLER, G. A., of Ithaca, N. Y. Determination of all the groups of order p^m which contain the abelian groups of type $(m-2, 1)$, p being any prime 259–272

Math. 11 Mar 02 Macmillan 5.00

8

50037

PAGES

MOORE, ELIAKIM HASTINGS, of Chicago, Ill. Concerning Harnack's theory of improper definite integrals 296–330

———— On the theory of improper definite integrals 459–475

OSGOOD, W. F., of Cambridge, Mass. On the existence of a minimum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')\,dx$$

when x_0 and x_1 are conjugate points, and the geodesics on an ellipsoid of revolution: a revision of a theorem of Kneser's . . 166–182

———— On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass's 273–295

PORTER, M. B., of New Haven, Conn. Sets of coincidence points on the non-singular cubics of a syzygetic sheaf 37– 42

PRINGSHEIM, A., of Munich, Germany. Ueber die Anwendung der Cauchy'schen Multiplicationsregel auf bedingt convergente oder divergente Reihen 404–412

———— Ueber den Goursat'schen Beweis des Cauchy'schen Integralsatzes 413–421

SMITH, PERCEY F., of New Haven, Conn. Geometry within a linear spherical complex 234–248

STECKER, HENRY FREEMAN, of Ithaca, N. Y. On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves 152–165

STRINGHAM, IRVING, of Berkeley, Cal. On the geometry of planes in a parabolic space of four dimensions 183–214

STRONG, WENDELL M., of New Haven, Conn. Note on non-quaternion number systems 43– 48

VAN VLECK, E. B., of Middletown, Conn. On the convergence of continued fractions with complex elements 215–233

———— On the convergence and character of a certain form of continued fraction 476–483

WILCZYNSKI, E. J., of Berkeley, Cal. Invariants of systems of linear differential equations 1– 24

———— Geometry of a simultaneous system of two linear homogeneous differential equations of the second order 343–362

Notes and errata: volumes 1 and 2 484–487

LIBRARY
OF THE
UNIVERSITY of ILLINOIS.

INVARIANTS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

BY

E. J. WILCZYNSKI

The theory of the invariants of a single linear homogeneous differential equation rests upon STAECKEL'S theorem, that the most general point-transformation which converts a general homogeneous linear differential equation of the m-th order $(m > 1)$

$$\frac{d^m y}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + P_m(x)y = 0\,, \tag{1}$$

into another equation of the same form and order, is

$$x = f(\xi)\,, \quad y = \eta\, g(\xi)\,, \tag{2}$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ.*

Those functions of P_1, P_2, $\cdots$, P_m and the derivatives of these quantities, which are the same for the equation (1) and for any equation obtained from (1) by a transformation (2), are called invariants of (1). Functions having this invariant property and containing also y, dy/dx, etc., are called covariants. The investigation of such invariants and covariants has led to many new and interesting results concerning the equation (1). This theory is associated with the names of COCKLE, MALET, LAGUERRE, HALPHEN, BRIOSCHI, FORSYTH, and others.

The author has recently shown that the most general point-transformation, which converts a system of n homogeneous linear differential equations into another of the same form and order, is

$$x = f(\xi)\,, \quad y_k = \sum_{i=1}^{n} a_{ki}(\xi)\eta_i \qquad (k = 1, 2, \cdots, n), \tag{3}$$

where $f(\xi)$ and $a_{ki}(\xi)$ are arbitrary functions of ξ, and the determinant $|a_{ki}(\xi)|$ does not vanish identically.†

We shall consider, in this paper, those combinations of the coefficients of a system of linear differential equations which remain invariant when the system

*Crelle's Journal, vol. 111.

†American Journal of Mathematics, January, 1901.

5 0 3 3 1

is transformed by the above transformation (3). These transformations obviously form an infinite continuous group, and we shall employ LIE'S theory throughout, as has been done in the case of a single differential equation by Dr. BOUTON.* We shall not, in this first paper, pay much attention to the applications of the theory, nor give more than a passing mention to covariants.

§ 1. *Finite transformations of the dependent variables.*

We shall at first confine ourselves to the transformation of the dependent variables. Those functions of the coefficients of the system which remain invariant for all such transformations, may be called *seminvariants*. These are of considerable importance in themselves, and besides furnish the basis for the theory of invariants under the general transformation.

Let the given system be

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} p_{ikl} y_k^{(l)} = 0 \qquad (i = 1, 2, \cdots, n), \tag{1}$$

and let this system be transformed by the equations

$$y_k = \sum_{\lambda=1}^{n} a_{k\lambda}(x) \eta_\lambda \qquad (k = 1, 2, \cdots, n), \tag{2}$$

where $a_{k\lambda}(x)$ are arbitrary functions of x, and where the determinant

$$|a_{k\lambda}(x)| \qquad (k, \lambda = 1, 2, \cdots, n),$$

does not vanish. As usual we denote derivatives by accents. Then we have from (2)

$$y_k^{(l)} = \sum_{\lambda=1}^{n} \sum_{\rho=0}^{l} \binom{l}{\rho} a_{k\lambda}^{(\rho)} \eta_\lambda^{(l-\rho)} \qquad (k = 1, 2, \cdots, n;\ l = 0, 1, 2, \cdots, m), \tag{3}$$

where in general $\binom{l}{\rho}$ denotes the coefficient of x^ρ in the expansion of $(1 + x)^l$.

Equations (1) then become

$$\sum_{\lambda=1}^{n} a_{i\lambda} \eta_\lambda^{(m)} + \sum_{\lambda=1}^{n} \sum_{\rho=1}^{m} \binom{m}{\rho} a_{i\lambda}^{(\rho)} \eta_\lambda^{(m-\rho)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} \sum_{\mu=1}^{n} \sum_{\sigma=0}^{l} \binom{l}{\sigma} p_{ikl} a_{k\mu}^{(\sigma)} \eta_\mu^{(l-\sigma)} = 0 \tag{4}$$

$$(i = 1, 2, \cdots, n).$$

The coefficient of $\eta_\mu^{(\nu)}$ in the double sum is

$$\binom{m}{m-\nu} a_{i\mu}^{(m-\nu)},$$

and in the quadruple sum, the coefficient of $\eta_\mu^{(\nu)}$ is

* American Journal of Mathematics, vol. 21, no. 2, 1899.

$$\sum_{k=1}^{n}\left[\binom{\nu}{0}p_{ik\nu}a_{k\mu}+\binom{\nu+1}{1}p_{i,k,\nu+1}a_{k\mu}^{(1)}+\cdots+\binom{m-1}{m-1-\nu}p_{i,k,m-1}a_{k\mu}^{(m-1-\nu)}\right],$$

i. e.,

$$\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}.$$

Thus, from (4) we obtain

$$(5)\quad \sum_{\lambda=1}^{n}a_{i\lambda}\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]=0 \qquad (i=1,2,\cdots,n),$$

or if we put

$$(6)\qquad \Delta=|a_{i\lambda}| \qquad (i,\lambda=1,2,\cdots,n),$$

and denote by $A_{i\lambda}$ the minor of $a_{i\lambda}$ in this determinant,

$$(7)\quad \Delta\eta_{\lambda}^{(m)}+\sum_{\mu=1}^{n}\sum_{\nu=0}^{m-1}\eta_{\mu}^{(\nu)}\sum_{i=1}^{n}A_{i\lambda}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]=0 \qquad (\lambda=1,2,\cdots,n).$$

If then we write this system in the form

$$(8)\qquad \eta_{\lambda}^{(m)}+\sum_{\nu=0}^{m-1}\sum_{\mu=1}^{n}\pi_{\lambda\mu\nu}\eta_{\mu}^{(\nu)}=0 \qquad (\lambda=1,2,\cdots,n),$$

we have

$$(9)\qquad \Delta\pi_{\lambda\mu\nu}=\sum_{i=1}^{n}A_{i\lambda}\left[\binom{m}{m-\nu}a_{i\mu}^{(m-\nu)}+\sum_{k=1}^{n}\sum_{\tau=0}^{m-1-\nu}\binom{\nu+\tau}{\tau}p_{i,k,\nu+\tau}a_{k\mu}^{(\tau)}\right]$$

$$(\lambda,\mu=1,2,\cdots,n;\ \nu=0,1,\cdots,m-1).$$

Thus, if (1) is transformed into (8) by transformations (2), the relations between the coefficients of (1) and (8) are the equations (9).

Equations (9) represent an infinite continuous group, isomorphic with the group represented by equations (2). For to every transformation of the latter group corresponds one of the former, and they obviously have the group property. Both groups can be defined by differential equations, so that LIE'S theory of infinite groups may be applied.

§ 2. *Infinitesimal transformations of the dependent variables.*

We proceed to consider the infinitesimal transformations of our infinite group. The variables $y_1, y_2, \cdots, y_n$ will undergo the most general infinitesimal transformation of form (2) if we put

$$a_{ii}(x) = 1 + \phi_{ii}(x)\delta t, \quad a_{ik}(x) = \phi_{ik}(x)\delta t \tag{10}$$

$$(i \neq k),\ (i, k = 1, 2, \cdots, n),$$

where δt is an infinitesimal and the ϕ_{ik}'s are arbitrary functions of x. We wish to find the corresponding infinitesimal transformations of the coefficients p_{ikl}.

Neglecting infinitesimals of order higher than the first, we find

$$\Delta = \begin{vmatrix} 1 + \phi_{11}\delta t, & \phi_{12}\delta t, & \cdots, & \phi_{1n}\delta t \\ \phi_{21}\delta t, & 1 + \phi_{22}\delta t, & \cdots, & \phi_{2n}\delta t \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{n1}\delta t, & \phi_{n2}\delta t, & \cdots, & 1 + \phi_{nn}\delta t \end{vmatrix} = 1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t, \tag{11}$$

and similarly

$$\begin{cases} A_{ii} = 1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn} - \phi_{ii})\delta t, \\ A_{ki} = -\phi_{ik}\delta t \end{cases} \qquad (i \neq k). \tag{12}$$

Substituting these values in (9), we have

$$\Delta\pi_{\lambda\mu\nu} = \sum_{i=1}^{n} -\phi_{\lambda i}\delta t \left[\binom{m}{m-\nu} \phi_{i\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{i,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{i\mu\nu} \right] + [1 + (\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t] \left[\binom{m}{m-\nu} \phi_{\lambda\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{\lambda,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{\lambda\mu\nu} \right],$$

or

$$\Delta\pi_{\lambda\mu\nu} = p_{\lambda\mu\nu} - \sum_{i=1}^{n} \phi_{\lambda i} p_{i\mu\nu}\delta t + \binom{m}{m-\nu} \phi_{\lambda\mu}^{(m-\nu)}\delta t + \sum_{k=1}^{n} \sum_{\tau=0}^{m-1-\nu} \binom{\nu+\tau}{\tau} p_{\lambda,k,\nu+\tau}\phi_{k\mu}^{(\tau)}\delta t + p_{\lambda\mu\nu}(\phi_{11} + \phi_{22} + \cdots + \phi_{nn})\delta t. \tag{13}$$

Dividing by $\Delta = 1 + (\phi_{11} + \cdots + \phi_{nn})\delta t$, and denoting the infinitesimal difference $\pi_{\lambda\mu\nu} - p_{\lambda\mu\nu}$ by $\delta p_{\lambda\mu\nu}$, we find

$$\frac{\delta p_{\lambda\mu\nu}}{\delta t} = \sum_{k=1}^{n} (\phi_{k\mu} p_{\lambda k\nu} - \phi_{\lambda k} p_{k\mu\nu}) + \sum_{k=1}^{n} \sum_{\tau=1}^{m-1-\nu} \binom{\nu+\tau}{\tau} \phi_{k\mu}^{(\tau)} p_{\lambda,k,\nu+\tau} + \binom{m}{m-\nu} \phi_{\lambda\mu}^{(m-\nu)}, \tag{14}$$

$$(\lambda, \mu = 1, 2, \cdots, n;\ \nu = 0, 1, 2, \cdots, m-1).$$

These are the required infinitesimal transformations of $p_{\lambda\mu\nu}$. Those of $p'_{\lambda\mu\nu}$, $p''_{\lambda\mu\nu}$, etc., may be obtained from (14) by differentiation.

§ 3. *Calculation of the seminvariants for* $m = n = 2$.

The complexity of the general problem is so great that it appears wise to limit the further discussion in this paper to the very special case $m = n = 2$. This will throw considerable light on the general case.

Let us put, in this case, for abbreviation

$$(15)\qquad \begin{cases} p_{\lambda,\mu,m-1} = p_{\lambda\mu 1} = p_{\lambda\mu}, \qquad p_{\lambda,\mu,m-2} = p_{\lambda\mu 0} = q_{\lambda\mu}, \\ \dfrac{\partial f}{\partial p_{\lambda\mu}} = P_{\lambda\mu}, \quad \dfrac{\partial f}{\partial p'_{\lambda\mu}} = P'_{\lambda\mu}, \quad \dfrac{\partial f}{\partial q_{\lambda\mu}} = Q_{\lambda\mu}, \quad \text{etc.} \end{cases}$$

We have then, from (14),

$$(16)\qquad \begin{cases} \dfrac{\delta p_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\phi_{k\mu} p_{\lambda k} - \phi_{\lambda k} p_{k\mu}) + 2\phi'_{\lambda\mu}, \\ \dfrac{\delta p'_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\phi_{k\mu} p'_{\lambda k} - \phi_{\lambda k} p'_{k\mu} + \phi'_{k\mu} p_{\lambda k} - \phi'_{\lambda k} p_{k\mu}) + 2\phi^{(2)}_{\lambda\mu}, \\ \dfrac{\delta q_{\lambda\mu}}{\delta t} = \sum_{k=1}^{2} (\phi_{k\mu} q_{\lambda k} - \phi_{\lambda k} q_{k\mu}) + \sum_{k=1}^{2} \phi'_{k\mu} p_{\lambda k} + \phi^{(2)}_{\lambda\mu}. \end{cases}$$

Now if f is a seminvariant depending only upon $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$, we must have

$$(17)\qquad \sum_{\lambda,\mu} (P_{\lambda\mu}\delta p_{\lambda\mu} + P'_{\lambda\mu}\delta p'_{\lambda\mu} + Q_{\lambda\mu}\delta q_{\lambda\mu}) = 0$$

for all values of ϕ_{rs}, ϕ'_{rs}, ϕ''_{rs}. Putting the coefficients of these twelve arbitrary functions equal to zero, we obtain the following system of partial differential equations for such seminvariants:

$$(18)\qquad \begin{cases} 2P'_{rs} + Q_{rs} = 0, \\ 2P_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r} P'_{\lambda s} - p_{s\lambda} P'_{r\lambda} + p_{\lambda r} Q_{\lambda s}) = 0, \qquad (r, s = 1, 2). \\ \sum_{\lambda=1}^{2} (p_{\lambda r} P_{\lambda s} - p_{s\lambda} P_{r\lambda} + p'_{\lambda r} P'_{\lambda s} - p'_{s\lambda} P'_{r\lambda} + q_{\lambda r} Q_{\lambda s} - q_{s\lambda} Q_{r\lambda}) = 0. \end{cases}$$

This is a complete system of twelve equations with twelve independent variables. But there are two relations among them, so that we shall have two independent solutions, i. e., two independent seminvariants containing only the variables $p_{\lambda\mu}$, $p'_{\lambda\mu}$, $q_{\lambda\mu}$.

The first four equations of the system tell us that the quantities p'_{rs} and q_{rs} can occur only in combinations

$$2p'_{rs} - 4q_{rs}.$$

The next four equations, written separately, are

$$(19)\quad \begin{cases} 2P_{12}+(p_{11}-p_{22})P'_{12}+p_{21}(P'_{22}-P'_{11})+p_{11}Q_{12}+p_{21}Q_{22}=0\,, \\ 2P_{11}+\ p_{21}P'_{21}-p_{12}P'_{12}+p_{11}Q_{11}+p_{21}Q_{21}=0\,, \\ 2P_{22}+\ p_{12}P'_{12}-p_{21}P'_{21}+p_{12}Q_{12}+p_{22}Q_{22}=0\,, \\ 2P_{21}+(p_{22}-p_{11})P'_{21}+p_{12}(P'_{11}-P'_{22})+p_{12}Q_{11}+p_{22}Q_{21}=0\,. \end{cases}$$

They show that the only possible combinations of p_{rs}, p'_{rs}, q_{rs} are

$$(20)\quad \begin{cases} u_{11}=2p'_{11}-4q_{11}+p_{11}^2+p_{12}p_{21}\,, \\ u_{12}=2p'_{12}-4q_{12}+p_{12}(p_{11}+p_{22})\,, \\ u_{21}=2p'_{21}-4q_{21}+p_{21}(p_{11}+p_{22})\,, \\ u_{22}=2p'_{22}-4q_{22}+p_{22}^2+p_{12}p_{21}\,, \end{cases}$$

so that the seminvariants, here considered, are functions of u_{11}, u_{12}, u_{21}, u_{22} only.

The last four equations (18) are

$$(21)\quad \begin{cases} U_1\equiv (p_{11}-p_{22})P_{12}+p_{21}(P_{22}-P_{11})+(p'_{11}-p'_{22})P'_{12} \\ \qquad\qquad +p'_{21}(P'_{22}-P'_{11})+(q_{11}-q_{22})Q_{12}+q_{21}(Q_{22}-Q_{11})=0\,, \\ U_2\equiv\ p_{21}P_{21}-p_{12}P_{12}+p'_{21}P'_{21}-p'_{12}P'_{12}+q_{21}Q_{21}-q_{12}Q_{12}=0\,, \\ U_3\equiv\ p_{12}P_{12}-p_{21}P_{21}+p'_{12}P'_{12}-p'_{21}P'_{21}+q_{12}Q_{12}-q_{21}Q_{21}=0\,, \\ U_4\equiv (p_{22}-p_{11})P_{21}+p_{12}(P_{11}-P_{22})+(p'_{22}-p'_{11})P'_{21} \\ \qquad\qquad +p'_{12}(P'_{11}-P'_{22})+(q_{22}-q_{11})Q_{21}+q_{12}(Q_{11}-Q_{22})=0\,, \end{cases}$$

with the obvious relation

$$(22)\qquad U_2+U_3=0\,.$$

But there is another relation between these four equations and the other eight, these latter being themselves obviously independent. If we compute from them P_{rs} and P'_{rs} and substitute the values of these quantities, thus obtained, in (21), that system becomes

$$(23)\quad \begin{cases} (u_{11}-u_{22})Q_{12}-u_{21}(Q_{11}-Q_{22})=0\,, \\ -(u_{11}-u_{22})Q_{21}+u_{12}(Q_{11}-Q_{22})=0\,, \\ u_{12}Q_{12}-u_{21}Q_{21}=0\,, \end{cases}$$

where the last equation is a consequence of the other two.

The two independent solutions of this system are

$$I = u_{11} + u_{22}, \quad J = u_{11}u_{22} - u_{12}u_{21}, \tag{24}$$

and these are the seminvariants required.

Let us proceed to obtain next those seminvariants which contain also the quantities p''_{ik} and q'_{ik}. They must satisfy the following system of partial differential equations:

$$\begin{cases} (a)\ 2P''_{rs} + Q'_{rs} = 0, \\ (b)\ 2P'_{rs} + Q_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P''_{r\lambda} + p_{\lambda r}Q'_{\lambda s}) = 0, \\ (c)\ 2P_{rs} + \sum_{\lambda=1}^{2} (p_{\lambda r}P'_{\lambda s} - p_{s\lambda}P'_{r\lambda} - 2p'_{s\lambda}P''_{r\lambda} \\ \qquad + q_{\lambda r}Q'_{\lambda s} - q_{s\lambda}Q'_{r\lambda} + p_{\lambda r}Q_{\lambda s}) = 0, \\ (d)\ \sum_{\lambda=1}^{2} (p_{\lambda r}P_{\lambda s} - p_{s\lambda}P_{r\lambda} + p'_{\lambda r}P'_{\lambda s} - p'_{s\lambda}P'_{r\lambda} + p''_{\lambda r}P''_{\lambda s} \\ \qquad - p''_{s\lambda}P''_{r\lambda} + q_{\lambda r}Q_{\lambda s} - q_{s\lambda}Q_{r\lambda} + q'_{\lambda r}Q'_{\lambda s} - q'_{s\lambda}Q'_{r\lambda}) = 0, \end{cases} \tag{25}$$

among which there is one relation. Thus there are 15 independent equations and 20 variables. There are therefore five seminvariants satisfying these equations. Of these we already know four, namely, I, J, dI/dx, dJ/dx, which are obviously independent.

Let us put

$$u'_{ik} = \frac{du_{ik}}{dx}, \quad \text{etc.}$$

Then, since according to (20), we have

$$u_{ik} = 2p'_{ik} - 4q_{ik} + \sum_{j=1}^{2} p_{ij}p_{jk}, \tag{26}$$

therefore

$$u'_{ik} = 2p''_{ik} - 4q'_{ik} + \sum_{j=1}^{2} (p_{ij}p'_{jk} + p'_{ij}p_{jk}). \tag{27}$$

It will be easily proved from equations (25) (a) and (b) that our seminvariants are functions of the twelve arguments

$$p_{ik}, \quad u_{ik}, \quad u'_{ik}. \tag{28}$$

Denoting the left members of (25) (c) by $\Omega_1 \cdots \Omega_4$, so that

$$\Omega_1 = 2P_{11} + \cdots, \quad \Omega_2 = 2P_{12} + \cdots, \quad \Omega_3 = 2P_{21} + \cdots, \quad \Omega_4 = 2P_{22} + \cdots,$$

we find

$$\Omega_1(u_{ik}) = \Omega_2(u_{ik}) = \Omega_3(u_{ik}) = \Omega_4(u_{ik}) = 0, \tag{29}$$

$$\begin{cases} \Omega_1(p_{11}) = 2, & \Omega_2(p_{11}) = 0, & \Omega_3(p_{11}) = 0, & \Omega_4(p_{11}) = 0, \\ \Omega_1(p_{12}) = 0, & \Omega_2(p_{12}) = 2, & \Omega_3(p_{12}) = 0, & \Omega_4(p_{12}) = 0, \\ \Omega_1(p_{21}) = 0, & \Omega_2(p_{21}) = 0, & \Omega_3(p_{21}) = 2, & \Omega_4(p_{21}) = 0, \\ \Omega_1(p_{22}) = 0, & \Omega_2(p_{22}) = 0, & \Omega_3(p_{22}) = 0, & \Omega_4(p_{22}) = 2, \end{cases} \tag{30}$$

and

$$(31)\quad \begin{cases} \Omega_1(u'_{11}) = 0\,, & \Omega_2(u'_{11}) = -u_{21}\,, & \\ & \Omega_3(u'_{11}) = +u_{12}\,, & \Omega_4(u'_{11}) = 0\,, \\ \Omega_1(u'_{12}) = -u_{12}\,, & \Omega_2(u'_{12}) = u_{11} - u_{22}\,, & \\ & \Omega_3(u'_{12}) = 0\,, & \Omega_4(u'_{12}) = +u_{12}\,, \\ \Omega_1(u'_{21}) = +u_{21}\,, & \Omega_2(u'_{21}) = 0\,, & \\ & \Omega_3(u'_{21}) = -(u_{11} - u_{22})\,, & \Omega_4(u'_{21}) = -u_{21}\,, \\ \Omega_1(u'_{22}) = 0\,, & \Omega_2(u'_{22}) = +u_{21}\,, & \\ & \Omega_3(u'_{22}) = -u_{12}\,, & \Omega_4(u'_{22}) = 0\,. \end{cases}$$

From these equations it is easily seen that the eight independent functions of the arguments (28) which verify the equations $\Omega_\lambda = 0$, are the quantities u_{ik} and

$$(32)\quad \begin{cases} v_{11} = 2u'_{11} + p_{12}u_{21} - p_{21}u_{12}\,, \\ v_{12} = 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22})\,, \\ v_{21} = 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22})\,, \\ v_{22} = 2u'_{22} - p_{12}u_{21} + p_{21}u_{12}\,. \end{cases}$$

Denoting by $X_1, \cdots, X_4$ the left members of (25) (d) we find:

$$(33)\quad \begin{cases} X_1(v_{11}) = 0\,, & X_2(v_{11}) = -v_{21}\,, & \\ & X_3(v_{11}) = +v_{12}\,, & X_4(v_{11}) = 0\,, \\ X_1(v_{12}) = -v_{12}\,, & X_2(v_{12}) = v_{11} - v_{22}\,, & \\ & X_3(v_{12}) = 0\,, & X_4(v_{12}) = +v_{12}\,, \\ X_1(v_{21}) = +v_{21}\,, & X_2(v_{21}) = 0\,, & \\ & X_3(v_{21}) = -(v_{11} - v_{22})\,, & X_4(v_{21}) = -v_{21}\,, \\ X_1(v_{22}) = 0\,, & X_2(v_{22}) = +v_{21}\,, & \\ & X_3(v_{22}) = -v_{12}\,, & X_4(v_{22}) = 0\,, \end{cases}$$

the equations for $X_1(u_{ik})$, etc., being precisely of the same form.

From (33) we find that

$$v_{11} + v_{22} \quad \text{and} \quad v_{11}v_{22} - v_{12}v_{21}$$

are solutions of the equations $X_\lambda = 0$. But

$$(34)\qquad v_{11} + v_{22} = 2I\,,$$

while

$$K = v_{11}v_{22} - v_{12}v_{21}, \tag{35}$$

is obviously a new seminvariant, independent of I, J, dI/dx, dJ/dx.

If we should now write down the differential equations satisfied by the seminvariants involving $p_{ik}^{(3)}$ and $q_{ik}^{(2)}$ besides the quantities already considered, there would be twenty such equations with only one relation between them, and twenty-eight independent variables. Hence there must be $28 - 19 = 9$ such seminvariants. But we know eight of these, viz.:

$$I,\ I',\ I'';\quad J,\ J',\ J'';\ K,\ K'; \tag{36}$$

these are independent, for it is easily seen that from the existence of a relation between them would follow the existence of a relation between I, I', J, J', K. But these quantities were independent.

We can obtain the ninth semivariant without writing down and integrating the last mentioned system of twenty equations. The process which we shall employ is much more instructive, and is capable of generalization.

We notice first this theorem: *the quantities u_{ik} and v_{ik} are cogredient.*

This follows from equations (33), together with the corresponding equations for $X_\lambda(u_{ik})$, as well as from the following formulæ, which express the infinitesimal transformations of u_{ik} and v_{ik}. It will be found from (16) and (20), that

$$\begin{cases} \dfrac{\delta u_{11}}{\delta t} = \phi_{21}u_{12} - \phi_{12}u_{21}, \\ \dfrac{\delta u_{12}}{\delta t} = (\phi_{22} - \phi_{11})u_{12} + \phi_{12}(u_{11} - u_{22}), \\ \dfrac{\delta u_{21}}{\delta t} = (\phi_{11} - \phi_{22})u_{21} + \phi_{21}(u_{22} - u_{11}), \\ \dfrac{\delta u_{22}}{\delta t} = -\phi_{21}u_{12} + \phi_{12}u_{21}, \end{cases} \tag{37}$$

and from (16), (32), and (37),

$$\begin{cases} \dfrac{\delta v_{11}}{\delta t} = \phi_{21}v_{12} - \phi_{12}v_{21}, \\ \dfrac{\delta v_{12}}{\delta t} = (\phi_{22} - \phi_{11})v_{12} + \phi_{12}(v_{11} - v_{22}), \\ \dfrac{\delta v_{21}}{\delta t} = (\phi_{11} - \phi_{22})v_{21} + \phi_{21}(v_{22} - v_{11}), \\ \dfrac{\delta v_{22}}{\delta t} = -\phi_{21}v_{12} + \phi_{12}v_{21}. \end{cases} \tag{38}$$

Now certain combinations of the u_{ik}'s and p_{ik}'s, viz. $v_{11} + v_{22}$ and $v_{11}v_{22} - v_{12}v_{21}$, were seminvariants. Since the v_{ik}'s are cogredient with the u_{ik}'s, the same combinations with v_{ik} in place of u_{ik} will also be seminvariants.

Let us, therefore, put

$$(39)\qquad \begin{cases} w_{11} = 2v'_{11} + p_{12}v_{21} - p_{21}v_{12}, \\ w_{12} = 2v'_{12} + (p_{11} - p_{22})v_{12} - p_{12}(v_{11} - v_{22}), \\ w_{21} = 2v'_{21} - (p_{11} - p_{22})v_{21} + p_{21}(v_{11} - v_{22}), \\ w_{22} = 2v'_{22} - p_{12}v_{21} + p_{21}v_{12}; \end{cases}$$

then we know that $w_{11} + w_{22}$ and $w_{11}w_{22} - w_{12}w_{21}$ are seminvariants. But

$$w_{11} + w_{22} = 2(v'_{11} + v'_{22}) = 4I',$$

while

$$(40)\qquad L = w_{11}w_{22} - w_{12}w_{21}$$

is the new seminvariant. That it is independent of the other eight can be easily seen by considering the special case in which $p_{ik} = 0$.

We now have all the seminvariants, viz.: I, J, K, L, and their derivatives. For, suppose we wish to find the seminvariants involving $p^{(4)}_{ik}$ and $q^{(3)}_{ik}$. They are determined by a system of $24 - 1 = 23$ independent equations with 36 independent variables. Therefore, there exist $36 - 23 = 13$ such seminvariants. But they are merely the 8 seminvariants (36) and L, I''', J''', K'', L'.

Thus all seminvariants of the system (1) *for* $m = n = 2$, *are functions of the quantities* I, J, K, L, *and of their derivatives.*

It is interesting to note what would be the result of continuing our above process for obtaining seminvariants. *Suppose we had formed*

$$t_{11} = 2w'_{11} + p_{12}w_{21} - p_{21}w_{12}, \text{ etc.}$$

Then would

$$\begin{cases} t_{11} - t_{22} = g_1(u_{11} - u_{22}) + g_2(v_{11} - v_{22}) + g_3(w_{11} - w_{22}), \\ t_{12} = g_1u_{12} + g_2v_{12} + g_3w_{12}, \\ t_{21} = g_1u_{21} + g_2v_{21} + g_3w_{21}, \end{cases}$$

where g_1, g_2, g_3 *are seminvariants.*

For g_1, g_2, g_3 are the quotients of determinants of the third order formed out of the matrix

$$\begin{vmatrix} t_{11} - t_{22}, & u_{11} - u_{22}, & v_{11} - v_{22}, & w_{11} - w_{22} \\ t_{12}, & u_{12}, & v_{12}, & w_{12} \\ t_{21}, & u_{21}, & v_{21}, & w_{21} \end{vmatrix},$$

and an account of the cogrediency of these quantities, such quotients are seminvariants. In fact, the determinants themselves are seminvariants.

§ 4. *An application of the theory of seminvariants.*

The theory, developed in the preceding paragraphs, was first suggested by the following considerations. Let

$$D(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \tag{41}$$

be a homogeneous, linear, differential equation, and let $y_1, \cdots, y_n$ form a fundamental system of (41), so that

$$D(y_i) = 0 \qquad (i = 1, 2, \cdots, n). \tag{41a}$$

Suppose that the coefficients of (41) are uniform functions of x, and let $x = a_\mu$ be a singular point of one or all of these coefficients. If the variable x describes a closed circuit around this singular point a_μ, $y_1, \cdots, y_n$ will, in general, undergo a linear substitution with constant coefficients, changing into

$$\bar{y}_k = \sum_{i=1}^{n} \lambda_{ki}^{(\mu)} y_i \qquad (k = 1, 2, \cdots, n),$$

where the determinant $|\lambda_{ki}^{(\mu)}| \neq 0$. Denote this substitution by A_μ, so that we may write

$$y_k = A_\mu y_k. \tag{42}$$

Now let us put in (41a)

$$y_k = \sum_{i=1}^{n} a_{ki} \eta_i = S\eta_k \qquad (k = 1, 2, \cdots, n), \tag{43}$$

where again the determinant

$$|a_{ki}| \neq 0,$$

and where a_{ki} are uniform functions of x. Then $\eta_1, \cdots, \eta_n$ will verify a system of n linear differential equations, obtained from (41a) by the transformation (43). This system is not a general system but has the special property that corresponding to a circuit of x around a_μ, $\eta_1, \cdots, \eta_n$ undergo the substitution

$$S^{-1} A_\mu S,$$

which is the transformed of A_μ by the substitution S, and has as its coefficients uniform functions of x.

Conversely, if a system of n linear differential equations has this property, it is obviously possible to find a substitution

$$y_k = S\eta_k$$

which reduces it to the form (41a), or what amounts to the same thing, n linear

combinations of $\eta_1, \cdots, \eta_n$ can be formed with variable coefficients which form a fundamental system of a single linear differential equation of the nth order. Such functions as $\eta_1, \cdots, \eta_n$ are a very special kind of what I have called Λ functions.*

The question then arises: what are the necessary and sufficient conditions in order that a system of linear differential equations may have the property in question?

In our particular case, $m = n = 2$, it must then be possible to transform the given system into one of the form

$$y_i'' + py_i' + qy_i = 0 \qquad (i = 1, 2), \tag{44}$$

by a transformation of the form (43). The seminvariants of the given system must therefore be equal to those of (44). But for (44) we have

$$p_{12} = p_{21} = q_{12} = q_{21} = 0, \quad p_{11} = p_{22} = p, \quad q_{11} = q_{22} = q, \tag{45}$$

so that

$$\begin{cases} u_{11} = u_{22} = 2p' - 4q + p^2, & u_{12} = u_{21} = 0, \\ v_{11} = v_{22} = 2u_{11}', & v_{12} = v_{21} = 0, \\ w_{11} = w_{22} = 2v_{11}' = 4u_{11}'', & w_{12} = w_{21} = 0, \end{cases} \tag{46}$$

whence

$$I = 2u_{11}, \quad J = u_{11}^2, \quad K = 4(u_{11}')^2, \quad L = 16(u_{11}'')^2. \tag{47}$$

We have then in this case the relations:

$$I^2 - 4J = 0, \quad K - \left(\frac{dI}{dx}\right)^2 = 0, \quad L - 4\left(\frac{d^2I}{dx^2}\right)^2 = 0. \tag{48}$$

As will be seen from the expression deduced for the invariants in § 5, *all of the invariants vanish in this case.*

But the vanishing of all of the invariants, or even the fulfillment of the equations (48), while necessary, are not sufficient conditions for this case. The conditions (46) however, from which the others were derived, are both necessary and sufficient. It suffices to write down the system of invariant equations

$$u_{11} - u_{22} = 0, \quad u_{12} = u_{21} = 0, \tag{49}$$

for the other equations (46) follow from these.

That the conditions (49) are sufficient follows from the results of § 6. It will there be shown that every system of linear differential equations of the second order can by a transformation of the form

$$y_i = a_{i1}\eta_1 + a_{i2}\eta_2 \qquad (i = 1, 2), \tag{50}$$

*American Journal of Mathematics, vol. 16, No. 2, 1899.

be reduced to a form called the semi-canonical form. The result of this reduction in general is the system (82), which in our case reduces to

$$\eta_i'' = \tfrac{1}{4} u_{11} \eta_i \qquad (i = 1, 2), \tag{51}$$

i. e., η_1 and η_2 satisfy the same linear differential equation of the second order.

Thus we have seen that a system of the form

$$y_i'' + p_{i1} y_1' + p_{i2} y_2' + q_{i1} y_1 + q_{i2} y_2 = 0 \qquad (i = 1, 2) \tag{52}$$

can, by a transformation of form (50), *be reduced to the form* (44), *if and only if the conditions* (49) *are fulfilled. In that case, the integration of the system* (52) *is equivalent to the integration of a single linear differential equation of the second order. The invariants are all zero, and by a change of both dependent and independent variables this equation may be reduced to the form*

$$\frac{d^2 u}{d\xi^2} = 0 . \tag{53}$$

§ 5. *Calculation of the invariants for* $m = n = 2$.

The invariants of our system must obviously be functions of the seminvariants which we have already found. We shall therefore first investigate how these seminvariants are affected by a transformation of the independent variable x.

As before, let

$$y_i'' + p_{i1} y_1' + p_{i2} y_2' + q_{i1} y_1 + q_{i2} y_2 = 0 \qquad (i = 1, 2), \tag{54}$$

be the given system. If we introduce the new independent variable

$$\xi = \xi(x),$$

this system becomes

$$y_i'' + \pi_{i1} y_1' + \pi_{i2} y_2' + \kappa_{i1} y_1 + \kappa_{i2} y_2 = 0 \qquad (i = 1, 2), \tag{55}$$

where

$$\left\{ \begin{array}{l} \pi_{11} = \dfrac{p_{11}}{\xi'} + \dfrac{\xi''}{(\xi')^2}, \quad \pi_{12} = \dfrac{p_{12}}{\xi'}, \quad \pi_{21} = \dfrac{p_{21}}{\xi'}, \quad \pi_{22} = \dfrac{p_{22}}{\xi'} + \dfrac{\xi''}{(\xi')^2}, \\ \qquad\qquad \kappa_{ik} = \dfrac{q_{ik}}{(\xi')^2}. \end{array} \right. \tag{56}$$

For an infinitesimal transformation, we put

$$\xi(x) = x + \phi(x)\delta t, \quad \delta x = \phi(x)\delta t, \tag{57}$$

where δt is an infinitesimal and $\phi(x)$ an arbitrary function. We shall then have

$$\left\{ \begin{array}{ll} \delta p_{11} = (-\phi' p_{11} + \phi'')\delta t, & \delta p_{12} = -\phi' p_{12} \delta t, \\ \delta p_{21} = -\phi' p_{21} \delta t, & \delta p_{22} = (-\phi' p_{22} + \phi'')\delta t, \\ \multicolumn{2}{c}{\delta q_{ik} = -2\phi' q_{ik} \delta t.} \end{array} \right. \tag{58}$$

If f is any function of x, and f' its derivative, we have

$$\delta f' = \delta\frac{df}{dx} = \frac{\delta(df)}{dx} - \frac{df}{dx^2}\delta(dx) = \frac{d}{dx}(\delta f) - f'\frac{d}{dx}(\delta x)$$

or

$$\delta f' = \frac{d}{dx}(\delta f) - f'\phi'(x)\delta t. \tag{59}$$

Thus we find from (58)

$$\begin{cases} \delta p'_{11} = (\phi^{(3)} - \phi'' p_{11} - 2\phi' p'_{11})\delta t, \\ \delta p'_{12} = \quad\ (- \phi'' p_{12} - 2\phi' p'_{12})\delta t, \\ \delta p'_{21} = \quad\ (- \phi'' p_{21} - 2\phi' p'_{21})\delta t, \\ \delta p'_{22} = (\phi^{(3)} - \phi'' p_{22} - 2\phi' p'_{22})\delta t. \end{cases} \tag{60}$$

Therefore, making use of (20), (58), and (60), we find

$$\begin{cases} \delta u_{11} = (2\phi^{(3)} - 2\phi' u_{11})\delta t, \quad \delta u_{12} = - 2\phi' u_{12}\delta t, \\ \delta u_{21} = - 2\phi' u_{21}\delta t \qquad , \quad \delta u_{22} = (2\phi^{(3)} - 2\phi' u_{22})\delta t. \end{cases} \tag{61}$$

Consequently we shall have, remembering the definition of I and J,

$$\begin{cases} \delta I = (4\phi^{(3)} \ - 2\phi' I\ )\delta t, \\ \delta J = (2\phi^{(3)} I - 4\phi' J\ )\delta t. \end{cases} \tag{62}$$

In the same way we find

$$\begin{cases} \delta v_{11} = (4\phi^{(4)} - 4\phi'' u_{11} - 3\phi' v_{11})\delta t, \\ \delta v_{12} = (- 4\phi'' u_{12} - 3\phi' v_{12})\delta t, \\ \delta v_{21} = (- 4\phi'' u_{21} - 3\phi' v_{21})\delta t, \\ \delta v_{22} = (4\phi^{(4)} - 4\phi'' u_{22} - 3\phi' v_{22})\delta t, \end{cases} \tag{63}$$

whence

$$\delta K = (8\phi^{(4)} I' - 8\phi'' J' - 6\phi' K)\delta t, \tag{64}$$

after a slight transformation, involving equation (67) for J'.

We find further

$$\begin{cases} \delta w_{11} = (8\phi^{(5)} - 8\phi^{(3)} u_{11} - 10\phi'' v_{11} - 4\phi' w_{11})\delta t, \\ \delta w_{12} = (- 8\phi^{(3)} u_{12} - 10\phi'' v_{12} - 4\phi' w_{12})\delta t, \\ \delta w_{21} = (- 8\phi^{(3)} u_{21} - 10\phi'' v_{21} - 4\phi' w_{21})\delta t, \\ \delta w_{22} = (8\phi^{(5)} - 8\phi^{(3)} u_{22} - 10\phi'' v_{22} - 4\phi' w_{22})\delta t; \end{cases} \tag{65}$$

whence follows

$$\delta L = [32\phi^{(5)}I'' - 16\phi^{(3)}(2J'' - K) - 20\phi''K' - 8\phi'L]\delta t\,, \tag{66}$$

if it be noted that

$$\begin{cases} u_{11}v_{22} + u_{22}v_{11} - u_{12}v_{21} - u_{21}v_{12} = 2J'\,, \\ v_{11}w_{22} + v_{22}w_{11} - v_{12}w_{21} - v_{21}w_{12} = 2K'\,, \\ w_{11}u_{22} + w_{22}u_{11} - w_{12}u_{21} - w_{21}u_{12} = 2(2J'' - K)\,. \end{cases} \tag{67}$$

Let us now, by an obvious extension of the theory of invariants of a single linear differential equation, assign to p_{ik} the weight -1, to q_{ik} the weight -2, while to $p_{ik}^{(\lambda)}$ and $q_{ik}^{(\mu)}$ are assigned the weights $-1-\lambda$, $-2-\mu$ respectively. Then it is easy to see that the following statements are true.

1. *Every absolute invariant is isobaric in the coefficients and of weight zero.*

2. *If an absolute invariant is a rational function of the coefficients and of their derivatives, it is the quotient of two relative invariants of the same weight.*

3. *A relative invariant is isobaric in the coefficients, and if the common weight of all of its terms is $-\nu$, it satisfies the equation*

$$\xi'(x)^{\nu}\theta_{\nu}(\xi) = \theta_{\nu}(x)\,, \tag{68}$$

or, for infinitesimal transformations,

$$\delta\theta_{\nu} = -\nu\phi'(x)\theta_{\nu}\delta t\,. \tag{68a}$$

We shall speak of such an invariant as being of weight ν, rather than $-\nu$. The negative weights have been introduced, following the example of FORSYTH, principally because they have a decided advantage when we consider *covariants*.

The proof of the above three statements is essentially the same as in the case of a single linear differential equation, and need not, therefore, be given.

It is now a simple matter to find the rational invariants of any assigned weight, or else to establish their non-existence.

First, it is clear that no such invariants of weights 1, 2, 3 exist. An invariant of weight 4 must verify the equation

$$\delta\theta_4 = -4\phi'(x)\theta_4\delta t\,.$$

The invariant is

$$\theta_4 = I^2 - 4J. \tag{69}$$

An invariant of weight 5 must be of the form

$$aII' + bI^{(3)} + cJ'.$$

We find that no such invariant exists, or that it vanishes identically.

An invariant of weight 6 must verify the equation

$$\frac{\delta\theta_6}{\delta t} = -6\phi'\theta_6 .$$

The most general expression of weight 6 is

$$\theta_6 = aI^3 + bIJ + cK + dI^{(4)} + eJ'' + fII'' + gI'^2 .$$

We find

$$\begin{aligned}\frac{\delta\theta_6}{\delta t} = {} & 3aI^2(4\phi^{(3)} - 2\phi' I) + bI(2\phi^{(3)}I - 4\phi' J) + bJ(4\phi^{(3)} - 2\phi' I) \\ & + c(8\phi^{(4)}I' - 8\phi'' J' - 6\phi' K) \\ & + d(4\phi^{(7)} - 2\phi^{(5)}I - 9\phi^{(4)}I' - 12\phi^{(3)}I'' - 10\phi'' I^{(3)} - 6\phi' I^{(4)}) \\ & + e(2\phi^{(5)}I + 4\phi^{(4)}I' + 2\phi^{(3)}I'' - 4\phi^{(3)}J - 9\phi'' J' - 6\phi' J'') \\ & + fI(4\phi^{(5)} - 2\phi^{(3)}I - 5\phi'' I' - 4\phi' I'') + fI''(4\phi^{(3)} - 2\phi' I) \\ & + 2gI'(4\phi^{(4)} - 2\phi'' I - 3\phi' I') ,\end{aligned}$$

and this must be equal to $-6\phi'\theta_6$ for all values of $\phi, \phi', \cdots, I, I', \cdots, J, J', \cdots, K$. We find therefore the equations:

$$\begin{cases} d = 0 , & e + 2f = 0 , \quad 2c + e + 2g = 0 , \quad 6a + b - f = 0 , \\ b - e = 0 , & 8c + 9e = 0 , \quad 5f + 4g = 0 , \end{cases}$$

whence

$$a = -\tfrac{1}{4}e , \quad b = e , \quad c = -\tfrac{9}{8}e , \quad d = 0 , \quad f = -\tfrac{1}{2}e , \quad g = \tfrac{5}{8}e .$$

Putting $e = -8$, we find

$$\theta_6 = 2I(I^2 - 4J) + 5(K - I'^2) + 4(K - 2J'' + II'') . \tag{70}$$

There is no invariant of weight 7, and there are two independent invariants of weight 8, one of which is θ_4^2, while the other is

$$\begin{aligned}\theta_8 = {} & 143(L - 4I''^2) - 54(I^2 + 4J)\theta_4 - 20I''\theta_4 + 25I'\theta_4' - 206I\theta_4'' \\ & - 20\theta_4^{(4)} - 902I(K - I'^2) - 220(K'' - 2I'I^{(3)} - 2I''^2) .\end{aligned} \tag{71}$$

We can easily find an invariant of weight 10, without going through this general process. We have

$$\frac{\delta(K - I'^2)}{\delta t} = 4\phi''(II' - 2J') - 6\phi'(K - I'^2) ,$$

$$\frac{\delta(II' - 2J')}{\delta t} = -2\phi''(I^2 - 4J) - 5\phi'(II' - 2J') ,$$

whence eliminating ϕ'', we find

$$\frac{\delta\theta_{10}}{\delta t} = -10\phi'\theta_{10},$$

where

$$\theta_{10} = (I^2 - 4J)(K - I'^2) + (II' - 2J')^2. \tag{72}$$

From any two invariants θ_λ and θ_μ, of weights λ and μ respectively, we can always deduce a third. For we have

$$\frac{\theta_\lambda^\mu(\xi)}{\theta_\mu^\lambda(\xi)} = \frac{\theta_\lambda^\mu(x)}{\theta_\mu^\lambda(x)}$$

an absolute invariant. Hence, by logarithmic differentiation,

$$\left\{\mu\frac{\theta_\lambda'(\xi)}{\theta_\lambda(\xi)} - \lambda\frac{\theta_\mu'(\xi)}{\theta_\mu(\xi)}\right\}\xi' = \mu\frac{\theta_\lambda'(x)}{\theta_\lambda(x)} - \lambda\frac{\theta_\mu'(x)}{\theta_\mu(x)}.$$

Therefore

$$\theta_{\lambda+\mu+1} = \mu\theta_\mu\theta_\lambda' - \lambda\theta_\lambda\theta_\mu' \tag{73}$$

is a new invariant of weight $\lambda + \mu + 1$. It is called by FORSYTH the *Jacobian* of θ_λ and θ_μ, in his theory of invariants of a single equation.

We thus obtain

$$\left\{\begin{aligned}
\theta_{11} &= 3\theta_6\theta_4' - 2\theta_4\theta_6', \\
\theta_{13} &= 2\theta_8\theta_4' - \theta_4\theta_8', \\
\theta_{15} &= 5\theta_{10}\theta_4' - 2\theta_4\theta_{10}', \\
\vartheta_{15} &= 4\theta_8\theta_6' - 3\theta_6\theta_8', \\
\theta_{17} &= 5\theta_{10}\theta_6' - 3\theta_6\theta_{10}', \\
\theta_{19} &= 5\theta_{10}\theta_8' - 4\theta_8\theta_{10}',
\end{aligned}\right. \tag{74}$$

from which still others can be derived. Of all of the invariants found so far

$$\theta_4,\ \theta_6,\ \theta_{10},\ \theta_{15}$$

are the only ones which involve no higher derivatives of p_{ik} than the third, and no higher derivatives of q_{ik} than the second, or what amounts to the same thing, these are the only invariants found so far which depend only on the seminvariants

$$I,\ I',\ I'';\ J,\ J',\ J'';\ K,\ K';\ L. \tag{75}$$

Moreover the invariants (74) are not all independent. For instance we have the syzygy

$$2\theta_4\theta_{17} + 5\theta_{10}\theta_{11} - 6\theta_6\theta_{15} = 0\,. \tag{76}$$

In order to obtain *all* of the invariants depending only upon the seminvariants (75), we write down the partial differential equations which such invariants must verify.

From the expressions for δI, $\delta I'$, $\delta I''$, δJ, etc., δL, we find that these partial differential equations are five in number, viz.:

$$\left\{\begin{aligned}
Y_1 f &= 2I\frac{\partial f}{\partial I} + 3I'\frac{\partial f}{\partial I'} + 4I''\frac{\partial f}{\partial I''} + 4J\frac{\partial f}{\partial J} + 5J'\frac{\partial f}{\partial J'} + 6J''\frac{\partial f}{\partial J''} \\
&\qquad + 6K\frac{\partial f}{\partial K} + 7K'\frac{\partial f}{\partial K'} + 8L\frac{\partial f}{\partial L} = 0\,, \\
Y_2 f &= -2I\frac{\partial f}{\partial I'} - 5I'\frac{\partial f}{\partial I''} - 4J\frac{\partial f}{\partial J'} - 9J'\frac{\partial f}{\partial J''} - 8J'\frac{\partial f}{\partial K} \\
&\qquad - (8J'' + 6K)\frac{\partial f}{\partial K'} - 20K'\frac{\partial f}{\partial L} = 0\,, \\
Y_3 f &= 4\frac{\partial f}{\partial I} - 2I\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J} + 2I'\frac{\partial f}{\partial J'} + 2(I'' - 2J)\frac{\partial f}{\partial J''} \\
&\qquad - 8J'\frac{\partial f}{\partial K'} - 16(2J'' - K)\frac{\partial f}{\partial L} = 0\,, \\
Y_4 f &= 4\frac{\partial f}{\partial I'} + 2I\frac{\partial f}{\partial J'} + 4I'\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K} + 8I''\frac{\partial f}{\partial K'} = 0\,, \\
Y_5 f &= 4\frac{\partial f}{\partial I''} + 2I\frac{\partial f}{\partial J''} + 8I'\frac{\partial f}{\partial K'} + 32I''\frac{\partial f}{\partial L} = 0\,,
\end{aligned}\right. \tag{77}$$

where $Y_i f = 0$ is obtained by equating to zero the coefficient of $\phi^{(i)}$ in δf.

They are independent and, therefore, have $9 - 5 = 4$ independent solutions, i. e., there are four independent absolute, or five independent relative invariants. We have already found four of these, viz.: θ_4, θ_6, θ_{10}, θ_{15}. The fifth invariant, found by integrating (77), is

$$\theta_{22} = \theta_{10}[(I^2 - 4J)(L - 4I''^2) + 4(II'' - 2J'' + K)^2] - \theta_{10}'^2\,.$$

But this can be shown to be divisible by θ_4, so that we may complete our system of invariants by taking θ_{22}/θ_4, which is

$$\begin{aligned}
\theta_{18} = {} & \theta_{10}(L - 4I''^2) + 4(K - I'^2)(II'' - 2J'' + K)^2 - \theta_4(K' - 2I'I'')^2 \\
& - 2\theta_4'(K' - 2I'I'')(II'' - 2J'' + K).
\end{aligned} \tag{78}$$

§ 6. *Canonical forms of a system of two linear differential equations of the second order.*

Our system of equations (54) can always be transformed into another which contains no first derivatives, by a transformation of the form

$$y_i = a_{i1}\eta_1 + a_{i2}\eta_2 \qquad (i=1, 2),$$

where a_{ik} are functions of x. For, on making this transformation, we find

$$(79)\quad \begin{cases} a_{i1}\eta_1'' + a_{i2}\eta_2'' + (2a_{i1}' + p_{i1}a_{11} + p_{i2}a_{21})\eta_1' + (2a_{i2}' + p_{i1}a_{12} + p_{i2}a_{22})\eta_2' \\ \qquad + (a_{i1}'' + p_{i1}a_{11}' + p_{i2}a_{21}' + q_{i1}a_{11} + q_{i2}a_{21})\eta_1 \\ \qquad + (a_{i2}'' + p_{i1}a_{12}' + p_{i2}a_{22}' + q_{i1}a_{12} + q_{i2}a_{22})\eta_2 = 0 \qquad (i=1, 2), \end{cases}$$

so that, if we take a_{ik} subject to the conditions

$$(80)\quad \begin{cases} a_{11}' = -\tfrac{1}{2}(p_{11}a_{11} + p_{12}a_{21}), \quad a_{12}' = -\tfrac{1}{2}(p_{11}a_{12} + p_{12}a_{22}), \\ a_{21}' = -\tfrac{1}{2}(p_{21}a_{11} + p_{22}a_{21}), \quad a_{22}' = -\tfrac{1}{2}(p_{21}a_{12} + p_{22}a_{22}), \end{cases}$$

equations (79) will contain no terms in η_1' and η_2'. The functions (a_{11}, a_{21}) and (a_{12}, a_{22}) must therefore be taken as solutions of the same system of linear differential equations, viz.,

$$(81)\quad \begin{cases} \alpha' = -\tfrac{1}{2}(p_{11}\alpha + p_{12}\beta), \\ \beta' = -\tfrac{1}{2}(p_{21}\alpha + p_{22}\beta). \end{cases}$$

Moreover (a_{11}, a_{21}) and (a_{12}, a_{22}) must be independent solutions of (81), for otherwise the determinant

$$a_{11}a_{22} - a_{12}a_{21}$$

would vanish. The functions a_{ik} can therefore be determined so as to effect the desired result, by integrating (81).

If one makes use of (80) and the equations obtained from (80) by differentiation the transformed system (79) becomes

$$(82)\quad \begin{cases} a_{11}\eta_1'' + a_{12}\eta_2'' = \tfrac{1}{4}(a_{11}u_{11} + a_{21}u_{12})\eta_1 + \tfrac{1}{4}(a_{12}u_{11} + a_{22}u_{12})\eta_2, \\ a_{21}\eta_1'' + a_{22}\eta_2'' = \tfrac{1}{4}(a_{11}u_{21} + a_{21}u_{22})\eta_1 + \tfrac{1}{4}(a_{12}u_{21} + a_{22}u_{22})\eta_2, \end{cases}$$

where the quantities u_{ik} have been previously defined.

Thus every binary system of homogeneous linear differential equations of the second order can be converted into another, involving no first derivatives, i. e., *into one for which* $p_{ik} = 0$. We will say that this transformed system has the *semi-canonical* form.

Suppose we have reduced a system to its semi-canonical form. *Let us find the most general sub-group G' of our general infinite continuous group G, which leaves the semi-canonical form of the system unaltered.*

In the semi-canonical form $p_{ik} = 0$. Let us make the most general infinitesimal transformation of the general group G. The coefficients of the first derivatives in the transformed system will be, according to (16) and (58),

$$\begin{cases} \delta p_{11} = (2\phi'_{11} - \phi'')\delta t, & \delta p_{12} = 2\phi'_{12}\delta t, \\ \delta p_{21} = 2\phi'_{21}\delta t, & \delta p_{22} = (2\phi'_{22} - \phi'')\delta t, \end{cases}$$

for the infinitesimal transformations of the coefficients due to infinitesimal transformations of both dependent and independent variables are equal to those due to the transformations of the dependent variables alone plus those due to the transformation of the independent variable alone.*

But for all of the transformations of our sub-group G', these quantities must vanish. Therefore must

$$\phi'_{12} = \phi'_{21} = 0, \quad 2\phi'_{11} - \phi'' = 0, \quad 2\phi'_{22} - \phi'' = 0,$$

or, if we denote by c_{ik} arbitrary constants,

$$(83) \qquad \begin{cases} \phi_{11} = c_{11} + \tfrac{1}{2}\phi', & \phi_{12} = c_{12}, \\ \phi_{21} = c_{21}, & \phi_{22} = c_{22} + \tfrac{1}{2}\phi', \end{cases}$$

so that the sub-group G' depends upon 4 arbitrary parameters c_{ik}, and an arbitrary function $\phi'(x)$.

We can determine *a sub-group G'' of the subgroup G', namely, that which leaves $q_{11} + q_{22}$ invariant.*

We have, in the same way as above, according to (16) and (58),

$$\delta(q_{11} + q_{22}) = [2\phi'(q_{11} + q_{22}) + \phi^{(2)}_{11} + \phi^{(2)}_{22}]\delta t,$$

so that for this sub-group we have the further relation

$$(84) \qquad \phi^{(2)}_{11} + \phi^{(2)}_{22} = -2\phi'(q_{11} + q_{22}),$$

or, using (83),

$$(85) \qquad \phi^{(2)}_{11} + \phi^{(2)}_{22} + 2[c_{11} + c_{22} - (\phi_{11} + \phi_{22})](q_{11} + q_{22}) = 0,$$

a linear differential equation of the second order for $\phi_{11} + \phi_{22}$. Its integration introduces two arbitrary constants; then ϕ' is found from (84), and the quadrature

$$\phi = \int \phi' dx$$

gives rise to a third constant. *Thus G'' is a seven-parameter group.* We can also represent G'' by (83), together with

$$(86) \qquad -\phi^{(3)} = 2\phi'(q_{11} + q_{22}),$$

* We have changed the sign of $\varphi(x)$, so as to make the infinitesimal transformation of the independent variable harmonize with those of the dependent variables.

an equation, whose integration gives rise to three constants. These together with the four constants c_{ik} are the parameters of the group.

The semi-canonical form of the system can be further simplified by the reduction to what we shall call the *canonical* form.

Suppose we have reduced our system by a first transformation to the semi-canonical form

$$y_i'' + q_{i1}y_1 + q_{i2}y_2 = 0 \qquad (i=1,2). \tag{87}$$

If now we make the transformation

$$\xi = \xi(x), \quad y_i = \beta_{i1}\eta_1 + \beta_{i2}\eta_2 \qquad (i=1,2),$$

we find

$$\begin{aligned}\beta_{i1}\xi'^2\frac{d^2\eta_1}{d\xi^2} + \beta_{i2}\xi'^2\frac{d^2\eta_2}{d\xi^2} + (\beta_{i1}\xi'' + 2\beta_{i1}'\xi')\frac{d\eta_1}{d\xi} + (\beta_{i2}\xi'' + 2\beta_{i2}'\xi')\frac{d\eta_2}{d\xi} \\ + (\beta_{i1}'' + q_{i1}\beta_{11} + q_{i2}\beta_{21})\eta_1 + (\beta_{i2}'' + q_{i1}\beta_{12} + q_{i2}\beta_{22})\eta_2 = 0 \qquad (i=1,2),\end{aligned} \tag{88}$$

which is again in the semi-canonical form if

$$\beta_{ik}\xi'' + 2\beta_{ik}'\xi' = 0,$$

or

$$\beta_{ik} = \frac{e_{ik}}{\sqrt{\xi'}},$$

where e_{ik} are arbitrary constants, whose determinant does not vanish. Put

$$e_{11} = e_{22} = 1, \quad e_{12} = e_{21} = 0,$$

or

$$\beta_{11} = \beta_{22} = \beta = \frac{1}{\sqrt{\xi'}}, \quad \beta_{12} = \beta_{21} = 0. \tag{89}$$

Then (88) becomes

$$\begin{cases}\beta\xi'^2\dfrac{d^2\eta_1}{d\xi^2} + (\beta'' + q_{11}\beta)\eta_1 + q_{12}\beta\eta_2 = 0, \\ \beta\xi'^2\dfrac{d^2\eta_2}{d\xi^2} + q_{21}\beta\eta_1 + (\beta'' + q_{22}\beta)\eta_2 = 0,\end{cases} \tag{90}$$

or

$$\frac{d^2\eta_i}{d\xi^2} + \rho_{i1}\eta_1 + \rho_{i2}\eta_2 = 0 \qquad (i=1,2). \tag{91}$$

Now β can be determined that in (91), $\rho_{11} + \rho_{22} = 0$; for this purpose it is only necessary to take for β a solution of the linear differential equation

$$2\beta'' + (q_{11} + q_{22})\beta = 0. \tag{92}$$

We have proved the following theorem:

Every system of linear, homogeneous differential equations of the second order can be converted into a system having the canonical form

$$\frac{d^2\eta_i}{d\xi^2} + \rho_{i1}\eta_1 + \rho_{i2}\eta_2 \qquad (i=1,2),$$

where

$$\rho_{11} + \rho_{22} = 0\,.$$

In order to effect this reduction, it is necessary to integrate a system of homogeneous linear differential equations of the first order (81), *a single homogeneous linear differential equation of the second order* (92), *and finally to effect the quadrature*

$$\xi(x) = \int \frac{dx}{\beta^2}\,.$$

This canonical form of the system corresponds to the LAGUERRE-FORSYTH form of a single linear differential equation.

The sub-group of G which leaves this canonical form unaltered is especially simple. It is the group G'' for the particular case that $q_{11} + q_{22} = 0$, whence, according to (86),

$$\phi^{(3)} = 0\,, \quad \text{or} \quad \phi = \lambda + \mu x + \nu x^2\,,$$

and from (83),

$$\begin{cases} \phi_{11} = c_{11} + \frac{1}{2}\mu + \nu x\,, & \phi_{12} = c_{12}\,, \\ \phi_{21} = c_{21}\,, & \phi_{22} = c_{22} + \frac{1}{2}\mu + \nu x\,. \end{cases}$$

The finite transformations of this group are

$$\xi = \frac{ax+\beta}{\gamma x+\delta}\,, \quad \eta_i = \frac{C_{i1}y_1 + C_{i2}y_2}{\gamma x+\delta} \qquad (i=1,2)\,. \tag{93}$$

The functions called, by Forsyth, *quadriderivatives* are invariants for this sub-group. If θ_σ is an invariant of weight σ, then

$$\theta_{\sigma,1} = 2\sigma\theta_\sigma\theta''_\sigma - (2\sigma+1)\theta'^2_\sigma \tag{94}$$

is what FORSYTH calls the quadriderivative of θ_σ.

We find that for the general infinitesimal transformations of the general group G,

$$\delta\theta_{\sigma,1} = -2\sigma^2\theta^2_\sigma\phi^{(3)}\delta t - 2(\sigma+1)\phi'\theta_{\sigma,1}\delta t\,.$$

For our sub-group $\phi^{(3)} = 0$, so that $\theta_{\sigma,1}$ is indeed an invariant of the sub-group. Its weight is $2\sigma + 2$.

§ 7. *Covariants.*

We shall not treat of covariants in an exhaustive manner in this paper. It will be sufficient to note a few simple theorems.

First we may observe that *no absolute covariant exists which depends only upon* y_1, y_2, dy_1/dx, dy_2/dx, *etc., i. e., which does not contain* x *explicitly, and makes no use of the fact that* y_1 *and* y_2 *are solutions of* (54).

For we have, taking infinitesimal transformations of the dependent variables only,

$$\begin{cases} \delta y_i = (\phi_{i1} y_1 + \phi_{i2} y_2)\delta t\,, \\ \delta y_i' = (\phi_{i1} y_1' + \phi_{i2} y_2' + \phi_{i1}' y_1 + \phi_{i2}' y_2)\delta t\,, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \delta y_i^{(m-1)} = (\phi_{i1} y_1^{(m-1)} + \phi_{i2} y_2^{(m-1)} + \cdots + \phi_{i2}^{(m-1)} y_2)\delta t\,. \end{cases}$$

But this group in $2m$ variables is always transitive. For if we take the special case obtained by putting ϕ_{ik}, ϕ_{ik}', ϕ_{ik}'', etc., equal to arbitrary constants, the group becomes a finite $4m$ parameter group in $2m$ variables which is a particular case of a general group of linear substitutions with mn^2 parameters and mn variables. This general group has been studied by the author and was found to be transitive.* Therefore the more general infinite group is also transitive, i. e., it has no invariants, and thus the theorem is proved.

It is even impossible to construct functions z_1 *and* z_2 *of* y_1, y_2 *and of their successive derivatives, not involving* x *explicitly, which shall be cogredient with* y_1 *and* y_2.

For, if it were not so, from z_1 and z_2 could then be constructed a third cogredient set, say η_1 and η_2, and the quotient

$$\frac{y_1 z_2 - y_2 z_1}{y_1 \eta_2 - y_2 \eta_1}$$

would be an absolute covariant of the form whose non-existence we have just shown.

An infinite number of covariants, containing x explicitly, can be constructed as follows. Put

$$\begin{cases} \bar{y}_i = y_i'' + p_{i1} y_1' + p_{i2} y_2' + q_{i1} y_1 + q_{i2} y_2, \\ \bar{\bar{y}}_i = \bar{y}_i'' + p_{i1} \bar{y}_1' + p_{i2} \bar{y}_2' + q_{i1} \bar{y}_1 + q_{i2} \bar{y}_2, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

Then $\bar{y}_i$, $\bar{\bar{y}}_i$, etc., are cogredient with y_i. Therefore the determinants

$$y_1 \bar{y}_2 - y_2 \bar{y}_1, \quad y_1 \bar{\bar{y}}_2 - y_2 \bar{\bar{y}}_1, \text{ etc.},$$

* Proceedings of the California Academy of Sciences, vol 1, no. 6.

are covariants. All of these vanish in consequence of the given system of differential equations.

Their covariant property is expressed by an equation of the form

$$C(\eta_1, \eta_2, \xi) = \frac{a_{11}a_{22} - a_{12}a_{21}}{\xi'(x)^2} C(y_1, y_2, x),$$

where

$$\eta_i = a_{i1}y_1 + a_{i2}y_2, \quad \xi = \xi(x) \qquad (i = 1, 2),$$

is the transformation of the variables.

Suppose we have reduced our given system to its canonical form. Then

$$y_1y_2' - y_2y_1'$$

is a covariant of the sub-group (93) which leaves this canonical form unchanged.

To find all of the covariants of the system we could proceed as in the case of the invariants by setting up the system of partial differential equations which they satisfy. We might also construct functions cogredient with y_1 and y_2. The determinant of two such systems would be a covariant.

UNIVERSITY OF CALIFORNIA, BERKELEY,
October 6, 1900.

DIVERGENT AND CONDITIONALLY CONVERGENT SERIES WHOSE PRODUCT IS ABSOLUTELY CONVERGENT*

BY

FLORIAN CAJORI

§ 1. *Introduction.*

It has been shown by ABEL that, if the product :

$$\sum_{n=0}^{n=\infty}(u_0v_n + u_1v_{n-1} + \cdots + u_nv_0),$$

of two conditionally convergent series :

$$\sum_{n=0}^{n=\infty}u_n, \qquad \sum_{n=0}^{n=\infty}v_n,$$

is convergent, it converges to the product of their sums. Tests of the convergence of the product of conditionally convergent series have been worked out by A. PRINGSHEIM,† A. VOSS,‡ and myself.§ There exist certain conditionally-convergent series which yield a convergent result when they are raised to a certain positive integral power, but which yield a divergent result when they are raised to a higher power. Thus,

$$\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{1}{n^r},$$

where $r = 7/9$, is a conditionally convergent series whose fourth power is convergent, but whose fifth power is divergent.‖ These instances of conditionally

* Presented to the Society April 28, 1900. Received for publication April 28, 1900.

† Mathematische Annalen, vol. 21 (1883), p. 327 ; vol. 26 (1886), p. 157.

‡ Mathematische Annalen, vol. 24 (1884), p. 42.

§ American Journal of Mathematics, vol. 15 (1893), p. 339 ; vol. 18 (1896), p. 195 ; Bulletin of the American Mathematical Society, (2) vol. 1 (1895), p. 180.

‖ This may be a convenient place to point out a slight and obvious extension of the results which I have published in the American Journal of Mathematics, vol. 18, p. 201. It was proved there that the conditionally convergent series :

$$\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{1}{n^r}, \qquad (0 < r \leqq 1),$$

when raised by Cauchy's multiplication rule to a positive integral power q, is convergent whenever $(q-1)/q < r$; but the power of the series is divergent, if $(q-1)/q > r$. The case for which $(q-1)/q = r$ was left untouched, but an examination of the formulæ appearing in that article readily yields the result that in this case the series is divergent. Hence the condition for divergency should be written $(q-1)/q \geqq r$.

convergent series, yielding powers of the series which converge less rapidly the higher the power and which for a sufficiently high power yield a divergent result, suggest the following questions: *Does the product of two conditionally convergent series never converge more rapidly than do one or both of the factor-series? Can the product of two conditionally convergent series or of a conditionally convergent and a divergent series in no case be absolutely convergent?*

The first doubt of the correctness of a negative reply arose in connection with the conditionally convergent series:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n}.$$

This yields convergent results, no matter to how high a positive integral power it may be raised.* If the series is squared, the constituents, $1/(n-r)(r+1)$, which go to make up the nth term of the product, namely,

$$\sum_{r=0}^{r=n-1} \frac{1}{n-r} \frac{1}{r+1},$$

are all positive or all negative for that term. Hence the numerical value of the nth term is the sum of the numerical values of all the products $1/(n-r)(r+1)$. If the square of the series is indicated by

$$\sum_{n=1}^{n=\infty} a_n,$$

then all the constituents, $a_{n-r} \cdot 1/(r+1)$, which enter into the composition of the nth term in the cube of the series have all like signs in that term. Similarly for higher powers. Now it has been shown that all the positive integral powers of the conditionally convergent series:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n}$$

continue to converge, even though each term of a given power of the series is numerically the sum of the numerical values of its constituents, and it is evident that, if the constituents of each term were partly positive and partly negative, each term would be numerically smaller. *Can the signs of the terms of two conditionally convergent series and the numerical values of its terms be so chosen that the product of the two series is absolutely convergent?*

In an important article on the multiplication of conditionally convergent series† A. Pringsheim states that he knows no *a priori* reason why there might not be special cases in which such a product is absolutely convergent. "Aller-

* American Journal of Mathematics, vol. 18, p. 204.

† Mathematische Annalen, vol. 21, p. 332.

dings," he continues, "war es mir bisher nicht möglich an irgend einem bestimmten derartigen Falle die unbedingte Convergenz zu constatiren, und es dürfte dies — wenn es überhaupt solche Fälle giebt, was ja immerhin als fraglich erscheinen muss — wegen der zusammengesetzten Beschaffenheit der Reihenglieder w_ν eben nur sehr schwer gelingen."

§ 2. *Two typical examples.*

We have succeeded in devising special cases in which an absolutely convergent series is obtained as the result of multiplying two conditionally convergent series together, or one conditionally convergent series by a divergent series. The series obtained by removing the parentheses from the series:

$$S_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{4p+1}-\frac{1}{4p+4}+\frac{1}{4p+1}-\frac{1}{4p+4}\right),$$

$$S_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{4p+1}+\frac{1}{4p+4}-\frac{1}{4p+1}-\frac{1}{4p+4}\right),$$

are both conditionally convergent. Let the four terms obtained in either series, or the terms in a similar series, for any one positive integral value of p, be called a "set" of terms. The $4n$-th term of the product of S_1 and S_2 is

$$\sum_{p=0}^{p=n-1}\left(-\frac{1}{4n-4p}\,\frac{1}{4p+1}+\frac{1}{4n-4p-3}\,\frac{1}{4p+4}+\frac{1}{4n-4p}\,\frac{1}{4p+1}-\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\right).$$

We see that the sum of each set of constituents for the $4n$-th term is identically zero; therefore the entire $4n$-th term is identically zero.

The $(4n+1)$-th term of the product series is as follows:

$$\sum_{p=0}^{p=n'}\left(\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p}\,\frac{1}{4p+4}-\frac{1}{4n-4p-3}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\right)$$
$$+\sum_{p=0}^{p=n'}\left(\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}-\frac{1}{4n-4p-3}\,\frac{1}{4p+1}-\frac{1}{4n-4p}\,\frac{1}{4p+4}\right)-a,$$

where $n'=\frac{1}{2}n-1$ or $\frac{1}{2}(n-1)$, according as n is even or odd. When n is odd,

$$a=\frac{1}{2n+2}\,\frac{1}{2n+2}-\frac{1}{2n-1}\,\frac{1}{2n-1}-\frac{1}{2n+2}\,\frac{1}{2n+2}=-\frac{1}{2n-1}\,\frac{1}{2n-1};$$

and when n is even,

$$a = -\frac{1}{2n+1}\,\frac{1}{2n+1}.$$

In the first summation we are adding $4n'+4$ constituents from one end of the column or series yielding the $(4n+1)$-th term of the product; in the second summation we are adding $4n'+4$ constituents from the other end. In the middle of the series the sets of constituents will overlap, if n is odd, and will not touch if n is even. When n is odd, then a represents the sum of the constituents which have been counted twice; when n is even, then a stands for the negative value of the constituent not counted at all. Each of these constituents is of the degree -2 with respect to n and as their number is finite, their sum cannot be of a higher degree with respect to n than the degree -2.

The $(4n+1)$-th term, simplified, is

$$2\sum_{p=0}^{p=n'}\left(\frac{1}{4p+1}\,\frac{-4}{(4n-4p+1)(4n-4p-3)}\right)-a.$$

For any positive integral value of n the fraction $4/(4n-4p+1)(4n-4p-3)$ has its maximum value when $p=n'$. Writing n' for p, we obtain a fraction which is less than $1/n^2$. Hence for any value of p,

$$\frac{1}{4p+1}\,\frac{4}{(4n-4p+1)(4n-4p-3)} < \frac{1}{4p+1}\,\frac{1}{n^2}.$$

It is easy to see that

$$\sum_{n=1}^{2n}\frac{1}{n} < a\log(2n),$$

where a is a constant greater than $3/2\log 2$. Hence

$$\sum_{p=0}^{p=n'}\frac{1}{4p+1} < a\log(2n),$$

and

$$\sum_{p=0}^{p=n'}\frac{1}{4p+1}\,\frac{4}{(4n-4p+1)(4n-4p-3)} < \sum_{p=0}^{p=n'}\frac{1}{4p+1}\,\frac{1}{n^2} < \frac{a\log(2n)}{n^2}.$$

But a number m can be found, such that, when $n>m$, we have

$$\frac{a\log(2n)}{n^2} < \frac{1}{n(\log n)^{\theta}},$$

where $\theta>1$. Therefore, for $n>m$, the $(4n+1)$-th term is numerically less than the nth term of a series known to be absolutely convergent.

The $(4n+2)$-th term of the product-series is

$$\sum_{p=0}^{p=n'}\Big(-\frac{1}{4n-4p+4}\,\frac{1}{4p+1}+\frac{1}{4n-4p+1}\,\frac{1}{4p+4}+\frac{1}{4n-4p}\,\frac{1}{4p+1}-\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\Big)$$
$$+\sum_{p=0}^{p=n'}\Big(+\frac{1}{4n-4p+4}\,\frac{1}{4p+1}-\frac{1}{4n-4p+1}\,\frac{1}{4p+4}-\frac{1}{4n-4p}\,\frac{1}{4p+1}+\frac{1}{4n-4p-3}\,\frac{1}{4p+4}\Big)-a,$$

where n' is defined as before and where

$$a=\frac{1}{2n+2}\,\frac{1}{2n-1}-\frac{1}{2n-1}\,\frac{1}{2n+2}=0\,,$$

when n is odd, and

$$a=+\frac{1}{2n+4}\,\frac{1}{2n+1}-\frac{1}{2n+1}\,\frac{1}{2n+4}=0\,,$$

when n is even. We see that the entire $(4n+2)$-th term is identically equal to zero.

The $(4n+3)$-th term of the product-series is

$$\sum_{p=0}^{p=n'}\Big(+\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p+4}\,\frac{1}{4p+4}-\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\Big)$$
$$+\sum_{p=0}^{p=n'}\Big(-\frac{1}{4n-4p+1}\,\frac{1}{4p+1}-\frac{1}{4n-4p+4}\,\frac{1}{4p+4}+\frac{1}{4n-4p+1}\,\frac{1}{4p+1}+\frac{1}{4n-4p}\,\frac{1}{4p+4}\Big)-a$$
$$=2\sum_{p=0}^{p=n'}\left(\frac{1}{4p+4}\,\frac{4}{(4n-4p+4)(4n-4p)}\right)-a\,,$$

where

$$a=\frac{1}{2n+2}\,\frac{1}{2n+2}\,,$$

when n is odd, and

$$a=-\frac{1}{2n+1}\,\frac{1}{2n+1}+\frac{1}{2n+4}\,\frac{1}{2n+4}+\frac{1}{2n+1}\,\frac{1}{2n+1}\,,$$

when n is even. The reasoning which we applied to the $(4n+1)$-th term will show that this term, too, when $n>m$, is numerically less than $1/n(\log n)^\theta$, $\theta>1$.

We have now shown that the $4n$-th term and the three immediately succeeding terms are either identically zero, or, for $n>m$, numerically less than $1/n(\log n)^\theta$. Hence the product of the series S_1 and S_2 converges absolutely.

Again, take the conditionally convergent series, obtained by dropping the parentheses in the series:

$$T_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}-\frac{1}{6p+2}+\frac{1}{6p+3}-\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right)$$

and the divergent series, obtained by dropping the parentheses in the series:

$$T_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}+\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right).$$

The $6n$-th term of their product, viz.,

$$\sum_{p=0}^{p=n-1}\left(-\frac{1}{6n-6p+3}\frac{1}{6p+1}+\frac{1}{6n-6p+2}\frac{1}{6p+2}+\frac{1}{6n-6p+1}\frac{1}{6p+3}\right.$$
$$\left.+\frac{1}{6n-6p+3}\frac{1}{6p+1}-\frac{1}{6n-6p+2}\frac{1}{6p+2}-\frac{1}{6n-6p+1}\frac{1}{6p+3}\right),$$

is seen to be identically zero.

The $(6n+1)$-th term of the product of T_1 and T_2 is

$$\sum_{p=0}^{p=n'}\left(+\frac{1}{6n-6p+1}\frac{1}{6p+1}-\frac{1}{6n-6p-3}\frac{1}{6p+2}-\frac{1}{6n-6p-4}\frac{1}{6p+3}\right.$$
$$\left.-\frac{1}{6n-6p-5}\frac{1}{6p+1}+\frac{1}{6n-6p-3}\frac{1}{6p+2}+\frac{1}{6n-6p-4}\frac{1}{6p+3}\right)$$
$$+\sum_{p=0}^{p=n'}\left(+\frac{1}{6n-6p+1}\frac{1}{6p+1}+\frac{1}{6n-6p-3}\frac{1}{6p+2}+\frac{1}{6n-6p-4}\frac{1}{6p+3}\right.$$
$$\left.-\frac{1}{6n-6p-5}\frac{1}{6p+1}-\frac{1}{6n-6p-3}\frac{1}{6p+2}-\frac{1}{6n-6p-4}\frac{1}{6p+3}\right)-a$$
$$=2\sum_{p=0}^{p=n'}\left(\frac{1}{6p+1}\frac{-6}{(6n-6p+1)(6n-6p-5)}\right)-a,$$

where n' is defined as before and where

$$a = -\frac{1}{3n}\frac{1}{3n-1} - \frac{1}{3n-1}\frac{1}{3n} - \frac{1}{3n-2}\frac{1}{3n-2} + \frac{1}{3n}\frac{1}{3n-1} + \frac{1}{3n-1}\frac{1}{3n}$$

$$= -\frac{1}{3n-2}\frac{1}{3n-2},$$

when n is odd, and $a = -1/(3n+1)(3n+1)$, when n is even. By repeating the reasoning used in previous cases we find that this term, for $n > m$, is numerically less than $1/n(\log n)^{\theta}$, $\theta > 1$.

The $(6n+3)$-th, $(6n+4)$-th, and $(6n+5)$-th terms in the product of T_1 and T_2, when subjected to this course of reasoning, all yield the same conclusion. The $(6n+2)$-th term is identically zero. Hence the product-series is absolutely convergent.

§ 3. *The general method.*

The method by which the above factor-series were constructed is simple. The harmonic series was modified by adopting some simple law of succession of signs. To render the series conditionally convergent, make the number of + terms in a set equal to the number of − terms. If these numbers are unequal, then the series is divergent. Then add suitable finite numbers to the denominators of some of the terms in the set, so that, when the product of the two series is formed, the constituents of any set of the nth term in the product, when added, yield a fractional sum having denominators whose degree in n is higher by two than the degree of the numerators. We were able to prove the absolute convergence from the fact that in each complete set of constitutents of the nth term in the product-series there are as many positive constituents as there are negative ones. To secure this even distribution of signs among the constituents for the sets in every term of the product, it is necessary that in one of the factor-series the number of + terms in any set be equal to the number of − terms in that set. This condition is not sufficient, because among the terms in the product some might have a preponderance of + constituents and others a preponderance of − constituents. This happens, for instance, when we multiply the series + + − − + + − − etc. by itself.

In the proper selection of signs the following rule, which we give without proof, may be of service. Let a, denoting the number of terms in each set of the series $\sum u_m$, be an even number and equal to $2c$; also let the first c terms in each set be +, the remaining terms −. Let the number of terms in each set of the second series, $\sum v_m$, be b. Then, in order to secure as many + constituents in each set of a term of the product of the two series $\sum u_m$ and $\sum v_m$, as there are negative constituents, it is sufficient that d should be a factor of c, where d is the smallest value, numerically, satisfying the congruence $b \equiv d \pmod{a}$.

In explaining the further process of constructing our series we shall start out with a particular selection of signs, but the process is the same as for any other selection, similarly made. Let $a=4$, $b=3$, then $c=2$ and $d=-1$, a factor of c. Accordingly, the signs of terms in each set of the first series are $++--$. Let the signs of the b terms in each set of $\sum v_m$ be $++-$. We have now the two factor-series obtained by dropping the parentheses from the series on the right hand sides of the formulas:

$$\sum u_m \equiv \sum_{p=1}^{p=\infty}\left(\frac{1}{4p+a_1}+\frac{1}{4p+1+a_2}-\frac{1}{4p+2+a_3}-\frac{1}{4p+3+a_4}\right)$$

and

$$\sum v_m \equiv \sum_{p=1}^{p=\infty}\left(\frac{1}{3p+\beta_1}+\frac{1}{3p+1+\beta_2}-\frac{1}{3p+2+\beta_3}\right),$$

in which the Greek letters are quantities to be chosen, if possible, so that the product of the two series shall be absolutely convergent.

For brevity, we shall, in the series $\sum u_m$, represent by $\{u_{ap}+a\}$ a fraction $1/(ap+a)$, and, similarly, in the series $\sum v_m$, by $\{v_{bp}+\beta\}$ a fraction $1/(bp+\beta)$. Accordingly we have, dropping the parentheses as before,

$$\sum u_m \equiv \sum_{p=1}^{p=\infty}(\{u_{4p}+a_1\}+\{u_{4p}+1+a_2\}-\{u_{4p}+2+a_3\}-\{u_{4p}+3+a_4\})$$

and

$$\sum v_m \equiv \sum_{p=1}^{p=\infty}(\{v_{3p}+\beta_1\}+\{v_{3p}+1+\beta_2\}-\{v_{3p}+2+\beta_3\}).$$

In the product of $\sum u_m$ and $\sum v_m$, a set will consist of 12 constituents. The *first* set in the $(12n)$-th term will be (since $\{v_{12}-9+\beta_1\}\equiv\{v_3+\beta_1\}$, and similarly in other cases)

$$\text{(I)}\quad\begin{cases}
-\{u_{12n}+3+a_4''\}\{v_{12}-9+\beta_1\}-\{u_{12n}+2+a_3''\}\{v_{12}-8+\beta_2\}\\
-\{u_{12n}+1+a_2''\}\{v_{12}-7+\beta_3\}+\{u_{12n}+a_1''\}\{v_{12}-6+\beta_1'\}\\
-\{u_{12n}-1+a_4'\}\{v_{12}-5+\beta_2'\}+\{u_{12n}-2+a_3'\}\{v_{12}-4+\beta_3'\}\\
+\{u_{12n}-3+a_2'\}\{v_{12}-3+\beta_1''\}+\{u_{12n}-4+a_1'\}\{v_{12}-2+\beta_2''\}\\
+\{u_{12n}-5+a_4\}\{v_{12}-1+\beta_3''\}-\{u_{12n}-6+a_3\}\{v_{12}+\beta_1'''\}\\
+\{u_{12n}-7+a_2\}\{v_{12}+1+\beta_2'''\}-\{u_{12n}-8+a_1\}\{v_{12}+2+\beta_3'''\}.
\end{cases}$$

Here each of the quantities β_1, β_2, β_3 occurs four times. We have broadened our assumptions by attaching a single accent to each letter the first time it is repeated and a double and triple accent the second and third time it is repeated. Accordingly β_1, β_1', β_1'', β_1''', β_2, β_2', β_2'', β_2''', β_3, β_3', β_3'', β_3''' may in general represent each a different number. Similarly, the letters a_1, a_2, a_3, a_4 receive a single accent the first time they are repeated in a set of constituents and a double accent the second time they are repeated.

We shall limit ourselves to the case where, in any term of the product series, each set, when multiplied by n, yields a product that converges toward zero when $n=\infty$. Accordingly, if the set I be multiplied by n, we get, for $n=\infty$ (supposing the Greek letters to be positive constants, or variables which approach positive finite limits), the first of the two following equations:

$$\text{(II)}\quad \begin{cases} -\{v_{12}-9+\beta_1\}-\{v_{12}-8+\beta_2\}-\{v_{12}-7+\beta_3\}+\{v_{12}-6+\beta'_1\} \\ -\{v_{12}-5+\beta'_2\}+\{v_{12}-4+\beta'_3\}+\{v_{12}-3+\beta''_1\}+\{v_{12}-2+\beta''_2\} \\ +\{v_{12}-1+\beta''_3\}-\{v_{12}+\beta'''_1\}+\{v_{12}+1+\beta'''_2\}-\{v_{12}+2+\beta'''_3\}=0, \\ +\{v_{12}-9+\beta_1\}-\{v_{12}-8+\beta_2\}+\{v_{12}-7+\beta_3\}+\{v_{12}-6+\beta'_1\} \\ +\{v_{12}-5+\beta'_2\}+\{v_{12}-4+\beta'_3\}-\{v_{12}-3+\beta''_1\}+\{v_{12}-2+\beta''_2\} \\ -\{v_{12}-1+\beta''_3\}-\{v_{12}+\beta'''_1\}-\{v_{12}+1+\beta'''_2\}-\{v_{12}+2+\beta'''_3\}=0. \end{cases}$$

The second equation in II is got from the $(12n+1)$-th term by the same procedure. All the even terms between the $(12n)$-th and the $(12n+12)$-th yield the same equation as the $(12n)$-th term; all the odd terms yield the same equation as the $(12n+1)$-th term.

The *last* set in the $(12n)$-th term of the product is

$$\text{(III)}\quad \begin{cases} -\{v_{12n}+2+\beta'''_3\}\{u_{12}-8+a_1\}+\{v_{12n}+1+\beta'''_2\}\{u_{12}-7+a_2\} \\ -\{v_{12n}-1+\beta'''_1\}\{u_{12}-6+a_3\}+\{v_{12n}-1+\beta''_3\}\{u_{12}-5+a_4\} \\ +\{v_{12n}-2+\beta''_2\}\{u_{12}-4+a'_1\}+\{v_{12n}-3+\beta''_1\}\{u_{12}-3+a'_2\} \\ +\{v_{12n}-4+\beta'_3\}\{u_{12}-2+a'_3\}-\{v_{12n}-5+\beta'_2\}\{u_{12}-1+a'_4\} \\ +\{v_{12n}-6+\beta'_1\}\{u_{12}-a''_1\}-\{v_{12n}-7+\beta_3\}\{u_{12}+1+a''_2\} \\ -\{v_{12n}-8+\beta_2\}\{u_{12}+2+a''_3\}-\{v_{12n}-9+\beta_1\}\{u_{12}+3+a''_4\}. \end{cases}$$

Assuming set III, when multiplied by n, to yield a product which converges toward zero, when $n=\infty$, we get the first of the following three equations:

$$\text{(IV)}\quad \begin{cases} -\{u_{12}-8+a_1\}+\{u_{12}-7+a_2\}-\{u_{12}-6+a_3\}+\{u_{12}-5+a_4\} \\ +\{u_{12}-4+a'_1\}+\{u_{12}-3+a'_2\}+\{u_{12}-2+a'_3\}-\{u_{12}-1+a'_4\} \\ +\{u_{12}+a''_1\}-\{u_{12}+1+a''_2\}-\{u_{12}+2+a''_3\}-\{u_{12}+3+a''_4\}=0, \\ +\{u_{12}-8+a_1\}-\{u_{12}-7+a_2\}-\{u_{12}-6+a_3\}-\{u_{12}-5+a_4\} \\ -\{u_{12}-4+a'_1\}+\{u_{12}-3+a'_2\}-\{u_{12}-2+a'_3\}+\{u_{12}-1+a'_4\} \\ +\{u_{12}+a''_1\}+\{u_{12}+1+a''_2\}+\{u_{12}+2+a''_3\}-\{u_{12}+3+a''_4\}=0, \\ +\{u_{12}-8+a_1\}+\{u_{12}-7+a_2\}+\{u_{12}-6+a_3\}-\{u_{12}-5+a_4\} \\ +\{u_{12}-4+a'_1\}-\{u_{12}-3+a'_2\}-\{u_{12}-2+a'_3\}-\{u_{12}-1+a'_4\} \\ -\{u_{12}+a''_1\}+\{u_{12}+1+a''_2\}-\{u_{12}+2+a''_3\}+\{u_{12}+3+a''_4\}=0. \end{cases}$$

The second and third equations in IV are derived from the last set in the $(12n+1)$-th and $(12n+2)$-th term, respectively. The $(12n+3)$-th,

$(12n+6)$-th, and $(12n+9)$-th terms yield the same equation in IV as the $(12n)$-th term; the $(12n+4)$-th, $(12n+7)$-th, $(12n+10)$-th terms the same as the $(12n+1)$-th term; the $(12n+5)$-th, $(12n+8)$-th, $(12n+11)$-th the same as the $(12n+2)$-th term.

The equations II, when simplified and by the substitution of v_{12p} for v_{12} made to apply to other sets in the terms of the product, are

$$\text{(V)}\quad \begin{cases} -\{v_{12p}-8+\beta_2\}+\{v_{12p}-6+\beta_1'\}+\{v_{12p}-4+\beta_3'\} \\ \qquad +\{v_{12p}-2+\beta_2''\}-\{v_{12p}+\beta_1'''\}-\{v_{12p}+2+\beta_3'''\}=0, \\ \{v_{12p}-9+\beta_1\}+\{v_{12p}-7+\beta_3\}+\{v_{12p}-5+\beta_2'\} \\ \qquad -\{v_{12p}-3+\beta_1''\}-\{v_{12p}-1+\beta_3''\}-\{v_{12p}+1+\beta_2'''\}=0. \end{cases}$$

The equations IV, when simplified, u_{12p} being written for u_{12}, become

$$\text{(VI)}\quad \begin{cases} \{u_{12p}-7+a_2\}+\{u_{12p}-4+a_1'\}-\{u_{12p}-1+a_4'\}-\{u_{12p}+2+a_3''\}=0, \\ \{u_{12p}-8+a_1\}-\{u_{12p}-5+a_4\}-\{u_{12p}-2+a_3'\}+\{u_{12p}+1+a_2''\}=0, \\ \{u_{12p}-6+a_3\}-\{u_{12p}-3+a_2'\}-\{u_{12p}+a_1''\}+\{u_{12p}+3+a_4''\}=0. \end{cases}$$

If we aim to assign to the Greek letters constant values, then, in every equation in V each fraction must admit of being associated singly with another fraction in that equation which is numerically equal to it, but of opposite sign. This relation is brought about by the following six equations:

$$\text{(VII)}\quad \begin{cases} \{v_{12p}-9+\beta_1\}=\{v_{12p}-3+\beta_1''\}, & \{v_{12p}-6+\beta_1'\}=\{v_{12p}+\beta_1'''\}, \\ \{v_{12p}-8+\beta_2\}=\{v_{12p}-2+\beta_2''\}, & \{v_{12p}-5+\beta_2'\}=\{v_{12p}+1+\beta_2''\}, \\ \{v_{12p}-7+\beta_3\}=\{v_{12p}-1+\beta_3''\}, & \{v_{12p}-4+\beta_3'\}=\{v_{12p}+2+\beta_3'''\}. \end{cases}$$

The same considerations applied to equations VI yield the following six relations:

$$\text{(VIII)}\quad \begin{cases} \{u_{12p}-8+a_1\}=\{u_{12p}-2+a_3'\}, & \{u_{12p}-5+a_4\}=\{u_{12p}+1+a_2''\}, \\ \{u_{12p}-7+a_2\}=\{u_{12p}-1+a_4'\}, & \{u_{12p}-4+a_1'\}=\{u_{12p}+2+a_3''\}, \\ \{u_{12p}-6+a_3\}=\{u_{12p}+a_1''\}, & \{u_{12p}-3+a_2'\}=\{u_{12p}+3+a_4''\}. \end{cases}$$

Applying relations VIII, one may write the series $\sum u_m$ thus, the parentheses being suppressed:

$$\text{(IX)}\quad \begin{cases} \sum u_m \equiv \sum_{p=1}^{p=\infty}(\{u_{12p}-8+a_1\}+\{u_{12p}-7+a_2\}-\{u_{12p}-6+a_3\} \\ \qquad -\{u_{12p}-5+a_4\}+\{u_{12p}-4+a_1'\}+\{u_{12p}-3+a_2'\} \\ \qquad -\{u_{12p}-8+a_1\}-\{u_{12p}-7+a_2\}+\{u_{12p}-6+a_3\} \\ \qquad +\{u_{12p}-5+a_4\}-\{u_{12p}-4+a_1'\}-\{u_{12p}-3+a_2'\}). \end{cases}$$

Applying relations VII, one may write the series $\sum v_m$ thus, the parentheses being suppressed:

$$(X)\quad \begin{cases} \sum v_m \equiv \sum_{p=1}^{p=\infty} (\{v_{12p}-9+\beta_1\}+\{v_{12p}-8+\beta_2\}-\{v_{12p}-7+\beta_3\} \\ \qquad +\{v_{12p}-6+\beta_1'\}+\{v_{12p}-5+\beta_2'\}-\{v_{12p}-4+\beta_3'\} \\ \qquad +\{v_{12p}-9+\beta_1\}+\{v_{12p}-8+\beta_2\}-\{v_{12p}-7+\beta_3\} \\ \qquad +\{v_{12p}-6+\beta_1'\}+\{v_{12p}-5+\beta_2'\}-\{v_{12p}-4+\beta_3'\}). \end{cases}$$

Series IX is conditionally convergent; series X divergent. In both series any constant positive values may be assigned to the Greek letters. The product of the two series is absolutely convergent. This may be shown by examining the $12n$-th term and the eleven immediately succeeding terms of the product-series in the same way as we did the terms in the product of S_1 and S_2. We shall here write down the expression for the $(12n+1)$-th term only, viz.,

$$\begin{aligned}
&\sum_{p=1}^{p=n'}(+\{u_{12n-12p}+16+\alpha_1\}\{v_{12p}-9+\beta_1\}-\{u_{12n-12p}+9+\alpha_2'\}\{v_{12p}-8+\beta_2\} \\
&\quad +\{u_{12n-12p}+8+\alpha_1'\}\{v_{12p}-7+\beta_3\}+\{u_{12n-12p}+7+\alpha_4\}\{v_{12p}-6+\beta_1'\} \\
&\quad +\{u_{12n-12p}+6+\alpha_3\}\{v_{12p}-5+\beta_2'\}+\{u_{12n-12p}+5+\alpha_2\}\{v_{12p}-4+\beta_3'\} \\
&\quad -\{u_{12n-12p}+4+\alpha_1\}\{v_{12p}-9+\beta_1\}+\{u_{12n-12p}+9+\alpha_2'\}\{v_{12p}-8+\beta_2\} \\
&\quad -\{u_{12n-12p}+8+\alpha_1'\}\{v_{12p}-7+\beta_3\}-\{u_{12n-12p}+7+\alpha_4\}\{v_{12p}-6+\beta_1'\} \\
&\quad -\{u_{12n-12p}+6+\alpha_3\}\{v_{12p}-5+\beta_2'\}-\{u_{12n-12p}+5+\alpha_2\}\{v_{12p}-4+\beta_3'\}) \\
&+\sum_{p=1}^{p=n'}(+\{v_{12n-12p}+15+\beta_1\}\{u_{12p}-8+\alpha_1\}-\{v_{12n-12p}+8+\beta_3'\}\{u_{12p}-7+\alpha_2\} \\
&\quad -\{v_{12n-12p}+7+\beta_2'\}\{u_{12p}-6+\alpha_3\}-\{v_{12n-12p}+6+\beta_1'\}\{u_{12p}-5+\alpha_4\} \\
&\quad -\{v_{12n-12p}+5+\beta_3\}\{u_{12p}-4+\alpha_1'\}+\{v_{12n-12p}+4+\beta_2\}\{u_{12p}-3+\alpha_2'\} \\
&\quad -\{v_{12n-12p}+3+\beta_1\}\{u_{12p}-8+\alpha_1\}+\{v_{12n-12p}+8+\beta_3'\}\{u_{12p}-7+\alpha_2\} \\
&\quad +\{v_{12n-12p}+7+\beta_2'\}\{u_{12p}-6+\alpha_3\}+\{v_{12n-12p}+6+\beta_1'\}\{u_{12p}-5+\alpha_4\} \\
&\quad +\{v_{12n-12p}+5+\beta_3\}\{u_{12p}-4+\alpha_1'\}-\{v_{12n-12p}+4+\beta_2\}\{u_{12p}-3+\alpha_2'\}) \\
&-\alpha \\
&\quad =-12\sum_{p=1}^{p=n'}\{v_{12p}-9+\beta_1\}\{u_{12n-12p}+16+\alpha_1\}\{u_{12n-12p}+4+\alpha_1\} \\
&\qquad -12\sum_{p=1}^{p=n'}\{u_{12p}-8+\alpha_1\}\{v_{12n-12p}+15+\beta_1\}\{v_{12n-12p}+3+\beta_1\}-\alpha,
\end{aligned}$$

where $n'=\frac{1}{2}n$ when n is even, and $n'=\frac{1}{2}(n+1)$ when n is odd, and where α is used in the same way as before. In this case α consists of one constituent when n is even, and of eleven constituents when n is odd. By the very same reasoning as that applied in the series S_1 and S_2 it follows that the $(12n+1)$-th

term of the product of IX and X, for $n > m$, has a numerical value less than $1/n(\log n)^\theta$, where $\theta > 1$. The two factor-series IX and X differ from the pairs S_1, S_2 and T_1, T_2 in this—that (suppressing in each series each parenthesis which encloses a set) corresponding terms in IX and X need not have the same numerical values. However, the three pairs of factor-series have in common one property, to which allusion has not yet been made; viz., in one of the series of each pair, the sum of the terms in each set is identically zero, so that the sum of the series is itself zero.

COLORADO COLLEGE, COLORADO SPRINGS.

SETS OF COINCIDENCE* POINTS ON THE NON-SINGULAR CUBICS OF A SYZYGETIC SHEAF†

BY

M. B. PORTER

I.

If $C_3 = 0$ be any non-singular cubic and

$$u = \int_{ab}^{xy} du$$

be an everywhere finite integral attached to this curve, the point (a, b) being a point of inflexion, the necessary and sufficient condition that $3m$ points $P_1 \cdots P_{3m}$ of C_3 lie also on an m-ic is given by ABEL'S theorem, namely:

$$\sum_1^{3m} u_i \equiv 0 \pmod{\omega, \omega'}, \tag{1}$$

where ω and ω' are the moduli of periodicity of u and

$$u_i = \int_{ab}^{x_i y_i} du \qquad (i = 1, \cdots, 3m),$$

the points P_i being the points (x_i, y_i).

If $3m - s - 1$ of these points P_i be taken as fixed and we suppose the m-ic to have a contact of order $s - 1$ at u_1, the m-ic will still cut C_3 at a point u_2 dependent on u_1 and by (1) we have:

$$su_1 + u_2 \equiv c \pmod{\omega, \omega'},$$

where c denotes the negative of the sum of the u's of the arbitrarily chosen *fixed* points.

We consider the following sequence of n such congruences:

$$\left.\begin{array}{lll} su_1 + u_2 & \equiv c \\ su_2 + u_3 & \equiv c \\ \vdots \quad\; \vdots & \vdots \\ su_n + u_{n+1} & \equiv c \end{array}\right\} \pmod{\omega, \omega'};$$

* Points where a cubic can have an eighth order contact with cubics of the syzygetic sheaf are called by HALPHEN coincidence points of the cubic.

† Presented to the Society at the New York meeting, June 29, 1900, under a different title. Received for publication November 16, 1900.

if we require that $u_{n+1} \equiv u_1$, i. e., that the curvilinear polygon of points of parameters $u_1 \cdots u_{n+1}$ be a closed polygon,* the necessary and sufficient condition will be:

$$[s^n - (-1)^n]u_1 \equiv c' \pmod{\omega, \omega'}. \tag{2}$$

In case $m = s = 1$, we have a rectilinear polygon of n sides inscribed and circumscribed in C_3, (2) becoming in this case,

$$[2^n - (-1)^n]\, u_1 \equiv 0 \pmod{\omega, \omega'}. \tag{2'}$$

The number of actual n-gons is given by a formula of PICQUET* and it is evident from the nine perspective transformations of C_3 into itself that any one of these n-gons will be perspective with some other n-gon relative to an inflexion point.

HALPHEN ‡ has investigated the points $u = (n'\omega + n''\omega')/3m$ which contain as special cases the vertices of these rectilinear n-gons for the non-singular cubics of the syzygetic sheaf $x^3 + y^3 + z^3 + 6axyz = 0$, and finds that the locus of these points as a varies, which is evidently a combinant of the sheaf, consists of eight irreducible curves when m is divisible by 3, and of nine such curves when m is not so divisible.

II.

It is the purpose of this section to consider certain geometrical relations that subsist between an inflexion triangle and its associated group of in- and circumscribed rectilinear triangles. By (2') the number of such triangles is 24.

---

*The *Schliessungsproblem* here suggested is a slight generalization of that considered by PICQUET (Journal de l'École Polytechnique, vol. 54, 1884). The method of enumeration used by PICQUET is of course applicable here, but the number-theoretic results arrived at are completely general. Thus for n prime the number of n-gons on the part of the curve corresponding to ω will be:

$$\frac{s^n - s}{n}, \tag{A}$$

since $s+1$ values of u must be rejected as corresponding to the s-contact points. In case n be not prime, to obtain the number of n-gons it would be necessary to reject besides the $s+1$ s-contact points all solutions corresponding to polygons whose number of sides divides n. Denoting by $\phi(s, m)$ the number of vertices of such m-gons on the part of the curve corresponding to ω we have:

$$s^n - (-1)^n - s - 1 = \sum_{m'} \phi(s, m') \qquad (1 < m' \leqq n), \tag{B}$$

where the summation is for the different factors m' of n.

By [B] for $n \neq 2$ we get:

$$\phi(s, n) = s^n - \sum s^{\overline{a_i}} + \sum s^{\overline{a_i a_j}} - \cdots \pm s^{\overline{a_1 a_2 \cdots a_\nu}} = F(s, n),$$

$a_1 a_2 \cdots a_\nu$ denoting the different prime factors of n. For $n = 2$ we have $\phi = F - 2$.

The expressions [A] and [B] may be regarded as establishing FERMAT'S theorem and its generalization.

‡ *Recherches sur les courbes planes du troisième degré*, Mathematische Annalen, vol. 15.

These twenty-four triangles can be divided into four groups of six each, each group being associated with one of the four inflexion triangles. Further the six triangles of any group can be paired in nine ways so that each pair shall be perspective in three ways, the centers of perspectivity for each pair being the three inflexion points lying on a side of the associated inflexion triangle.

Thus for the inflexion triangle whose sides may be denoted by

$$\left(\omega + \frac{n\omega'}{3}, \quad \frac{\omega + n\omega'}{3}, \quad \frac{2\omega + n\omega'}{3}\right) \qquad (n = 0, 1, 2),$$

the scheme is as follows:

ω	$\frac{\omega}{3}$	$\frac{2\omega}{3}$
[A] $\frac{\omega}{9} + \frac{n\omega'}{3}$	[B] $-\frac{2\omega}{9} + \frac{n\omega'}{3}$	[C] $\frac{4\omega}{9} + \frac{n\omega'}{3}$
[A′] $-\frac{\omega}{9} - \frac{n\omega'}{3}$	[B′] $\frac{2\omega}{9} - \frac{n\omega'}{3}$	[C′] $-\frac{4\omega}{9} - \frac{n\omega'}{3}$

$\omega + \frac{\omega'}{3}$	$\frac{\omega + \omega'}{3}$	$\frac{2\omega + \omega'}{3}$
$\frac{\omega}{9} + \frac{n\omega'}{3}$	$-\frac{2\omega}{9} + \frac{n\omega'}{3}$	$\frac{4\omega}{9} + \frac{n\omega}{3}$
$-\frac{\omega}{9} - \frac{(n+1)\omega'}{3}$	$\frac{2\omega}{9} - \frac{(n+1)\omega'}{3}$	$-\frac{4\omega}{9} - \frac{(n+1)\omega'}{3}$

$\omega + \frac{2\omega'}{3}$	$\frac{\omega + 2\omega'}{3}$	$\frac{2\omega + 2\omega'}{3}$
$\frac{\omega}{9} + \frac{n}{3}\omega'$	$-\frac{2\omega}{9} + \frac{n\omega'}{3}$	$\frac{4\omega}{9} + \frac{n\omega'}{3}$
$-\frac{\omega}{9} - \frac{(n-1)\omega'}{3}$	$\frac{2\omega}{9} - \frac{(n-1)\omega'}{3}$	$-\frac{4\omega}{9} - \frac{(n-1)\omega'}{3}$

where ABC is perspective with $A'B'C'$, with $B'C'A'$, and with $C'A'B'$, the centers of perspectivity being ω, $2\omega/3$, and $\omega/3$ respectively. Thus the nine intersections of the sides of these two triangles lie three by three on the harmonic polar of ω, $2\omega/3$, $\omega/3$.

Referred to one of these in-circumscribed triangles, as $\omega/9$, $-2\omega/9$, $4\omega/9$, the equation of C_3 can, by a suitable choice of the constants of projection, be written:

[A]* $$x^2z + y^2x + z^2y + 2\eta xyz = 0,$$

the other sheaf of cubics in-circumscribed in this same triangle being

[A′] $$y^2z + z^2x + x^2y + 2\eta xyz = 0,$$

* This equation will be referred to indifferently as sheaf [A] or cubic (of sheaf) [A].

where [A′] is derived from [A] by the involutory transformation

$$x : y : z = y'z' : x'z' : x'y' .$$

The two cubics [A] and [A′] are evidently transformed into themselves by the three transformations:

$$[B] \qquad x : y : z = y' : z' : x' , \quad z' : x' : y' , \quad x' : y' : z' ;$$

the remainder of the 18 linear transformations of C_3 into itself depend on the parameter η. If a straight line be drawn through any vertex of the triangle of reference and the two variable points of intersection with C_3 be put in correspondence we have the three involutory transformations of [A] or [A′] into themselves given by

$$[C] \qquad x : y : z = y'z' : x'y' : x'z' , \quad x'y' : x'z' : y'z' , \quad x'z' : y'z' : x'y' .$$

Under transformation [C] the nine inflexion points will be interchanged with the vertices of three in-circumscribed triangles and all the remaining vertices of in-circumscribed triangles will be merely interchanged. Thus, of the covariants whose vanishing gives the vertices of the triangles in question, three will go over into the sides of the associated inflexion triangle and must therefore be conics circumscribing the triangle of reference, while the remaining covariants will be merely interchanged or transformed into themselves under the transformation [C].

To obtain these conics it will thus only be necessary to determine the equation of the sides of the associated inflexion triangle.

The Hessian of [A] is

$$x^3 + y^3 + z^3 - 3xyz - \eta^2(x^2z + y^2x + z^2y + 2\eta xyz) = 0 ;$$

thus all the cubics of either sheaf [A] or [A′] have the fixed inflexion triangle:

$$x^3 + y^3 + z^3 - 3xyz = \begin{vmatrix} y & z & x \\ x & y & z \\ z & x & y \end{vmatrix} = (x + y + z)(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z) ,$$

where $\epsilon^3 = 1$. By the application to this inflexion triangle of the transformation [C] we find that the vertices of three of the in-circumscribed triangles of the sheaves [A], [A′] lie on the conics:

$$(\alpha) \qquad xy + yz + xz = 0 ,$$

$$(\beta) \qquad xy + \epsilon\, yz + \epsilon^2 xz = 0 ,$$

$$(\gamma) \qquad xy + \epsilon^2 yz + \epsilon\, xz = 0 .$$

Each of these conics has a double contact with two sides of the associated inflexion triangle, the third side being the chord of contact. The Hessian of the net determined by (α), (β), (γ) being the sides of the triangle of reference.*

* The conics α, β, γ may be obtained directly without recourse to the Hessian; for if $ax + by + cz = 0$ be the equation of a side of the associated inflexion triangle we must have, applying [C], $axz + byz + cxy \equiv \rho(ayz + bxy + cxz)$, so that $\rho = \sqrt[3]{1}$, i. e., $a = b = c$ or $a = \varepsilon b = \varepsilon^2 c$ or $a = \varepsilon^2 b = \varepsilon c$.

The vertex of any in-circumscribed triangle is the tangential of three points whose affixes are given by $\bar{u} = (n\omega + n'\omega')/18$. The loci of the nine points associated with the vertices of the triangle of reference on which the u points lie three by three are:

(α') $\qquad x^2 - yz = 0\,,$

(β') $\qquad y^2 - xz = 0\,,$

(γ') $\qquad z^2 - xy = 0\,.$

These conics are the first polars of the vertices of the fundamental triangle relative to the inflexion triangle, and have precisely the same relation to the fundamental triangle that (α), (β), (γ) have to the inflexion triangle. The Hessian of their net is thus the sides of the inflexion triangle. If the inflexion triangle be taken as the triangle of coördinates, [A] can be written:

$$\bar{x}^3 + \bar{y}^3 + \bar{z}^3 + 6b\bar{x}\bar{y}\bar{z} = 0\,,$$

where

$$\bar{x} = \sqrt[3]{3 + 2\eta}\,(x + y + z)\,,$$

$$\bar{y} = \sqrt[3]{3\epsilon^2 + 2\eta}\,(x + \epsilon^2 y + \epsilon z)\,,$$

$$\bar{z} = \sqrt[3]{3\epsilon + 2\eta}\,(x + \epsilon y + \epsilon^2 z)\,,$$

$$b = -\frac{\eta}{\sqrt[3]{27 + 8\eta^3}}\,.$$

Writing

$$\lambda_1 = \sqrt[3]{3 + 2\eta}\,, \quad \lambda_2 = \sqrt[3]{3\epsilon^2 + 2\eta}\,, \quad \lambda_3 = \sqrt[3]{3\epsilon + 2\eta}\,,$$

we have, as the equation of a side of one of the variable inflexion triangles of [A],

$$(\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_1 + \epsilon^2\lambda_2 + \epsilon\lambda_3)y + (\lambda_1 + \epsilon\lambda_2 + \epsilon^2\lambda_3)z = 0\,,$$

which envelopes the locus:

$$\rho u = \lambda_1 + \lambda_2 + \lambda_3\,,$$

$$\rho v = \lambda_1 + \epsilon^2\lambda_2 + \epsilon\lambda_3\,,$$

$$\rho w = \lambda_1 + \epsilon\lambda_2 + \epsilon^2\lambda_3\,,$$

or

$$wv^2 + vu^2 + uw^2 = 0\,.$$

This equianharmonic curve of class three is inscribed in all three of the variable inflexion triangles of [A].

The locus of the variable vertices of these inflexion triangles is found to be three sextics.

The equations of the nine joins of the vertices of the triangle of reference and those of the associated inflexion triangle are:

$$x - y = 0\,, \qquad x - \epsilon y = 0\,, \qquad x - \epsilon^2 y = 0\,,$$

$$y - z = 0\,, \qquad y - \epsilon z = 0\,, \qquad y - \epsilon^2 z = 0\,,$$

$$z - x = 0\,, \qquad z - \epsilon x = 0\,, \qquad z - \epsilon^2 x = 0\,.$$

Since any three of these equations, picked out after the manner of a term of a three-row determinant, are simultaneous we have the theorem:

Each in-circumscribed triangle is in six ways perspective with its associated inflexion triangle. *

There remain two more triangles associated with the inflexion triangle under consideration. The loci of these vertices are easily determined by the method employed by HALPHEN. Let $x'y'z'$ be the current coördinates; the tangent at xyz is

$$x'(2xz + y^2 + 2\eta yz) + y'(2xy + z^2 + 2\eta xz) + z'(2yz + x^2 + 2\eta xy) = 0 .$$

If we make the substitution $x' : y' : z' = z : x : y$ which transforms [A] into itself, it is clear that the locus:

$$z(2xz + y^2 + 2\eta yz) + x(2xy + z^2 + 2\eta xz) + y(2yz + x^2 + 2\eta xy) = 0 ,$$

will pass through the vertices of an in-circumscribed triangle. Eliminating η by means of [A] we get:

$$y^4z^2 + z^4x^2 + x^4y^2 - z^3xy^2 - x^2y^3z - x^3yz^2 = 0 ,$$

that is,

$$(y^2z + \epsilon z^2x + \epsilon^2x^2y)(y^2z + \epsilon^2z^2x + \epsilon x^2y) = 0 . \dagger$$

These two equianharmonic cubics are in-circumscribed in three of the inflexion triangles of the sheaf:

[D] $$x^3 + y^3 + z^3 + 6bxyz = 0 ,$$

and have the fourth inflexion triangle for an inflexion triangle.

It is to be noted that the conics a, β, γ, a', β', γ' are taken together a combinant of the sheaf of cubics [D], there being in all thirty-six such conics, each of which circumscribes one inflexion triangle and has a double contact with two sides of another, the third side of the second triangle being the chord of contact. The rôle played by these conics in the geometry of the sheaf [D] has not, so far as the writer is aware, been investigated.

YALE UNIVERSITY, NEW HAVEN, CONN.

* The same thing is well known to be true for any two inflexion triangles, the six centers of perspectivity here being the vertices of the other two inflexion triangles. The theorem in question may be deduced from the fact that the vertices of the inflexion triangles lie three by three on the harmonic polars.

† These cubics are together with $y^2z + z^2x + x^2y = 0$ a combinant of the sheaf of curves [D] below and determine on it the vertices of nine in-circumscribed triangles. These cubics belong to the set of eight obtained by HALPHEN, loc. cit., and were given by Professor H. S. WHITE, Transactions of the American Mathematical Society, April, 1900.

NOTE ON NON-QUATERNION NUMBER SYSTEMS*

BY

WENDELL M. STRONG

SCHEFFERS† has divided all number systems into the quaternion and non-quaternion systems and has shown that the n fundamental units of a non-quaternion system may be so chosen that the multiplication table takes a particularly simple form, which is in turn characteristic of the non-quaternion systems. In this paper I shall show that the choice of the units may be so regulated that the multiplication table becomes still simpler.

SCHEFFER'S form, which we shall call the *regular form*, has the following characteristic properties:

1°. The units are divided into two essentially different classes, the e's and the η's, with the notation:

$$e_1, e_2, \cdots, e_r; \quad \eta_1, \eta_2, \cdots, \eta_s \qquad (r+s=n).$$

2°. The sum of the η's is the modulus (or idemfactor).

3°. As to the products $\eta_\lambda \eta_\mu$, we have the formulas:

$$\eta_\lambda^2 = \eta_\lambda, \quad \eta_\lambda \eta_\mu = 0 \qquad (\lambda, \mu = 1, \cdots, s; \lambda \neq \mu).$$

4°. For each e_k there is one and only one η_λ such that

$$\eta_\lambda e_k = e_k;$$

and for each e_k there is one and only one η_μ such that

$$e_k \eta_\mu = e_k.$$

The particular units η_λ, η_μ thus related to e_k may or may not be distinct. The unit e_k is said to have the *character* (λ, μ).

Every other product of an η with e_k is zero.

5°. The product of two e's contains only e's and, moreover, only e's of lower index than either of the factors—that is, with certain constants γ_{ijs},

* Presented to the Society at the Columbus meeting, August 25, 1899. Received for publication December 6, 1900.

† Mathematische Annalen, vol. 39, pp. 307–313, 1891.

$$\begin{cases} e_i e_j = \sum_{s=1}^{i-1} \gamma_{ijs} e_s & (i \gtrless j), \\ e_i e_j = \sum_{s=1}^{j-1} \gamma_{ijs} e_s & (i > j). \end{cases}$$

We note as of especial importance the particular case:

$$e_1 e_i = e_i e_1 = 0 \qquad (i = 1, \cdots, r).$$

6°. (Corollary of 4° and 5°.) If e_i is of character (λ, μ) and e_j of character (μ, ν), the product $e_i e_j$ can contain units of character (λ, ν) only. If e_i is of character (λ, μ) and e_j of character (ρ, σ) where $\rho \neq \mu$, the product $e_i e_j$ is zero.

We shall for the present consider a regular system from which the η's have been deleted, that is, the part consisting of the e's and their products with each other only—this is called a *degenerate system.** A degenerate system is said to be regular if it is in a form to satisfy 5°.

The sequel depends essentially upon the following linear ordering of the products† of the multiplication table.

(*a*) The products are divided into groups and the groups themselves ordered, as follows:

The first group contains all products $e_1 e_j$ and $e_j e_1$ $(j = 1, 2, \cdots, r)$:
The second group contains all products $e_2 e_j$ and $e_j e_2$ $(j = 2, 3, \cdots, r)$:
The k-th group contains all products $e_k e_j$ and $e_j e_k$ $(j = k, k+1, \cdots, r)$:
The $(r-1)$-th group contains all products $e_{r-1} e_j$ and $e_j e_{r-1}$ $(j = r-1, r)$:
The r-th group contains $e_r e_r$.‡

(*b*) Within a group, say the i-th, the products are ordered thus:

$$e_i e_i, \ e_i e_{i+1}, \ e_{i+1} e_i, \ e_i e_{i+2}, \ e_{i+2} e_i, \ \cdots, \ e_i e_r, \ e_r e_i.$$

In any system § there are certain products, in number m $(0 \leqq m < r)$, each of which is linearly independent of the products preceding it in order. These products, which are of fundamental importance to us, will be called *independent products;* they will be denoted in order of precedence by

$$\pi_1, \ \pi_2, \ \pi_3, \ \cdots, \ \pi_m.$$

* SCHEFFERS'S "ausgeartetes System," loc. cit., p. 308.

† "Product" will henceforth be used to denote the product of two e's unless otherwise stated.

‡ If the multiplication table be written in rectangular array, the first group consists of the first line and column, the second group of the part of the second line and column not contained in the first, etc.

§ "System" will mean "degenerate system" while such systems alone are being considered.

To repeat: π_1 is the first product different from zero; π_2 is the first product which cannot be expressed in terms of π_1; π_3 is the first product which cannot be expressed in terms of π_1 and π_2; and so on.

The set of "independent products" is a particular set of linearly independent quadratic products of the system, of which every set then contains m elements. Of course, under a transformation of the system the number m is invariant but the set of "independent products" need not be.

A system whose multiplication table contains m independent products will be called an *m-product* system.

THEOREM I. *Any m-product degenerate non-quaternion number system can be so transformed that* (1) *the transformed system will be regular;* (2) *the m independent products of the transformed system will be the first m units of that system;* (3) *the order of equivalence of independent products and units will be:*

$$\pi_1 = e_1,\ \pi_2 = e_2,\ \cdots,\ \pi_m = e_m.$$

The form indicated by this theorem will be called the *normal form.*

Proof. Assuming that any $(m-1)$-product system can be reduced to the normal form, we shall prove that then any m-product system can be reduced to that form.

Let any regular m-product system be transformed by taking the independent products as the first m units; that is, let

$$\bar{e}_1 = \pi_1,\ \bar{e}_2 = \pi_2,\ \cdots,\ \bar{e}_m = \pi_m,$$

the choice of the remaining new units,

$$\bar{e}_{m+1},\ \bar{e}_{m+r},\ \cdots,\ \bar{e}_r,$$

being subject to the usual restrictions in the transformation of a number system, and also to the restriction* that only units of like character shall be united in a transformed unit. We proceed to prove that *the transformed system is regular.*

In the first place, in the transformed system the first m units, and these only, will enter the products. For, the transformed system is an m-product system; hence its products must contain at least m units; again the transformed units are linear combinations of the original units and consequently any product $\bar{e}_i \bar{e}_j$ can be expressed in terms of $\pi_1, \pi_2, \cdots, \pi_m$, that is, in terms of $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_m$. Hence, regularity is preserved in so far as products $\bar{e}_i \bar{e}_j$ $(i > m, j > m)$ are concerned.

It remains to consider the products $\bar{e}_i \bar{e}_j$, where in each product one index is less than or equal to m, that is, the products

$$\pi_i \bar{e}_j,\ \bar{e}_j \pi_i \qquad (i = 1, \cdots, m;\ j = 1, \cdots, r;\ i \leqq j)$$

* For the units $\bar{e}_{m+1}, \cdots, \bar{e}_r$ one may indeed choose certain $r - m$ of the original units.

We consider one such pair of products $\pi_i \bar{e}_j$, $\bar{e}_j \pi_i$. In the original system, we have

$$\pi_i = e_s e_t = a_1 e_1 + a_2 e_2 + \cdots + a_{u-1} e_{u-1} \qquad (u \leqq s,\ u \leqq t).$$

Any product in the original system, one of whose factors consists of units of index lower than u precedes the independent product

$$\pi_i = e_s e_t \qquad (u \leqq s,\ u \leqq t),$$

and so, by the definition of independent products, is expressible linearly in terms of $\pi_1, \pi_2, \cdots, \pi_{i-1}$. Hence either product of $\bar{e}_i = \pi_i$ with $\bar{e}_j\ (j \gtreqless i)$ can contain only $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{i-1}$. Hence we see that the transformed system is regular.

We now have an m-product regular system in which the units entering products are m in number and are the first m units of the system.* The unit $\bar{e}_1$ being deleted, a regular system remains in which there are $m-1$ units entering products, and $m-1$ independent products. In accordance with our assumption the deleted system may be reduced to the normal form; let the reduction be made and denote the new units by $e_2, e_3 \cdots e_r$, the new independent products by $\pi_1', \pi_2', \cdots, \pi_{m-1}'$. Since the deleted system is normal,

$$\pi_1' = e_2,\ \pi_2' = e_3,\ \cdots,\ \pi_{m-1}' = e_m.$$

On the restoration of $\bar{e}_1$, which will now be denoted by e_1, we have the original m-product system which we had before deletion but in a new form; it will evidently still be regular.

Since the products $e_1 e_i$, $e_i e_1$ all vanish, if $e_j e_k = e_i$ was an independent product in the deleted system,

$$e_j e_k = e_i + a_i e_1$$

will be an independent product now. Then $m-1$ of the independent products will be

$$e_2 + a_2 e_1,\ e_3 + a_3 e_1,\ \cdots,\ e_m + a_m e_1.$$

Before searching for the remaining independent product let us make the transformation:

$$\bar{e}_2 = e_2 + a_2 e_1,\ \bar{e}_3 = e_3 + a_3 e_1,\ \cdots,\ \bar{e}_m = e_m + a_m e_1,$$

the remaining units being unchanged. The regularity has not been disturbed and the independent products remain unchanged.

Denoting the transformed units by $e_2, e_3, \cdots, e_r$, we have as $m-1$ of the independent products the units $e_2, e_3, \cdots, e_m$. The only other unit which enters any product is e_1; hence, the first product which contains e_1 is the remaining independent product; denote this product $e_g e_h$ by π_t.

* We have by this first transformation made a twofold simplification of the regular form of the system.

If e_i is the independent product immediately preceding π_t,

$$e_g e_h = \pi_t = a_1 e_1 + a_2 e_2 + \cdots + a_i e_i \qquad (a_1 \neq 0;\ i < g,\ i < h).$$

Now we make the transformation:

$$\bar{e}_1 = e_2,\ \bar{e}_2 = e_3,\ \cdots,\ \bar{e}_{i-1} = e_i,\ \bar{e}_i = \pi_t,\ \bar{e}_{i+1} = e_{i+1},\ \cdots,\ \bar{e}_r = e_r,$$

and prove that the transformed system will be regular and that its independent products will be

$$\pi_1 = \bar{e}_1,\ \pi_2 = \bar{e}_2,\ \cdots,\ \pi_{m-1} = \bar{e}_{m-1},\ \pi_m = \bar{e}_m,$$

that is, that the transformed system is in the normal form.

The transformation in effect omits the unit e_1 and leaving the remaining units in their original order introduces π_t as the new unit $\bar{e}_i$. The products of the new system are the products of the old (apart from the vanishing e_1-products) and the products involving $\pi_t = \bar{e}_i$. Of these all not involving $\bar{e}_i$ obviously have regular expressions, since in the original system the first product involving e_1 was $e_g e_h = \pi_t = \bar{e}_i$ $(g > i,\ h > i)$, and so all products $\bar{e}_k \bar{e}_l$ involving e_1 have $k > i$, $l > i$ and may be expressed regularly, the e_1 being replaced by $\bar{e}_i, \bar{e}_1, \cdots, \bar{e}_{i-1}$. And it is easy to see also that the products involving $\bar{e}_i$ are expressible regularly.

Further, in the new system $\bar{e}_1, \cdots, \bar{e}_{i-1}$ arise as before as sequential independent products from the products preceding the product $e_g e_h$, the products involving $\bar{e}_i$ as a factor obviously being expressible in terms of the products involving $\bar{e}_1, \cdots, \bar{e}_{i-1}$. Then, as the next independent product,

$$\pi_t = e_g e_h = \bar{e}_i = \bar{e}_g \bar{e}_h \qquad (i < g,\ i < h)$$

enters, and by the intervention of the $\bar{e}_i$, with $\bar{e}_1, \cdots, \bar{e}_{i-1}$ to replace the original e_1, the $\bar{e}_{i+1}, \cdots, \bar{e}_m$ enter as before as sequential independent products. Thus after the transformation the independent products in order are $\bar{e}_1, \cdots, \bar{e}_m$.

Thus the general step of the induction proof is established, and it is evident that a zero-product system is normal; hence we have proved that *every system can be brought into the normal form.*

THEOREM II. *The product of two independent products is not an independent product.*

Let e_i and e_j $(i < j)$ be independent products in a normal system and let $e_p e_q = e_j$.

Then

$$e_i e_j = (e_i e_p) e_q = (a_1 e_1 + a_2 e_2 + \cdots + a_{i-1} e_{i-1}) e_q.$$

Hence $e_i e_j$ is equal to a sum of products which precede it, and cannot be an independent product.

We have thus far considered the degenerate system, and have defined the normal form for that alone. We call the complete non-quaternion system normal if the system is regular and the degenerate system normal. It is necessary to show that the transformations employed have not destroyed regularity with respect to the products of the e's with the η's in order to extend our results to the complete system.

The regularity of the complete system could be destroyed by transformations of the degenerate system which maintain its own regularity in one way only; that is by uniting in a transformed unit, two or more units of different characters.

In the first transformation of the degenerate system it was explicitly stated that the units united in a transformed unit should all be of the same character.

The other transformations which united two or more units in a transformed unit were made by means of equations of the form:

$$\bar{e}_i = e_j e_k,$$

and all such transformations are permissible, for one sees by 6° that the units contained in a product $e_i e_j$ must all be of the same character. Hence we have:

THEOREM III. *Any non-quaternion system may be reduced to the normal form.*

YALE UNIVERSITY, *June*, 1900.

ON THE REDUCTION OF THE GENERAL ABELIAN INTEGRAL*

BY

J. C. FIELDS

§ 1.

In their exceedingly interesting book *Théorie des Fonctions algébriques et de leurs Intégrales*, pp. 344–345, APPELL and GOURSAT have given a brief sketch of HERMITE'S method for obtaining by rational operations the reduced form for a hyperelliptic integral, prefacing the same by a remark which would seem to imply that the more general problem in the case of the Abelian integrals was still awaiting its solution.

To this remark was due an attempt on my part to solve the indicated problem, and the method by which I succeeded in effecting its solution will be found in the present paper. The results here given, it may be remarked, have been in my possession now for about four years and were presented at the annual meeting of the American Mathematical Society held in Toronto in 1897 but have not hitherto been published.

Let

$$F(z, u) = \sum e_{rs} z^r u^s = 0 \tag{1}$$

be the equation to an irreducible algebraic curve of degree n.

The form of this equation we shall suppose to be the standard one in which the only multiple points are double points separate from the branch points and with distinct tangents. We shall throughout regard u as the dependent variable and shall further assume that no line parallel to the axis of u is an inflexional tangent or includes more than one point of the nature of a branch or double point or is tangent at a double point, the singular points too being all supposed to be finite and the asymptotes all distinct and none of them parallel to an axis and no two parallel to each other.

That the most general algebraic equation can be reduced to such a form by a birational transformation is a well-known fact, and in such a reduced form it is customary to treat the equation.

* Presented to the Society at the Toronto meeting, August 16-17, 1897, under a different title. Received for publication November 28, 1900.

On the above hypotheses a term in u^n will present itself in the equation to our curve and we shall find it convenient to assume that its coefficient e_{0n} has the value 1.

Also as a consequence of the said hypotheses z and u can only become infinite together, and further for any given value $z = a$ the equation $F(a, u) = 0$ will never furnish less than $n - 1$ distinct values of u.

On indicating any arbitrary rational function of z, u by $R(z, u)$ the general Abelian integral corresponding to our curve $F(z, u) = 0$ will be

$$\int R(z, u)dz. \tag{2}$$

This integral we know may be expressed as the sum of a rational function of z, u, of an integral of the first kind, of a linear function with constant coefficients of p elementary integrals of the second kind, and of a number of elementary integrals of the third kind, where

$$p = \tfrac{1}{2}(n - 1)(n - 2) - d$$

is the number known as the "deficiency" of the curve, d indicating the number of the double points.

Starting out now with any arbitrary Abelian integral we purpose in this paper to reduce the same to a sum of the character in question, employing thereto a succession of processes, none of which involves other than rational operations in our variables.

By aid of the relation $F(z, u) = 0$ it is known that any rational function $R(z, u)$ may be reduced to the form of a polynomial in u of the degree $n - 1$, whose coefficients are rational functions of z, and on applying the principle of partial fractions to these functions of z we may express $R(z, u)$ in the form:

$$R(z, u) = \sum_{\lambda} R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) + R_0(z, u), \tag{3}$$

where $R_{\lambda}(1/(z - a_{\lambda}), u)$ denotes a polynomial in $1/(z - a_{\lambda})$, u and $R_0(z, u)$ a polynomial in z, u, the degrees of these polynomials in the variable u not being greater than $n - 1$.

Our Abelian integral then takes the form:

$$\int R(z, u)dz = \sum_{\lambda} \int R_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right)dz + \int R_0(z, u)dz. \tag{4}$$

We shall assume that $R_{\lambda}(1/(z - a_{\lambda}), u)$ contains only negative powers of $z - a_{\lambda}$, as all terms involving other than such powers may be supposed to be included under $R_0(z, u)$.

§ 2.

For the further reduction of our integral we have need of a formula which we shall now derive. The formula in question is the following:

$$\frac{du}{dz} = -\frac{F'_z}{F'_u} = \sum_e \frac{\phi_e(u)}{z - a_e} = \sum_e \frac{c_e F(a_e, u)}{(z - a_e)(u - b_e)}, \tag{5}$$

where the summation is supposed to extend to all branch and double points (a_e, b_e), and where for the constant coefficients c we have

$$c_e = \left[\frac{1}{F''_u(a_e, u)}\right]_{u=b_e} \text{ or } \left[\frac{2}{F''_u(a_e, u)}\right]_{u=b_e}$$

according as (a_e, b_e) is a branch or double point.

Starting out from the identity

$$-\frac{F'_z}{F'_u} = -\frac{F'_z[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}}{F'_u[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}},$$

where $u_1, u_2, \cdots, u_{n-1}$ are the $n-1$ roots of $F(z, u) = 0$ conjugate to u, we see that the denominator of the expression on the right is a symmetric function of the n quantities $u, u_1, \cdots, u_{n-1}$ and therefore a rational function of z. It is, in fact, the discriminant, an integral rational function of z of degree $n(n-1)$ whose factors $z - a_e$ correspond to the several branch and double points. In case a double point corresponds to the value $z = a_e$ the factor $z - a_e$ will appear to the second power in the discriminant and in the case of a branch point to the first power, as is well known.

The factor $[F'_u]_{u=u_1} \cdots [F'_u]_{u=u_{n-1}}$ in the numerator of the above expression is symmetric in the $n-1$ quantities $u_1 \cdots u_{n-1}$ and may evidently be expressed as a polynomial in z, u. The numerator itself may then be written in the form:

$$A_1 u^{n-1} + A_2 u^{n-2} + \cdots + A_n,$$

where the coefficients A are polynomials in z.

On indicating the discriminant by the letter D we then obtain for du/dz the expression

$$\frac{du}{dz} = \frac{A_1 u^{n-1} + A_2 u^{n-2} + \cdots + A_n}{D}, \tag{6}$$

where, as may readily be shown, the degree of the numerator in z, u is not greater than $n(n-1)$ and where in particular the degree of the term A_n is less than this number.

Suppose, namely, that the directions of the n points at ∞ on the curve $F(z, u) = 0$ are given by the equations

$$u - \kappa_1 z = 0\,, \quad u - \kappa_2 z = 0, \cdots, u - \kappa_n z = 0\,, \tag{7}$$

where, according to our original hypothesis in regard to the asymptotes of our curve, the n coefficients κ all have different values. The differential coefficient du/dz will then have the values κ_1, κ_2, $\cdots$, κ_n, respectively, for the several branches at ∞.

Indicating the degree of the numerator in the expression on the right of (6) by ν and representing by a_1, a_2, $\cdots$, a_n the coefficients of the terms of this degree in the elements $A_1 u^{n-1}$, $A_2 u^{n-2}$, $\cdots$, A_n respectively, we shall have

$$a_1 \kappa_r^{n-1} + a_2 \kappa_r^{n-2} + \cdots + a_n \tag{8}$$

for the coefficient of z^ν in the numerator in question, on substituting for u in terms of z from the rth branch of the curve. Now in case we had $\nu > n(n-1)$ the differential coefficient in (6) would be infinite for some branch of the curve at ∞ unless the expression (8) be equal to 0 for each of the n values κ_1, κ_2, $\cdots$, κ_n of κ_r. In this case, however, the expression in question being only of the degree $n-1$ must vanish identically and we should have $a_1 = a_2 = \cdots = a_n = 0$. It follows therefore that ν cannot be greater than $n(n-1)$.

Assuming then that we have $\nu = n(n-1)$, substituting for u in terms of z from the equation to one of the n branches at ∞, and putting $z = \infty$ in (6), we derive

$$\kappa_r = \frac{a_1}{\delta} \kappa_r^{n-1} + \frac{a_2}{\delta} \kappa_r^{n-2} + \cdots + \frac{a_n}{\delta}\,, \tag{9}$$

where δ is the coefficient of $z^{n(n-1)}$ in D.

Now this equation is of the degree $n-1$ in κ_r and must be satisfied by the n quantities κ_1, κ_2, $\cdots$, κ_n. It must therefore be an identity, and we consequently have

$$a_1 = 0\,,\ a_2 = 0\,,\ \cdots,\ a_{n-2} = 0\,,\ a_{n-1} = \delta\,,\ a_n = 0\,.$$

We see therefore that the degree of the coefficient A_n, as well as that of the other coefficients A in the numerator of the expression to the right of (6), must be less than $n(n-1)$. On applying to this expression the principle of partial fractions, the various parts will evidently all be fractional in their character.

The expression in (6) may then be written in the form:

$$\frac{du}{dz} = \sum_e \left\{ \frac{\phi_e(u)}{z - a_e} + \frac{\psi_e(u)}{(z - a_e)^2} \right\}, \tag{10}$$

where the summation extends to all those values $z = a_e$ to which correspond branch or double points, and where the numerators $\phi(u)$ and $\psi(u)$ are polynomials in u of degree $n-1$.

We have already remarked that $z - a_e$ is a quadratic factor of the discriminant D in case a double point corresponds to the value $z = a_e$, whereas in the case of a branch point the corresponding factor enters only to the first power.

In the summation above, then, the numerator $\psi_\varepsilon(u)$ in an element corresponding to a branch point must evidently be 0, and we shall prove that this also is so in the case of an element corresponding to a double point.

Consider, namely, an element

$$\frac{\phi(u)}{z-a}+\frac{\psi(u)}{(z-a)^2},$$

where to $z=a$ corresponds a double point. The other points of our curve corresponding to this value of the variable will then be ordinary points by virtue of our original hypothesis, and the differential coefficient du/dz will evidently be finite for each of the $n-1$ points corresponding to the value $z=a$. Its values for the two branches at the double point I shall indicate by m_1 and m_2 respectively.

For the $n-1$ points corresponding to the value $z=a$ the summation in (10) and therewith the element here in question must be finite. It evidently follows that $\psi(u)$ must be equal to 0 for each of the $n-1$ values of u corresponding to $z=a$, and on writing

$$F(a,\ u)=(u-b)^2(u-b')(u-b'')\cdots(u-b^{(n-2)}),$$

we must have

$$\psi(u)=c(u-b)(u-b')\cdots(u-b^{(n-2)}),$$

where c is a constant, for $\psi(u)$ has the degree $n-1$.

We may therefore write

$$\psi(u)=\frac{cF(a,\ u)}{u-b},$$

and our element will have the form

$$\frac{\phi(u)}{z-a}+\frac{\psi(u)}{(z-a)^2}=\frac{1}{z-a}\left\{\phi(u)+\frac{cF(a,\ u)}{(z-a)(u-b)}\right\}.$$

Now in order that this element may be finite for the two branches at the double point it is necessary that the expression in parentheses should vanish for both these branches.

Writing this expression in the form

$$\phi(u)+\frac{u-b}{z-a}\frac{cF(a,\ u)}{(u-b)^2},$$

we see then that we must have

$$\phi(b)+m_1\left[\frac{cF(a,\ u)}{(u-b)^2}\right]_{u=b}=0,$$

$$\phi(b)+m_2\left[\frac{cF(a,\ u)}{(u-b)^2}\right]_{u=b}=0.$$

From this it follows that

$$c(m_1 - m_2)\left[\frac{F(a, u)}{(u-b)^2}\right]_{u=b} = 0,$$

and as we have

$$m_1 \neq m_2, \quad \left[\frac{F(a, u)}{(u-b)^2}\right]_{u=b} \neq 0,$$

we must have $c = 0$.

It follows therefore that $\psi(u) = 0$ and the summation in (10) reduces to the form

$$\frac{du}{dz} = \sum_e \frac{\phi_e(u)}{z - a_e},$$

where, as before, the summation is supposed to extend to every value $z = a_e$ to which corresponds a branch or double point. Now the differential coefficient du/dz can only become infinite for the branch points, and for a branch point too it can only become infinite like $(z-a)^{-\frac{1}{2}}$, so that we see that in any case the numerator $\phi_e(u)$ of an element in the above summation must equal 0 for each of the $n-1$ values of u corresponding to the value $z = a_e$. It follows therefrom that $\phi_e(u)$ must have the form

$$\phi_e(u) = \frac{c_e F(a_e, u)}{u - b_e},$$

and for the differential coefficient du/dz we obtain the expression

$$\frac{du}{dz} = \sum_e \frac{c_e F(a_e, u)}{(z-a_e)(u-b_e)},$$

where it only remains now to determine the value of the constants c_e.

First considering the case of a double point (a_e, b_e) we may evidently write the expression for our differential coefficient in the form

$$\frac{du}{dz} = \frac{c_e F(a_e, u)}{(u-b_e)^2} \cdot \frac{u - b_e}{z - a_e} + f(z, u),$$

where the value of the function $f(z, u)$ for the point (a_e, b_e) is unambiguous.

On considering the respective values of this expression for the two branches passing through the double point, we shall have

$$m_1 = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\right]_{u=b_e} m_1 + f(a_e, b_e) = \tfrac{1}{2} c_e m_1 [F''_u]_{a_e b_e} + f(a_e, b_e),$$

$$m_2 = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\right]_{u=b_e} m_2 + f(a_e, b_e) = \tfrac{1}{2} c_e m_2 [F''_u]_{a_e b_e} + f(a_e, b_e);$$

whence by subtraction

$$(m_1 - m_2) = \tfrac{1}{2}(m_1 - m_2) c_e [F''_u]_{a_e b_e},$$

and therefore

$$c_e = \frac{2}{[F'''_u]_{a_e b_e}}.$$

In the case of a branch point (a_e, b_e) we evidently have

$$\left[\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e} = \left[\frac{d}{dz}(u-b_e)^2\right]_{a_e b_e} = \left[2(u-b_e)\frac{du}{dz}\right]_{a_e b_e}.$$

We have however also:

$$\left[(u-b_e)\frac{du}{dz}\right]_{a_e b_e} = \left[\frac{c_e F(a_e, u)}{z-a_e}\right]_{a_e b_e} = \left[\frac{c_e F(a_e, u)}{(u-b_e)^2}\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e};$$

and from these two formulæ combined we derive:

$$\left[\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e} = \left[\frac{2c_e F(a_e, u)}{(u-b_e)^2}\frac{(u-b_e)^2}{z-a_e}\right]_{a_e b_e}.$$

Here the expression to the left is evidently finite, and it is also different from 0 since by our original hypothesis the branch point (a_e, b_e) is not at the same time a double point. We may therefore divide both sides of our equation by the expression in question, and obtain

$$1 = c_e\left[\frac{2F(a_e, u)}{(u-b_e)^2}\right]_{a_e b_e} = c_e[F'''_u]_{a_e b_e};$$

whence in the case of a branch point (a_e, b_e) we have

$$c_e = \frac{1}{[F'''_u]_{a_e b_e}}.$$

We have therefore established our formula (5).

§ 3.

Returning now to the reduction of our integral in (4) we shall first consider an individual element,

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz,$$

in the summation appearing on the right.

Supposing the highest exponent of $1/(z-a_\lambda)$ in the expression under the integral sign to be greater than 1 and indicating the same by r we may write:

$$R_\lambda\left(\frac{1}{z-a_\lambda}, u\right) = \frac{f_r(u)}{(z-a_\lambda)^r} + R'\left(\frac{1}{z-a_\lambda}, u\right),$$

where $f(u)$ is a polynomial in u of degree $n-1$ and where $R'(1/(z-a_\lambda), u)$ is a polynomial in $1/(z-a_\lambda)$, u of degrees $r-1$ and $n-1$ respectively in $1/(z-a_\lambda)$ and u.

We shall then have:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz = \int \frac{f_r(u)}{(z-a_\lambda)^r}\,dz + \int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

$$= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int\left\{\frac{f_r(u)}{(z-a_\lambda)^r} - \frac{d}{dz}\,\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}\right\}dz$$

$$+ \int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

(11)

$$= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int\left\{\frac{f_r(u)+(r-1)g_{r-1}(u)}{(z-a_\lambda)^r}\right.$$

$$\left. - \sum_e \frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz + \int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz;$$

where $g_{r-1}(u)$ is a polynomial in u of degree $n-1$, as yet undetermined, and where for du/dz we have substituted its expression as given in (5), so that the summation with regard to e which here appears is supposed to extend to all branch and double points. We shall first suppose that none of the points of the curve corresponding to the value $z = a_\lambda$ is a branch or double point. In this case then we shall have $a_\lambda \neq a_e$ for all values of the a_e's. For the polynomial $g_{r-1}(u)$ then choosing

$$g_{r-1}(u) = -\frac{f_r(u)}{r-1},$$

we reduce our integral (11) to the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} - \int\left\{\sum_e \frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz$$

$$+ \int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz.$$

Here, in the expression to the right, $(z-a_\lambda)^{r-1}$ is the highest power of $z-a_\lambda$ which appears in a denominator under the integral sign, and on reducing the order to which u appears in the products $g'_{r-1}(u)\cdot\phi_e(u)$ by aid of the equation $F(z,\ u)=0$ and on separating the factors $(z-a_\lambda)^{r-1}$ and $z-a_e$ from one another in the denominators by the method of partial fractions, we express our integral in the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int R'_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

(12)

$$+ \sum_e \int S'_e\left(\frac{1}{z-a_e},\ u\right)dz + \int S'_0(z,\ u)dz,$$

where $R'_\lambda(1/(z-a_\lambda),\ u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree $r-1$ in $1/(z-a_\lambda)$, $S'_e(1/(z-a_e),\ u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $S'_0(z,\ u)$ a polynomial in z, u, none of these polynomials having in u a degree which is greater than $n-1$.

By the above process we have reduced by 1 the degree to which $1/(z-a_\lambda)$ appears under the integral sign and by application of the same process to

$$\int R'_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

we should obtain a formula:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}}+\int R''_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$
$$+\sum_e\int S''_e\left(\frac{1}{z-a_e},\ u\right)dz+\int S''_0(z,\ u)dz,$$

where $r-2$ is the degree of $R''_\lambda(1/(z-a_\lambda),\ u)$ in $1/(z-a_\lambda)$.

By successive applications of the process we should ultimately arrive at an expression for our integral in the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}}+\cdots+\frac{g_1(u)}{z-a_\lambda}$$
$$+\int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz+\sum_e\int \bar{S}_e\left(\frac{1}{z-a_e},\ u\right)dz+\int \bar{S}_0(z,\ u)dz,\tag{13}$$

where $\bar{R}_\lambda(1/(z-a_\lambda),\ u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree 1 in $1/(z-a_\lambda)$, $\bar{S}_e(1/(z-a_e),\ u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $\bar{S}_0(z,\ u)$ a polynomial in z, u.

§ 4.

We have obtained formula (13) on the assumption that no branch or double point corresponds to the value $z=a_\lambda$. Assuming on the contrary now that we have a branch or double point $(a_\lambda,\ b_\lambda)$ so that $z-a_\lambda$ is included among the factors $z-a_e$, we derive from (11):

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz=\frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}}+\int\left\{\frac{f_r(u)+(r-1)g_{r-1}(u)-g'_{r-1}(u)\cdot\phi_\lambda(u)}{(z-a_\lambda)^r}\right.$$
$$\left.-\sum_e{}'\frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z-a_\lambda)^{r-1}(z-a_e)}\right\}dz+\int R'\left(\frac{1}{z-a_\lambda},\ u\right)dz,\tag{14}$$

where the accent in the sign of summation $\sum'_e$ is intended to signify that the term corresponding to $a_e=a_\lambda$ is omitted from the summation.

Here again we wish by a proper choice of $g_{r-1}(u)$ to rid ourselves of terms under the integral sign which involve $(z - a_\lambda)^r$ in their denominator.

For this it will be necessary that the expression

$$f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u) \tag{15}$$

be divisible by $z - a_\lambda$, after it has been reduced to the normal form in which u does not appear to an order higher than $n - 1$. Now if this expression in its reduced form is to be divisible by $z - a_\lambda$ it must, as it stands, be equal to 0 for each of the $n - 1$ values $b_\lambda, b'_\lambda, \cdots, b^{(n-2)}_\lambda$ of u corresponding to the value $z = a_\lambda$, and must therefore be divisible by

$$\phi_\lambda(u) = c_\lambda(u - b_\lambda)(u - b'_\lambda)\cdots(u - b^{(n-2)}_\lambda) = \frac{c_\lambda F(a_\lambda, u)}{u - b_\lambda}.$$

It follows then that

$$f_r(u) + (r-1)g_{r-1}(u)$$

also must be divisible by $\phi_\lambda(u)$, and since the degree of this expression in u is $n - 1$ we may write:

$$f_r(u) + (r-1)g_{r-1}(u) = \kappa\phi_\lambda(u),$$

where κ is a constant which is yet to be determined.

For our function $g_{r-1}(u)$ we must therefore have

$$g_{r-1}(u) = \frac{\kappa\phi_\lambda(u) - f_r(u)}{r-1}, \tag{16}$$

and on substituting this in (15) we obtain:

$$\begin{aligned} f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u) &= \big(\kappa - g'_{r-1}(u)\big)\phi_\lambda(u) \\ &= \frac{1}{r-1}\{(r-1)\kappa - \kappa\phi'_\lambda(u) + f'_r(u)\}\phi_\lambda(u), \end{aligned} \tag{17}$$

which expression in its reduced form must be divisible by $z - a_\lambda$. Now we have:

$$(u - b_\lambda)\phi_\lambda(u) = c_\lambda F(a_\lambda, u) = c_\lambda\{F(a_\lambda, u) - F(z, u)\},$$

which is evidently divisible by $z - a_\lambda$. Also in order that the expression in (17) should be divisible by $(u - b_\lambda)\phi_\lambda(u)$ it is only necessary that the factor

$$(r-1)\kappa - \kappa\phi'_\lambda(u) + f'_r(u)$$

should be divisible by $u - b_\lambda$, which will be the case if we choose κ so as to satisfy the condition

$$\{(r-1) - \phi'_\lambda(b_\lambda)\}\kappa + f'_r(b_\lambda) = 0. \tag{18}$$

On so choosing κ the expression in (17) may be put into a form which is divisible by $z - a_\lambda$ and must therefore evidently also, in its reduced form, be

divisible by this factor. Such a choice of κ is of course always possible, excepting in the case where the coefficient of κ in (18) is equal to 0.

We have:

$$\phi'_\lambda(b_\lambda) = c_\lambda[(u - b'_\lambda)\cdots(u - b^{(n-2)}_\lambda)]_{u=b_\lambda} = \tfrac{1}{2}c_\lambda[F'''_u(a_\lambda, u)]_{u=b_\lambda}.$$

It follows then that

$$\phi'_\lambda(b_\lambda) = \tfrac{1}{2} \quad \text{or} \quad 1,$$

and that we therefore have:

$$\kappa = -\frac{f'_r(b_\lambda)}{r - 1 - \phi'_\lambda(b_\lambda)} = \frac{-f'_r(b_\lambda)}{r - \frac{3}{2}} \quad \text{or} \quad \frac{-f'_r(b_\lambda)}{r - 2},$$

according as (a_λ, b_λ) is a branch or double point. The one case here excepted is evidently that in which we have to do with a double point (a_λ, b_λ) and where at the same time $r = 2$, in which case the coefficient of κ in (18) is 0. Substituting the values just obtained for κ in the expression already given above for $g_{r-1}(u)$ we obtain:

$$g_{r-1}(u) = -\left\{\frac{4f'_r(b_\lambda)\cdot\phi_\lambda(u)}{(2r-2)(2r-3)} + \frac{f_r(u)}{r-1}\right\}$$

$$= -\left\{\frac{4}{(2r-2)(2r-3)}\frac{f'_r(b_\lambda)}{F'''_{b_\lambda}(a_\lambda, b_\lambda)}\frac{F(a_\lambda, u)}{u - b_\lambda} + \frac{f_r(u)}{r-1}\right\} \quad \text{or}$$

(19)

$$g_{r-1}(u) = -\left\{\frac{f'_r(b_\lambda)\cdot\phi_\lambda(u)}{(r-1)(r-2)} + \frac{f_r(u)}{r-1}\right\}$$

$$= -\left\{\frac{2}{(r-1)(r-2)}\frac{f'_r(b_\lambda)}{F'''_{b_\lambda}(a_\lambda, b_\lambda)}\frac{F(a_\lambda, u)}{u - b_\lambda} + \frac{f_r(u)}{r-1}\right\},$$

according as (a_λ, b_λ) is a branch point or a double point, the exceptional case being here of course excluded.

The substitution of these expressions for $g_{r-1}(u)$ in (14) will enable us to rid ourselves of the r-th power of $z - a_\lambda$ in the denominator under the integral sign, the numerator

$$f_r(u) + (r-1)g_{r-1}(u) - g'_{r-1}(u)\cdot\phi_\lambda(u)$$

in its reduced form being then divisible by $z - a_\lambda$.

By applying the principle of partial fractions to the terms of form

$$\frac{g'_{r-1}(u)\cdot\phi_e(u)}{(z - a_\lambda)^{r-1}(z - a_e)},$$

we may, as in the section preceding, separate the factors $(z - a_\lambda)^{r-1}$ and $z - a_e$ in the denominators, and on reducing the powers of u occurring in the various numerators we may put (14) into the form:

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \int R'_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz + \sum_e{}' \int S'_e\left(\frac{1}{z-a_e}, u\right)dz + \int S'_0(z, u)\,dz,$$

where $R'_\lambda(1/(z-a_\lambda), u)$ is a polynomial in $1/(z-a_\lambda)$, u of degree $r-1$ in $1/(z-a_\lambda)$, $S'_e(1/(z-a_e), u)$ a polynomial in $1/(z-a_e)$, u of degree 1 in $1/(z-a_e)$, and $S'_0(z, u)$ a polynomial in z, u, the degrees in u of these several polynomials being not greater than $n-1$.

By successive applications of the above process we shall evidently arrive at a reduced form,

$$\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz = \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}} + \cdots + \frac{g_1(u)}{z-a_\lambda} + \int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz + \sum_e \int \bar{S}_e\left(\frac{1}{z-a_e}, u\right)dz + \int \bar{S}_0(z, u)dz, \tag{20}$$

which is a form of the same type as that obtained in (13) for the case of the section preceding, excepting that here $1/(z-a_\lambda)$ may present itself to the second power in the polynomial $R_\lambda(1/(z-a_\lambda), u)$ where to $z=a_\lambda$ corresponds a double point.

We shall now consider more in detail the special case $r=2$ in which we have failed to get rid of the power $1/(z-a_\lambda)^2$ under the sign of integration.

The difficulty here arose in connection with the equation (18) which takes the form

$$0\cdot\kappa + f'_2(b_\lambda) = 0,$$

when $r=2$ and when at the same time (a_λ, b_λ) is a double point. This equation cannot be satisfied by any finite value of κ unless we happen to have $f'_2(b_\lambda)=0$, in which case κ may have any arbitrary value. Supposing then that we have to do with this case we shall take $\kappa=0$ when from (16) we obtain

$$g_{r-1}(u) = -\frac{f_r(u)}{r-1},$$

and the expression in (17) becomes

$$\frac{1}{r-1}f'_r(u)\cdot\phi_\lambda(u),$$

which is divisible by $(u-b_\lambda)\phi_\lambda(u)$ and in its reduced form is therefore divisible by $z-a_\lambda$.

The numerator of $(z-a_\lambda)^r = (z-a_\lambda)^2$ in (14) is then divisible by $z-a_\lambda$ and $1/(z-a_\lambda)$ will therefore appear only to the first power under the sign of integration.

If however we have $f_2'(b_\lambda) \neq 0$, I shall define a function $\bar{f}_2(u)$ by the equation

$$f_2(u) = \bar{f}_2(u) + \bar{h}\phi_\lambda(u),$$

where $\bar{h} = f_2'(b_\lambda)$.

Since $\phi_\lambda'(b_\lambda) = 1$, it follows then that $\bar{f}_2'(b_\lambda) = 0$.

On writing $r = 2$ in (11) we shall have:

$$\begin{aligned}\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz &= \int \frac{f_2(u)}{(z-a_\lambda)^2}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz \\ &= \int \frac{\bar{f}_2(u)}{(z-a_\lambda)^2}dz + \int \frac{\bar{h}\phi_\lambda(u)}{(z-a_\lambda)^2}dz + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz \\ &= \int \frac{\bar{f}_2(u)}{(z-a_\lambda)^2}dz + \int \frac{hF(a_\lambda, u)}{(z-a_\lambda)^2(u-b_\lambda)}dz \\ &\qquad + \int R'\left(\frac{1}{z-a_\lambda}, u\right)dz.\end{aligned}$$

Apart from the second integral on the right the expression here appearing is of the type just considered, in which we saw that we could get rid of the second power of $1/(z-a_\lambda)$ under the sign of integration.

Hence in the case where to $z = a_\lambda$ corresponds a double point we may evidently replace (20) by a formula of the type:

$$\begin{aligned}\int R_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz &= \frac{g_{r-1}(u)}{(z-a_\lambda)^{r-1}} + \frac{g_{r-2}(u)}{(z-a_\lambda)^{r-2}} + \cdots + \frac{g_1(u)}{z-a_\lambda} \\ &+ h_\lambda \int \frac{F(a_\lambda, u)}{(z-a_\lambda)^2(u-b_\lambda)}dz + \sum_e \int \bar{S}_e\left(\frac{1}{z-a_e}, u\right)dz + \int \bar{S}_0(z, u)dz;\end{aligned} \tag{21}$$

and under this formula, on putting 0 for the constant coefficient h_λ, may be supposed to be included that also for the case in which a branch point corresponds to $z = a_\lambda$, for in this case the integral

$$\int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda}, u\right)dz$$

in (20) may be supposed to be included under the summation

$$\sum_e \int \bar{S}\left(\frac{1}{z-a_e}, u\right)dz,$$

the first power only of $1/(z-a_\lambda)$ here presenting itself under the sign of integration.

§ 5.

By the processes of the two preceding sections, therefore, we can reduce any integral

$$\int R_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz$$

to the form given on the right of (21) or (13) respectively, according as a singular point does or does not correspond to the value $z=a_\lambda$. After the substitution of the reduced forms so obtained for the several integrals in the summation to the right of (4) this equation will assume the form:

$$\int R(z,u)dz = P(z,u) + \sum_\lambda \int \frac{h_\lambda F(a_\lambda,u)}{(z-a_\lambda)^2(u-b_\lambda)}\,dz + \sum_\lambda \int \bar{R}_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz + \sum_e \int S''_e\left(\frac{1}{z-a_e},\ u\right)dz + \int S''_0(z,u)dz,$$

where the first summation on the right-hand side is supposed to extend to the double points (a_λ, b_λ) only, the second summation to those values of a_λ for which negative powers of $z-a_\lambda$ occur in $R(z,u)$, from which however we may exclude elements corresponding to branch or double points, and the third summation to those values a_e to which correspond branch or double points.

The last two summations we may fuse into one, reducing the above formula to the form:

$$\int R(z,u)dz = P(z,u) + \sum_\lambda \int \frac{h_\lambda F(a_\lambda,u)}{(z-a_\lambda)^2(u-b_\lambda)}\,dz + \sum_\lambda \int S''_\lambda\left(\frac{1}{z-a_\lambda},\ u\right)dz + \int S''_0(z,u)dz, \tag{22}$$

where the first summation on the right still extends to the double points alone while the second summation extends to those values of a_λ to which correspond branch or double points and over and above these to all those values for which negative powers of the corresponding factor $z-a_\lambda$ appear in the expression of the function $R(z,u)$.

The function

$$\frac{F(a_\lambda,\ u)}{(z-a_\lambda)^2(u-b_\lambda)}$$

becomes infinite to the first order for each of the $n-1$ points corresponding to the value $z=a_\lambda$ and it is evident in fact that for a finite value of z none of the functions under the sign of integration to the right of (22) becomes infinite to an order greater than 1.* Apart from the points at ∞, therefore, the integrals

* While employing the form of expression "infinity of the first, second, etc., order" in the accepted sense of RIEMANN, I shall find it convenient for the temporary purpose of this paper to

on the right of (22) can only become infinite logarithmically. The residues corresponding to the several infinities are immediately determinable and we may suppose the logarithmic portion of the expression in (22) to be represented by an integral of the third kind which we shall indicate by (III) and which is made up of a sum with constant coefficients of a number of elementary integrals of the third kind.

Formula (22) may then be reduced to the form

$$\int R(z, u)dz = P(z, u) + \text{III} + \int H(z, u)dz, \tag{23}$$

where the integral $\int H(z, u)dz$ does not become infinite for any finite value of z. The function $H(z, u)$ itself will then not become infinite to an order as great as 1 for any finite value of the variable. What the form of our function $H(z, u)$ is would appear immediately from the forms of the expressions under the signs of integration to the right of (22).

We shall find it convenient however to here determine directly once for all the general form of a rational function $H(z, u)$ which does not become infinite to an order as great as 1 for a finite value of z.

As in (3), by the application of the principle of partial fractions, we may suppose our function to be expressed as a sum in the form

$$H(z, u) = \sum_{\lambda} H_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) + H_0(z, u). \tag{24}$$

The function $H(z, u)$ will or will not be infinite for a point corresponding to the value $z = a_{\lambda}$ according as this is or is not the case for the corresponding element $H_{\lambda}(1/(z - a_{\lambda}), u)$ on the right. Writing the element in the form

$$H_{\lambda}\left(\frac{1}{z - a_{\lambda}}, u\right) = \frac{\sigma_1(u)}{z - a_{\lambda}} + \cdots + \frac{\sigma_{s-1}(u)}{(z - a_{\lambda})^{s-1}} + \frac{\sigma_s(u)}{(z - a_{\lambda})^s}, \tag{25}$$

where the numerators $\sigma(u)$ are polynomials in u of degree $n - 1$, we see that it will become infinite to the order s for some point corresponding to the value $z = a_{\lambda}$ unless we have

$$\sigma_s(u) = 0$$

for each of the values u corresponding to $z = a_{\lambda}$. The number of these values therefore cannot be greater than $n - 1$.

It follows then that in (24) the summation may be confined to values a_{λ} to which correspond branch or double points. Since for each of the $n - 1$ values of u corresponding to $z = a_{\lambda}$ we have $\sigma_s(u) = 0$, we may write

make use of the expression "infinite to the order $\frac{1}{2}$, 1, etc.," to denote an infinity of the character indicated by $(z - a)^{-\frac{1}{2}}$, $(z - a)^{-1}$, etc. The two forms of expression then differ in their signification when employed relatively to a branch point, but it is hoped that this word of explanation will prevent any confusion from arising in the mind of the reader.

$$\sigma_s(u) = \frac{hF(a_\lambda, u)}{u - b_\lambda},$$

where h is a constant.

In the case of a branch point (a_λ, b_λ) the function

$$\frac{\sigma_s(u)}{(z - a_\lambda)^s} = \frac{hF(a_\lambda, u)}{(z - a_\lambda)^s(u - b_\lambda)},$$

and therewith the element H_λ, here in question also, would evidently be infinite to the order $s - \frac{1}{2}$ so that we cannot have $s \geqq 2$. It follows therefore that $s = 1$.

In the case where a branch point corresponds to the value $z = a_\lambda$ then we shall have for our element:

$$H_\lambda\left(\frac{1}{z - a_\lambda}, u\right) = \frac{hF(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)}, \tag{26}$$

where h may have any arbitrary constant value. In the case where to $z = a_\lambda$ corresponds a double point let us suppose, if possible, that we have $s \geqq 2$. In this case the expression

$$\frac{\sigma_{s-1}(u)}{(z - a_\lambda)^{s-1}} + \frac{\sigma_s(u)}{(z - a_\lambda)^s}$$

must evidently not be infinite to an order greater than $s - 2$, for then it would be infinite to the order 1 at least, and this would hold also in regard to the element on the left of (25). The expression

$$\sigma_{s-1}(u) + \frac{\sigma_s(u)}{(z - a_\lambda)} = \sigma_{s-1}(u) + \frac{hF(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)}$$

must therefore vanish for each of the $n - 1$ points corresponding to the value $z = a_\lambda$ and in particular for each of the branches through the double point (a_λ, b_λ).

On indicating the directions of the tangents at the double point by m_1 and m_2 respectively, we must then have:

$$\sigma_{s-1}(b_\lambda) + \tfrac{1}{2}hm_1[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0,$$

$$\sigma_{s-1}(b_\lambda) + \tfrac{1}{2}hm_2[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0,$$

and therefore

$$h(m_1 - m_2)[F''_u(a_\lambda, u)]_{u=b_\lambda} = 0.$$

Since, however, $m_1 \neq m_2$ and $[F''_u(a_\lambda, u)]_{u=b_\lambda} \neq 0$, this is impossible unless we have $h = 0$ and consequently $\sigma_s(u) = 0$.

It follows therefore that we cannot have $s \geqq 2$.

For $s = 1$ we shall have

$$\frac{\sigma_1(u)}{z-a_\lambda} = \frac{hF(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)},$$

a function which is finite for all the $n-1$ points corresponding to $z=a_\lambda$.

In the case where a double point corresponds to the value $z=a_\lambda$, as in the case of a branch point, the element in (25) takes the form given in (26), and for our function $H(z, u)$ in (24) we may write

$$H(z, u) = \sum_e \frac{h_e F(a_e, u)}{(z-a_e)(u-b_e)} + H_0(z, u), \tag{27}$$

where the summation is supposed to extend to all those values a_e to which correspond branch or double points, where the coefficients h_e may have any arbitrary constant values, and where $H_0(z, u)$ is an arbitrary polynomial in z, u. This then represents the most general function $H(z, u)$ which does not become infinite to an order as great as 1 for any finite value of z.

The most general function of z, u which becomes infinite only at ∞ we should evidently derive from (27) on equating to 0 therein all those coefficients h_e corresponding to the branch points, or in other words, on supposing the summation in (27) to be extended only to those values a_e to which correspond double points.

Returning now to formula (23) we may by (27) express the same in the form:

$$\int R(z, u)dz = P(z, u) + \text{III} + \int \left\{ \sum_e \frac{h_e F(a_e, u)}{(z-a_e)(u-b_e)} + H_0(z, u) \right\} dz. \tag{28}$$

§ 6.

We shall now occupy ourselves with the reduction of the integral

$$\int H_0(z, u)dz.$$

Supposing the degree of the polynomial $H_0(z, u)$ in z, u to be $n+r-1$ where $r \geqq 0$, we shall indicate this by the suffix $n+r-1$ and write

$$H_0(z, u) = T_{n+r-1}(z, u).$$

We may evidently express $T_{n+r-1}(z, u)$ in the form

$$T_{n+r-1}(z, u) = z^r t(z, u) + T'_{n+r-2}(z, u), \tag{29}$$

where the function $t(z, u)$ is a homogeneous polynomial in z, u of degree $n-1$ and where $T'_{n+r-2}(z, u)$ is a polynomial in z, u in general non-homogeneous and of degree $n+r-2$.

It is evident that the function $T_{n+r-1}(z, u)$ cannot be infinite to an order higher than $n+r-1$ for any of the n branches at ∞, and that it must be infinite to this order for one at least of these branches we may readily show.

Namely, on substituting in (29) for u in terms of z from the equation to a given one of the branches at ∞, the coefficient of z^{n+r-1} will be furnished by the element $z^r t(z, u)$ and will be the same as that obtained on substituting $u = \kappa z$ from the corresponding equation in (7). The term in z^{n+r-1} so obtained is

$$z^r t(z, \kappa z) = z^{n+r-1} t(1, \kappa),$$

and the coefficient $t(1, \kappa)$ of z^{n+r-1} will be different from 0 in the case of one at least of the n branches, as otherwise we should have

$$t(1, \kappa) = 0$$

for each of the n values $\kappa_1, \kappa_2, \cdots, \kappa_n$ and this equation would therefore be an identity since its degree in κ is $n - 1$. It would follow then also that we must have

$$t(z, u) = 0$$

identically, which however is not consistent with our hypothesis that our function $T_{n+r-1}(z, u)$ has the degree $n + r - 1$. We conclude then that a polynomial $T_{n+r-1}(z, u)$ becomes infinite to the order $n + r - 1$, that is, to an order equal to its degree, for one or more of the branches at ∞.

Now consider the integral

$$(30) \qquad \int z^r t(z, u) dz = \int \left\{ z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} \right\} dz + \frac{z^{r+1} t(z, u)}{n + r}.$$

The expression under the integral sign to the right of this identity evidently does not become infinite to the order $n + r - 1$ for any of the branches at ∞. It may be written in the form:

$$z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} = \frac{(n - 1) z^r t(z, u)}{n + r} - \frac{z^{r+1}}{n + r} \left(\frac{\partial t}{\partial z} + \frac{\partial t}{\partial u} \frac{du}{dz} \right)$$

$$= \frac{(n - 1) z^r t(z, u)}{n + r} - \frac{z^{r+1}}{n + r} \left(\frac{\partial t}{\partial z} + \frac{\partial t}{\partial u} \sum_e \frac{\phi_e(u)}{z - a_e} \right),$$

where for du/dz we have substituted its expression given in (5). From the last written form we immediately derive:

$$z^r t(z, u) - \frac{d}{dz} \frac{z^{r+1} t(z, u)}{n + r} = T''(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e},$$

where $T''(z, u)$ is a polynomial in (z, u) of degree in u not greater than $n - 1$ and where the functions $\theta_e(u)$ are polynomials in u alone whose degrees are equal to or less than $n - 1$.

Further, the expression to the left of the formula just written, as we have seen above, does not become infinite to the order $n + r - 1$ for any of the branches at ∞ and as none of the elements in the summation

$$\sum_e \frac{\theta_e(u)}{z - a_e}$$

on the right becomes infinite to an order which is greater than $n - 2$, it follows that the polynomial $T''(z, u)$ also does not become infinite to the order $n + r - 1$ and that its degree in z, u therefore is not greater than $n + r - 2$. Indicating this fact by the suffix $n + r - 2$, we may write:

$$(31) \qquad z^r t(z, u) - \frac{d}{dz}\frac{z^{r+1}t(z, u)}{n + r} = T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e};$$

and substituting the form so obtained for the function under the integral sign to the right of (30) we derive:

$$(32) \qquad \int z^r t(z, u)dz = \int \left\{ T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e} \right\} dz + \frac{z^{r+1}t(z, u)}{n + r}.$$

Combining this result with (29) we obtain:

$$\int T_{n+r-1}(z, u)dz = \int \left\{ T'_{n+r-2}(z, u) + T''_{n+r-2}(z, u) + \sum_e \frac{\theta_e(u)}{z - a_e} \right\} dz + \frac{z^{r+1}t(z, u)}{n + r},$$

and this we may more concisely write in the form

$$\int T_{n+r-1}(z, u)dz = g_{n+r}(z, u) + \sum_e \int \frac{\theta_e(u)}{z - a_e} dz + \int T_{n+r-2}(z, u)dz,$$

where

$$g_{n+r}(z, u) = \frac{z^{r+1}t(z, u)}{n + r}, \quad T_{n+r-2}(z, u) = T'_{n+r-2}(z, u) + T''_{n+r-2}(z, u).$$

Repeated application of the above process of reduction gives rise to a succession of integrals

$$\int T_{n+r-2}(z, u)dz, \quad \int T_{n+r-3}(z, u)dz, \cdots, \int T_{n-2}(z, u)dz,$$

and enables us to reduce our integral to an expression of the form

$$(33) \qquad \int T_{n+r-1}(z, u)dz = G_{n+r}(z, u) + \sum_e \int \frac{\sigma_e(u)}{z - a_e} dz + \int T_{n-2}(z, u)dz,$$

where $G_{n+r}(z, u)$ and $T_{n-2}(z, u)$ are polynomials in z, u of degrees $n + r$ and $n - 2$ respectively, and where the functions $\sigma_e(u)$ are polynomials in u of degree $n - 1$.

From the manner in which the functions

$$\frac{\sigma_e(u)}{z - a_e}$$

arise in the process above it is further self-evident that they do not become infinite to an order as great as 1 for the values a_e, so that their sum will constitute a function of the type represented in (27) and may be expressed in the form:

$$\sum_e \frac{\sigma_e(u)}{z - a_e} = \sum_e \frac{a_e F(a_e, u)}{(z - a_e)(u - b_e)},$$

where the coefficients a_e are constants and the summation is supposed to extend to those values a_e to which correspond branch or double points.

For formula (33) we may therefore write:

$$(34) \quad \int T_{n+r-1}(z, u)dz = G_{n+r}(z, u) + \sum_e \int \frac{a_e F(a_e, u)}{(z - a_e)(u - b_e)} dz + \int T_{n-2}(z, u)dz;$$

and on substituting for $\int H_0(z, u)dz = \int T_{n+r-1}(z, u)dz$ in (28) the expression here obtained, we arrive at a reduced form for our original integral $\int R(z, u)dz$ which may be written.

$$(35) \quad \int R(z, u)dz = \bar{G}(z, u) + \text{III} + \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz,$$

where III is an integral of the third kind, $\bar{G}(z, u)$ a rational function of z, u, $T_{n-2}(z, u)$ a polynomial in z, u of degree $n - 2$ and the β_e's constants, the summation with regard to e being extended to all the branch and double points of our curve.

§ 7.

Our reductions of the integral $\int R(z, u)dz$ in the above have so far consisted in getting rid of various infinities of the function under the sign of integration and accounting for the same by a rational function $\bar{G}(z, u)$, its logarithmic infinities at the same time being separated from the main integral and incorporated in an integral of the third kind III.

We shall now consider the remaining integral to the right of (35) with a view to a further reduction. The expression under the integral sign is of the type represented in (27), which for no finite value of z becomes infinite to an order which is as great as 1. The integral itself,

$$(36) \quad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz,$$

will therefore be finite for all finite values of z and for $z = \infty$ will not become

infinite to an order which is greater than $n-1$ since the function under the sign of integration is evidently not infinite to an order which is greater than $n-2$ for any of the points at ∞. If now we would attempt a further elimination of infinities from under the sign of integration by the aid of a rational function of z, u and that without introducing any new infinities, the most general function at our disposal is included under the type (27) and will evidently have the form

$$\sum_e \frac{\gamma_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-1}(z, u), \tag{37}$$

where $T_{n-1}(z, u)$ is a polynomial in z, u of degree $n-1$, the summation with regard to e being supposed to extend to all the double points and the coefficients γ_e having any arbitrary values; for this is the most general rational function whose infinities are all at ∞ and which does not there become infinite to an order which is greater than $n-1$.

The number of coefficients which the polynomial $T_{n-1}(z, u)$ places at our disposal is $\frac{1}{2}n(n+1)$, or we may say $\frac{1}{2}n(n+1)-1$ on rejecting the constant term which could only serve to modify the arbitrary constant of integration in (36), and the number of the coefficients γ_e in the expression (37) is d, the number of the double points of our curve. In all, then, the number of undetermined coefficients which the expression (37) places at our disposal is

$$d + \tfrac{1}{2}n(n+1) - 1.$$

Now the integral in (36) will evidently in general become infinite to the order $n-1$ for each of the n branches at ∞, possessing therefore in all $n(n-1)$ infinities all situated at ∞. If then each of these infinities requires an extra available constant in order to eliminate it from under the sign of integration, the number of available constants placed at our disposal by the expressions (37) will fall short of that requisite by at least

$$n(n-1) - d - \tfrac{1}{2}n(n+1) + 1,$$

that is, by at least

$$p = \tfrac{1}{2}(n-1)(n-2) - d.$$

In order to make good this deficit I shall introduce p more constants, introducing however with each one of them at the same time a new infinity. Namely instead of (37) I shall consider the expression

$$\sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + T_{n-1}(z, u), \tag{38}$$

in which the summation is supposed to extend not only to the double points, but also to p additional points (a_λ, b_λ) which do not coincide with double points but may otherwise be regarded as arbitrarily chosen. For these p points,

in case the corresponding coefficients γ are not zero, the function in (38) will evidently become infinite to the first order. For all other finite points it will be finite and for the n points at ∞ it will in general become infinite to the order $n-1$.

In the sequel we shall find that the $n(n-1)$ constants placed at our disposal by the expression (38) are all available and none of them superfluous for the purpose of ridding ourselves of all the infinities at ∞ of the general integral of the type (36).

§ 8.

Consider the expression

$$(39)\quad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z-a_e)(u-b_e)} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + T_{n-1}(z, u) \right\},$$

obtained on subtracting the function in (38) from the integral in (36). Regarding the function under the sign of integration as given and the coefficients in the subtracted function as undetermined, we shall attempt to determine the latter so that the expression in (39) may have no infinities at ∞.

From the equation to our curve in (1) we have:

$$\frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \frac{F(a_\lambda, u) - F(a_\lambda, b_\lambda)}{(z-a_\lambda)(u-b_\lambda)} = \frac{\sum_{rs} e_{rs} a_\lambda^r u^s - \sum_{rs} e_{rs} a_\lambda^r b_\lambda^s}{(z-a_\lambda)(u-b_\lambda)}$$

$$= \frac{\sum_{rs} e_{rs} a_\lambda^r (u^s - b_\lambda^s)}{(z-a_\lambda)(u-b_\lambda)},$$

whence

$$(40)\quad \frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = (z-a_\lambda)^{-1} \sum_{rs} e_{rs} a_\lambda^r (u^{s-1} + u^{s-2} b_\lambda + \cdots + b_\lambda^{s-1})$$

$$= (z-a_\lambda)^{-1} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s},$$

where

$$\{a_\lambda, b_\lambda\}_s = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho, n-\sigma} a_\lambda^\rho b_\lambda^{s-\sigma-1}.$$

We may then write:

$$\frac{F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} = \left\{ \frac{1}{z} + \frac{a_\lambda}{z^2} + \cdots + \frac{a_\lambda^{n-2}}{z^{n-1}} + \frac{a_\lambda^{n-1}}{z^{n-1}(z-a_\lambda)} \right\} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s}$$

$$= \left\{ \frac{1}{z} + \frac{a_\lambda}{z^2} + \cdots + \frac{a_\lambda^{n-2}}{z^{n-1}} \right\} \sum_{s=1}^{n} \{a_\lambda, b_\lambda\}_s u^{n-s} + P_\lambda(z, u),$$

where $P_\lambda(z, u)$ is a rational function of z, u which is equal to 0 for $z = \infty$. This we may evidently further reduce to the form:

$$\frac{F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} = \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + \bar{P}_\lambda(z, u), \tag{41}$$

where $\bar{P}_\lambda(z, u)$ is a rational function of z, u which is equal to 0 for $z = \infty$. The second of the two summations in (39) may then be written in the form:

$$\begin{aligned} \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} &= \sum_{\lambda=1}^{d+p} \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} \gamma_\lambda a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s z^{-r} u^{n-s} + \bar{P}(z, u) \\ &= \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} c_{n-s-r,\, n-s} z^{-r} u^{n-s} + \bar{P}(z, u) \\ &= \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} c_{n-s-r,\, n-s} z^{-r} u^{n-s} + P(z, u), \end{aligned} \tag{42}$$

where $\bar{P}(z, u)$ and $P(z, u)$ are rational functions of z, u which are finite for $z = \infty$, the former function in particular having the value 0 for this value of the variable, and where for the coefficients c we have

$$c_{n-s-r,\, n-s} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^{r-1} \{a_\lambda, b_\lambda\}_s.$$

For the suffixes of the coefficients c we shall find it convenient to choose, as we have done here, the sum of the exponents of z and u and the exponent of u respectively.

On writing

$$\gamma_{i\kappa} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^i b_\lambda^\kappa, \tag{43}$$

we shall evidently have

$$c_{n-s-r,\, n-s} = \sum_{\sigma=0}^{s-1} \sum_{\rho=0}^{\sigma} e_{\rho,\, n-\sigma} \gamma_{r-1+\rho,\, s-1-\sigma}. \tag{44}$$

Since the exponents of z which appear in the summation to the right of (42) are all negative whereas in the polynomial $T_{n-1}(z, u)$ in (39) no such exponents make their appearance, it will not be inconsistent with the notation employed above if we write:

$$T_{n-1}(z, u) = \sum_{s=1}^{n} \sum_{r=0}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s}. \tag{45}$$

Combining (42) and (45) we have:

$$T_{n-1}(z, u) + \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} = \sum_{s=1}^{n} \sum_{r=0}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s}$$

$$+ \sum_{s=1}^{n-1} \sum_{r=1}^{n-s} c_{n-s-r,\, n-s} z^{-r} u^{n-s} + \bar{P}(z, u)$$

$$= \sum_{s=1}^{n} \sum_{r=-(n-s)}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s} + \bar{P}(z, u). \tag{46}$$

This again we may write in the form:

$$\sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u) = \sum_{\nu=0}^{n-1} z^{-\nu} p_\nu(z, u) + \bar{P}(z, u)$$

$$= \sum_{\nu=0}^{n-2} z^{-\nu} p_\nu(z, u) + P(z, u), \tag{47}$$

where

$$p_\nu(z, u) = \sum_{s=1}^{n} c_{n-1-\nu,\, n-s} z^{s-1} u^{n-s},$$

is a homogeneous polynomial in (z, u) of degree $n - 1$.

The expression under the sign of integration in (39) we can evidently also reduce to a form like that on the right of (47). In this form further we may here omit terms of degree $n - 1$ since in the expression under the sign of integration terms of order higher than $n - 2$ do not present themselves.

We may then write

$$\sum_{e} \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) = z^{-1} q_1(z, u) + z^{-2} q_2(z, u)$$

$$+ \cdots + z^{-(n-1)} q_{n-1}(z, u) + Q(z, u), \tag{48}$$

where $q_1(z, u), \cdots, q_{n-1}(z, u)$ are homogeneous polynomials in z, u of degree $n - 1$ and where $Q(z, u)$ is a rational function of z, u which becomes 0 for $z = \infty$.

From (47) and (48) the expression (39) may be written in the form:

$$\int \{z^{-1} q_1(z, u) + z^{-2} q_2(z, u) + \cdots + z^{-(n-1)} q_{n-1}(z, u) + Q(z, u)\} dz$$

$$- \{p_0(z, u) + z^{-1} p_1(z, u) + \cdots + z^{-(n-2)} p_{n-2}(z, u) + P(z, u)\}. \tag{49}$$

Supposing now that the expression under the sign of integration is given while the subtracted expression is as yet undetermined, we shall attempt to determine the coefficients c of the polynomials $p_0(z, u), p_1(z, u), \cdots, p_{n-2}(z, u)$ in such way that the expression in (49) may be finite at ∞.

§ 9.

We have seen in § 6 that a function $z^r t(z, u)$, where $t(z, u)$ is a homogeneous polynomial in z, u of degree $n-1$, becomes infinite to the order $n+r-1$ but to no higher order, for one or more of the branches at ∞. We had there assumed $r \geq 0$ but the same evidently holds also for negative values of r.

The only portion of the expression (49) which might possibly involve infinities at ∞ of order $n-1$ is then evidently

$$\int z^{-1} q_1(z, u) dz - p_0(z, u). \tag{50}$$

We can however so choose $p_0(z, u)$ that the expression just written does not become infinite to an order higher than $n-2$. For, on taking

$$p_0(z, u) = \frac{q_1(z, u)}{n-1}$$

we shall have:

$$\int z^{-1} q_1(z, u) dz - p_0(z, u) = \int \left\{ z^{-1} q_1(z, u) - \frac{1}{n-1} \frac{d}{dz} q_1(z, u) \right\} dz \tag{51}$$

and the expression under the integral sign to the right evidently does not become infinite to the order $n-2$ for any of the branches at ∞. On substituting for du/dz its expression given in (5) we obtain the function

$$\frac{1}{n-1} \frac{d}{dz} q_1(z, u) = \frac{1}{n-1} \left\{ \frac{\partial q_1}{\partial z} + \frac{\partial q_1}{\partial u} \frac{du}{dz} \right\}$$

as a rational function of z, u which does not become infinite at ∞ to an order higher than $n-2$ nor elsewhere to an order as high as 1. It is then a function of the type represented in (48) and may therefore be similarly represented. The total expression under the sign of integration to the right of (51) may then also be represented in that form which appears to the right of (48) and we may consequently write:

$$z^{-1} q_1(z, u) - \frac{1}{n-1} \frac{d}{dz} q_1(z, u) = z^{-1} q_1'(z, u) + z^{-2} q_2'(z, u) + \cdots + z^{-(n-1)} q_{n-1}'(z, u) + Q'(z, u),$$

where $q_1'(z, u), \cdots, q_{n-1}'(z, u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q'(z, u)$ is a rational function of z, u which becomes 0 for $z = \infty$. Further, since the expression here represented does not become infinite to the order $n-2$ for any of the branches at ∞, we must have identically

$$q_1'(z, u) = 0.$$

From (51) we shall then have:

$$(52)\quad \int z^{-1}q_1(z, u)dz - p_0(z, u) = \int \{z^{-2}q_2'(z, u) + \cdots + z^{-(n-1)}q_{n-1}'(z, u) + Q'(z, u)\}dz,$$

and with the aid of this formula we may write our expression in (49) in the form:

$$(53)\quad \int \{z^{-2}q_2''(z, u) + \cdots + z^{-(n-1)}q_{n-1}''(z, u) + Q''(z, u)\}dz - \{z^{-1}p_1(z, u) + z^{-2}p_2(z, u) + \cdots + z^{-(n-2)}p_{n-2}(z, u) + P(z, u)\},$$

where

$$q_2''(z, u) = q_2(z, u) + q_2'(z, u),$$
$$\cdots\cdots\cdots$$
$$q_{n-1}''(z, u) = q_{n-1}(z, u) + q_{n-1}'(z, u);$$
$$Q''(z, u) = Q(z, u) + Q'(z, u).$$

The expression (53) evidently will not become infinite to an order greater than $n - 2$ for any of the branches at ∞ and we may further so choose $p_1(z, u)$ that it will not become infinite even to this order.

For the only portion of the expression (53) which might possibly involve infinities at ∞ of order $n - 2$ is evidently

$$\int z^{-2}q_2''(z, u)dz - z^{-1}p_1(z, u),$$

and on choosing

$$p_1(z, u) = \frac{q_2''(z, u)}{n - 2},$$

we shall have:

$$(54)\quad \int z^{-2}q_2''(z, u)dz - z^{-1}p_1(z, u) = \int \left\{z^{-2}q_2''(z, u) - \frac{d}{dz}\frac{z^{-1}q_2''(z, u)}{n - 2}\right\}dz,$$

where, on substituting for u in terms of z from the equation to a branch at ∞, the expression under the sign of integration on the right will evidently not be infinite to an order higher than $n - 4$ and the integral itself therefore to an order not higher than $n - 3$.

This expression we may write in the form:

$$z^{-2}q_2''(z, u) - \frac{1}{n - 2}\frac{d}{dz}z^{-1}q_2''(z, u) = \frac{n - 1}{n - 2}z^{-2}q_2''(z, u) - \frac{z^{-1}}{n - 2}\frac{dq_2''(z, u)}{dz}.$$

As in the case of $dq_1(z, u)/dz$ so here $dq_2''(z, u)/dz$—and in fact the derivative of any homogeneous polynomial in z, u of degree $n - 1$—is of the type of function represented in (48), and the expression just written may therefore evidently be written in the form:

$$\frac{n-1}{n-2}z^{-2}q_2''(z,u)-\frac{z^{-1}}{n-2}\frac{dq_2''(z,u)}{dz}=z^{-2}q_2'''(z,u)+z^{-3}q_3'''(z,u)\cdots + z^{-(n-1)}q_{n-1}'''(z,u)+Q'''(z,u),$$

where $q_2'''(z,u),\cdots,q_{n-1}'''(z,u)$ are homogeneous polynomials in z,u of degree $n-1$ and where $Q'''(z,u)$ is a rational function of z,u which becomes 0 for $z=\infty$. Further, since the expression here represented does not become infinite to the order $n-3$ for any of the branches at ∞, we must have identically

$$q_2'''(z,u)=0.$$

From (54) we shall then have:

$$\int z^{-2}q_2''(z,u)dz-z^{-1}p_1(z,u)=\int\{z^{-3}q_3'''(z,u)+z^{-4}q_4'''(z,u)+\cdots + z^{-(n-1)}q_{n-1}'''(z,u)+Q'''(z,u)\}dz, \tag{55}$$

and with the aid of this formula we may write the expression in (53) and therefore that in (49) in the form

$$\int\{z^{-3}q_3^{(4)}(z,u)+\cdots+z^{-(n-1)}q_{n-1}^{(4)}(z,u)+Q^{(4)}(z,u)\}dz-\{z^{-2}p_2(z,u) + \cdots+z^{-(n-2)}p_{n-2}(z,u)+P(z,u)\}, \tag{56}$$

where

$$q_3^{(4)}(z,u)=q_3''(z,u)+q_3'''(z,u),$$

. . . . . .

$$q_{n-1}^{(4)}(z,u)=q_{n-1}''(z,u)+q_{n-1}'''(z,u);$$

$$Q^{(4)}(z,u)=Q''(z,u)+Q'''(z,u).$$

The expression (56) will evidently not become infinite to an order greater than $n-3$ for any of the branches at ∞ and we might so choose $p_2(z,u)$ that it would not become infinite even to this order. Suppose now that by proper choice of the polynomials $p_2(z,u),\cdots,p_{r-1}(z,u)$ we have reduced (56) and therewith (49) to the form

$$\int\{z^{-(r+1)}q_{r+1}^{(2r)}(z,u)+\cdots+z^{-(n-1)}q_{n-1}^{(2r)}(z,u)+Q^{(2r)}(z,u)\}dz - \{z^{-r}p_r(z,u)+\cdots+z^{-(n-2)}p_{n-2}(z,u)\}+P(z,u), \tag{57}$$

where $q_{r+1}^{(2r)}(z,u),\cdots,q_{n-1}^{(2r)}(z,u)$ are homogeneous polynomials in z,u of degree $n-1$ and where $Q^{(2r)}(z,u)$ is a rational function of z,u which becomes 0 for $z=\infty$. This expression will evidently not become infinite to an order greater than $n-r-1$ for any of the branches at ∞ and we may so choose $p_r(z,u)$ that it will not become infinite to an order greater than $n-r-2$.

For the only portion of the expression (57) which might possibly involve infinities at ∞ of order $n-r-1$ is evidently

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)dz - z^{-r} p_r(z, u),$$

and on choosing

$$p_r(z, u) = \frac{q_{r+1}^{(2r)}(z, u)}{n-r-1}$$

we shall have:

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)dz - z^{-r} p_r(z, u) = \int \left\{ z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{d}{dz} \frac{z^{-r} q_{r+1}^{(2r)}(z, u)}{n-r-1} \right\} dz, \tag{58}$$

where, on substituting for u in terms of z from the equation to a branch at ∞, the expression under the sign of integration on the right will evidently not be infinite to an order higher than $n-r-3$ and the integral itself therefore to an order not higher than $n-r-2$.

This expression we may write in the form:

$$z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{1}{n-r-1} \frac{d}{dz} z^{-r} q_{r+1}^{(2r)}(z, u) = \frac{n-1}{n-r-1} z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{z^{-r}}{n-r-1} \frac{dq_{r+1}^{(2r)}(z, u)}{dz}.$$

The derivative $dq_{r+1}^{(2r)}(z, u)/(dz)$ is of the type of function represented in (48) and the expression with which we here have to do may evidently therefore be represented in the form:

$$\frac{n-1}{n-r-1} z^{-(r+1)} q_{r+1}^{(2r)}(z, u) - \frac{z^{-r}}{n-r-1} \frac{dq_{r+1}^{(2r)}(z, u)}{dz} = z^{-(r+1)} q_{r+1}^{(2r+1)}(z, u) + z^{-(r+2)} q_{r+2}^{(2r+1)}(z, u) + \cdots + z^{-(n-1)} q_{n-1}^{(2r+1)}(z, u) + Q^{(2r+1)}(z, u),$$

where $q_{r+1}^{(2r+1)}(z, u), \cdots, q_{n-1}^{(2r+1)}(z, u)$ are homogeneous polynomials in z, u of degree $n-1$ and where $Q^{(2r+1)}(z, u)$ is a rational function of z, u which becomes 0 for $z = \infty$. Further since, as we have just seen, the expression here represented does not become infinite to an order higher than $n-r-3$ for any of the branches at ∞, we must have identically

$$q_{r+1}^{(2r+1)}(z, u) = 0.$$

From (58) we shall then have:

$$\int z^{-(r+1)} q_{r+1}^{(2r)}(z, u)dz - z^{-r} p_r(z, u) = \int \{ z^{-(r+2)} q_{r+2}^{(2r+1)}(z, u) + \cdots + z^{-(n-1)} q_{n-1}^{(2r+1)}(z, u) + Q^{(2r+1)}(z, u) \} dz, \tag{59}$$

and with the aid of this formula we may write the expression in (57) and therewith that in (49) in the form

$$\int \{z^{-(r+2)}q_{r+2}^{(2r+2)}(z, u) + \cdots + z^{-(n-1)}q_{n-1}^{(2r+2)}(z, u) + Q^{(2r+2)}(z, u)\}dz$$
$$- \{z^{-(r+1)}p_{r+1}(z, u) + \cdots + z^{-(n-2)}p_{n-2}(z, u) + P(z, u)\} \tag{60}$$

where

$$q_{r+2}^{(2r+2)}(z, u) = q_{r+2}^{(2r)}(z, u) + q_{r+2}^{(2r+1)}(z, u),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$q_{n-1}^{(2r+2)}(z, u) = q_{n-1}^{(2r)}(z, u) + q_{n-1}^{(2r+1)}(z, u);$$
$$Q^{(2r+2)}(z, u) = Q^{(2r)}(z, u) + Q^{(2r+1)}(z, u).$$

By induction then we see that a proper choice of the homogeneous polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ will enable us to reduce (49) to the form

$$\int Q^{(2n-2)}(z, u)dz - P(z, u), \tag{61}$$

where $Q^{(2n-2)}(z, u)$ is a rational function of (z, u) which becomes equal to 0 for $z = \infty$. Since also $P(z, u)$ is finite for $z = \infty$ it follows that the expression (61) cannot become algebraically infinite at ∞. That it cannot become logarithmically infinite is also evident, for the integral in the equivalent expression (49), being only another form for the integral in (39), does not become infinite logarithmically.

We see then that the $n(n-1)$ coefficients in the $n-1$ polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ may be so chosen that the expression in (49) does not become infinite for $z = \infty$, and further we may show that there is only one way of so choosing them. For if we had two systems of functions $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$, $P(z, u)$ and $p_0'(z, u)$, $p_1'(z, u)$, $\cdots$, $p_{n-2}'(z, u)$, $P'(z, u)$ corresponding to two determinations of these coefficients, the expression

$$\int \{z^{-1}q_1(z, u) + z^{-2}q_2(z, u) + \cdots + z^{-(n-1)}q_{n-1}(z, u) + Q(z, u)\}dz$$
$$- \{p_0'(z, u) + z^{-1}p_1'(z, u) + \cdots + z^{-(n-2)}p_{n-2}'(z, u) + P'(z, u)\},$$

and that in (49) would both be finite for $z = \infty$. The difference between these two expressions would then be finite for $z = \infty$ and, since the functions $P(z, u)$ and $P'(z, u)$ are finite for this value of z, this also would be the case for the difference,

$$\{p_0(z, u) + z^{-1}p_1(z, u) + \cdots + z^{-(n-2)}p_{n-2}(z, u)\}$$
$$- \{p_0'(z, u) + z^{-1}p_1'(z, u) + \cdots + z^{-(n-2)}p_{n-2}'(z, u)\}.$$

From this however evidently follows that we must have identically:

$$p_0(z, u) = p_0'(z, u), \quad p_1(z, u) = p_1'(z, u), \quad \cdots, \quad p_{n-2}(z, u) = p_{n-2}'(z, u).$$

We see then that there is one and only one determination of the coefficients in the $n-1$ polynomials $p_0(z, u)$, $p_1(z, u), \cdots, p_{n-2}(z, u)$, such that the expression in (49) does not become infinite for $z = \infty$.

§ 10.

The $n(n-1)$ coefficients of the polynomials $p_0(z, u)$, $\cdots$, $p_{n-2}(z, u)$ in (47) are those which have been indicated by

$$\begin{array}{llll} c_{n-1,n-1}, & c_{n-1,n-2}, & \cdots, & c_{n-1,0}, \\ c_{n-2,n-1}, & c_{n-2,n-2}, & \cdots, & c_{n-2,0}, \\ c_{n-3,n-1}, & c_{n-3,n-2}, & \cdots, & c_{n-3,0}, \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ c_{1,n-1}, & c_{1,n-2}, & \cdots, & c_{1,0}. \end{array}$$

To these coefficients we shall suppose to be assigned that system of values for which the expression (49) does not become infinite at ∞.

With those coefficients in which the first suffix is equal to or greater than the second, we shall construct a polynomial

$$T_{n-1}(z, u) = \sum_{s=1}^{n} \sum_{r=0}^{s-1} c_{n-s+r,\, n-s} z^r u^{n-s}, \tag{62}$$

in which the constant term $c_{0\,0}$ may have any arbitrary value.

Corresponding to the $\frac{1}{2}(n-1)(n-2)$ coefficients

$$\begin{array}{l} c_{n-2,n-1}; \\ c_{n-3,n-1}, c_{n-3,n-2}; \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{1,n-1}, c_{1,n-2}, \cdots, c_{1,2}, \end{array} \tag{63}$$

in which the first suffix is smaller than the second, we shall have as many equations of the type given in (44). They will be obtained on giving to r and s respectively in (44) the values

$$r = 1, 2, \cdots, n-s-1; \quad s = 1, 2, \cdots, n-2. \tag{64}$$

This system of $\frac{1}{2}(n-1)(n-2)$ equations we shall attempt to satisfy by a proper choice of the quantities $\gamma_{i\kappa}$ on the right. The number of these quantities which present themselves in the $\frac{1}{2}(n-1)(n-2)$ equations is just equal to the number of the equations as we may readily show. For the greatest value for the sum of the suffixes i and κ in a term on the right of (44) is equal to $r+s-2$, and $r+s-2$, as we see from (64), cannot exceed $n-3$ in value. It follows therefore that the sum of the suffixes in the quantities $\gamma_{i\kappa}$ which ap-

pear in our equations, is in no case greater than $n-3$. Now the number of possible pairs of integers i, κ whose sum is equal to or less that $n-3$ is $\frac{1}{2}(n-1)(n-2)$, and that all the $\frac{1}{2}(n-1)(n-2)$ corresponding quantities $\gamma_{i\kappa}$ actually present themselves in our system of equations is immediately apparent. In fact each one of them appears in some one of the equations (44) with $e_{0,n}$ as coefficient as we see on taking $\rho=0$, $\sigma=0$ and substituting for r, s in the corresponding term $e_{0,n}\gamma_{r-1,s-1}$ the several pairs of values furnished by (64).

The number of the undetermined quantities $\gamma_{i\kappa}$ which present themselves in the system of equations (44) is then just equal to $\frac{1}{2}(n-1)(n-2)$, the number of these equations.

From the equations (44) we shall now seek to determine the quantities $\gamma_{i\kappa}$ in terms of the quantities $c_{n-s-r,n-s}$. For this purpose we shall write our equations in the form:

$$(65)\quad c_{n-s-r,n-s} = e_{0,n}\gamma_{r-1,s-1} + e_{1,n-1}\gamma_{r,s-2} + e_{2,n-2}\gamma_{r+1,s-3} + \cdots + e_{s-2,n-s+2}\gamma_{r+s-3,1} + e_{s-1,n-s+1}\gamma_{r+s-2,0} + \Gamma_{r-1,s-1},$$

where $\Gamma_{r-1,s-1}$ depends only on quantities $\gamma_{i\kappa}$, the sum of whose suffixes is less than $r+s-2$.

The number of equations of the type (65), in which the sum of the suffixes of the quantities γ in evidence on the right has a given value $r+s+2=\sigma$, is $\sigma+1$ and these $\sigma+1$ equations may evidently be written in the form:

$$\begin{aligned}
e_{0,n}\gamma_{0,\sigma} + e_{1,n-1}\gamma_{1,\sigma-1} + e_{2,n-2}\gamma_{2,\sigma-2} + \cdots + e_{\sigma-1,n-\sigma+1}\gamma_{\sigma-1,1} + e_{\sigma,n-\sigma}\gamma_{\sigma,0} &= c_{n-\sigma-2,n-\sigma-1} - \Gamma_{0,\sigma},\\
e_{0,n}\gamma_{1,\sigma-1} + e_{1,n-1}\gamma_{2,\sigma-2} + \cdots + e_{\sigma-2,n-\sigma+2}\gamma_{\sigma-1,1} + e_{\sigma-1,n-\sigma+1}\gamma_{\sigma,0} &= c_{n-\sigma-2,n-\sigma} - \Gamma_{1,\sigma-1},\\
e_{0,n}\gamma_{2,\sigma-2} + \cdots + e_{\sigma-3,n-\sigma+3}\gamma_{\sigma-1,1} + e_{\sigma-2,n-\sigma+2}\gamma_{\sigma,0} &= c_{n-\sigma-2,n-\sigma+1} - \Gamma_{2,\sigma-2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
e_{0,n}\gamma_{\sigma-1,1} + e_{1,n-1}\gamma_{\sigma,0} &= c_{n-\sigma-2,n-2} - \Gamma_{\sigma-1,1},\\
e_{0,n}\gamma_{\sigma,0} &= c_{n-\sigma-2,n-1} - \Gamma_{\sigma,0}.
\end{aligned}\qquad (66)$$

If now we suppose that the quantities $\gamma_{i\kappa}$ in which the sum of the suffixes is less than σ—and by consequence also the quantities Γ to the right of (66)—have been obtained in terms of the quantities c, the system of equations (66) will

determine in terms of these latter quantities the quantities $\gamma_{i\kappa}$ also, in which the sum of the suffixes is equal to σ. For the determinant of the coefficients on the left of (66) is different from 0 and has in fact the value $e_{0,n}^{\sigma+1} = 1$, since in our original equation $F(z, u) = 0$ we assumed $e_{0,n} = 1$.

More precisely still, on assuming that the quantities $\gamma_{i\kappa}$, in which the sum of the suffixes is less than σ, are linearly expressible in terms of the quantities c, it follows from the system of equations (66) that the quantities $\gamma_{i\kappa}$ in which the sum of the suffixes is equal to σ are also so expressible, and these equations furnish us with their expressions in terms of the c's.

Consider now the case $\sigma = 0$. In this case the system of equations (66) reduces to the single equation

$$e_{0,n}\gamma_{0,0} = c_{n-2,n-1},$$

so that the quantity $\gamma_{0\,0}$, in accord with our hypothesis above, is linearly expressible in terms of the quantities c and in fact is equal to one of them, namely $c_{n-2,n-1}$. It follows therefrom by our inductive reasoning above, that the quantities $\gamma_{0\,1}$ and $\gamma_{1\,0}$ corresponding to the case $\sigma = 1$ are also linearly expressible in terms of the quantities c, and by successive induction we see that the quantities $\gamma_{i\kappa}$ for which $\sigma(= i + \kappa)$ has one of the values $2, 3, \cdots, n-3$, are all linearly expressible in terms of the quantities c, the expressions for the same being furnished by the system of equations (44) supposed for that purpose to be grouped in a series of sets of the type given in (66).

The values of the $\frac{1}{2}(n-1)(n-2)$ quantities $\gamma_{i\kappa}$, so obtained, we may suppose to be substituted on the left in the corresponding equations of the type (43), when expressions for the $d + p = \frac{1}{2}(n-1)(n-2)$ quantities γ_λ in terms of our quantities $\gamma_{i\kappa}$ will be furnished by these equations in case we have

$$|a_\lambda^i b_\lambda^\kappa| \neq 0 \qquad (i+\kappa = 0, 1, 2, \cdots, n-3), \tag{67}$$

where by $|a_\lambda^i b_\lambda^\kappa|$ we indicate the determinant of the coefficients of the quantities γ_λ to the right of our equations (43). That the determination so made of the quantities γ_λ is unique is self-evident.

In the case where we have $|a_\lambda^i b_\lambda^\kappa| = 0$ it will be impossible to satisfy the system of equations (43) by any set of values γ_λ unless it happens that a certain relation exists between the quantities $\gamma_{i\kappa}$. Such a relation however does not in general exist, as we shall show later on.

Assuming then that our system of points (a_λ, b_λ) has been so chosen that the determinant above has a value other than 0, or, what evidently amounts to the same thing, in such way that the points do not all lie on a curve of degree $n + 3$, we shall suppose that the function

$$\sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u)$$

in (39) has been constructed with the values of the $d + p$ quantities γ_λ furnished by the equations (43) and with the polynomial $T_{n-1}(z, u)$ obtained in (62). Apart from the arbitrary constant in $T_{n-1}(z, u)$ the expression (39) so determined is unique.

We have shown that if the expression (39) is to be finite for all the branches at ∞ it must be determined as just stated. Conversely in case it has the determination here in question it will be finite for $z = \infty$.

For the coefficients γ_λ in (39) have been derived through the equations (43) and (44) from certain of the coefficients c in the polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$, these polynomials having been at the same time assumed by us to be those for which the expression (49) is finite at ∞; and conversely, from the values thus obtained for the coefficients γ_λ through the equations (43) and (44) in the reverse order, we should evidently arrive at the original coefficients c in question. Also the remaining coefficients c in the polynomials $p_0(z, u)$, $p_1(z, u)$, $\cdots$, $p_{n-2}(z, u)$ are identical with those in $T_{n-1}(z, u)$—the constant term $c_{0\,0}$ in the last named function being of course left out of consideration—so that the polynomials $p_0(z, u)$, $\cdots$, $p_{n-2}(z, u)$ in (49), regarded as a reduced form of the expression (39), are those for which the expression (49) was originally assumed to be finite at ∞. It follows therefore that the uniquely determined expression (39) is finite for $z = \infty$.

To recapitulate—we have proved in the above that in the expression (39) the coefficients γ_λ and the coefficients in the polynomial $T_{n-1}(z, u)$ can be chosen, and that uniquely, in such way that this expression does not become infinite for $z = \infty$ in case we have so selected our $d + p$ points (a_λ, b_λ) that they do not lie on a curve of degree $n - 3$, and we have furnished a method for determining the coefficients in question.

We have assumed from the outset that the d double points are included under the $d + p$ points (a_λ, b_λ) in the expression (39). It might however be remarked in this connection that all our reasoning above in regard to this expression holds good for any $d + p$ arbitrary points. That we can choose $d + p$ points (a_λ, b_λ) which do not lie on a curve of degree $n - 3$ is self-evident, for $d + p = \frac{1}{2}(n - 1)(n - 2)$ is the number of coefficients in the general equation to a curve of degree $n - 3$.

That however, among $d + p$ points so chosen we may include the d double points is not self-evident. It may however be proved that the passage through the double points imposes d conditions on our curve, or in other words that the conditions of adjointness are independent of one another.

I shall not stop here to prove the theorem in question but merely remark that the methods of the present paper furnish an extremely simple algebraic proof of it, which I propose to give in another paper at an early date. Meantime I shall accept the principle of the independence of the conditions of adjointness as

demonstrated, and we may therefore assume that the d double points are included among the $d + p$ points (a_λ, b_λ); and in (39), whatever the values of the constants in the expression under the sign of integration, we may regard the coefficients γ_λ and the coefficients in the polynomial $T_{n-1}(z, u)$ in the subtracted function as so determined that the difference does not become infinite for $z = \infty$.

§ 11.

Instead of (39) I shall now consider the expression

$$(68)\quad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \left(\frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + m_\lambda \Pi(a_\lambda, b_\lambda) \right) + T_{n-1}(z, u) \right\},$$

where by $\Pi(a_\lambda, b_\lambda)$ we indicate an elementary integral of the second kind possessing a single infinity of the first order at the point (a_λ, b_λ), and where the coefficients m_λ may be supposed to be so determined that the expressions

$$\frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + m_\lambda \Pi(a_\lambda, b_\lambda)$$

do not become infinite at the points (a_λ, b_λ).

The residues of the integrals $\Pi(a_\lambda, b_\lambda)$ we may assume to have the value 1, and the coefficients m_λ will then evidently be determined by the equations

$$\gamma_\lambda F'_{b_\lambda}(a_\lambda, b_\lambda) + m_\lambda = 0.$$

In the case of the double points the coefficients m_λ will of course have the value 0.

The expression (68) does not possess infinities other than those of (39) and it does not possess even these, for the infinities of the latter expression are all to be found among the points (a_λ, b_λ), for all of which (68) is finite. The expression (68) is then everywhere finite and must therefore be an integral of the first kind. On indicating this integral by I and on supposing (a_{p+1}, b_{p+1}), (a_{p+2}, b_{p+2}), $\cdots$, (a_{p+d}, b_{p+d}) to denote the d double points we may write (68) in the form:

$$(69)\quad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz - \left\{ \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u) \right\} - \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda) = \mathrm{I}.$$

Returning now to formula (35) and substituting therein for the integral

$$\int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz$$

the expression derived from (69), we obtain:

$$\int R(z, u)dz = \overline{G}(z, u) + \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + T_{n-1}(z, u) + \mathrm{I} + \sum_{\lambda=1}^{p} m_\lambda \mathrm{II}(a_\lambda, b_\lambda) + \mathrm{III}.$$

Indicating the aggregate rational portion of this expression by $G(z, u)$ we have:

$$\int R(z, u)dz = G(z, u) + \mathrm{I} + \sum_{\lambda=1}^{p} m_\lambda \mathrm{II}(a_\lambda, b_\lambda) + \mathrm{III}. \tag{70}$$

Our original Abelian integral $\int R(z, u)dz$ is herewith reduced to the sum of a rational function, an integral of the first kind, a linear function of p elementary integrals of the second kind and an integral of the third kind, the several steps in the process thereto employed being all of algebraic character. From a glance at the process also, it is plain that the coefficients m_λ and the function $G(z, u)$, apart from the arbitrary constant of integration, are completely determined when the system of points $(a_1, b_1), \cdots, (a_p, b_p)$ is given.

§ 12.

We have effected the reduction of the general Abelian integral indicated in (70), on the hypothesis that the p points $(a_1, b_1), \cdots, (a_p, b_p)$ do not lie on an adjoint curve of degree $n - 3$. In the case where they do lie on an adjoint curve of degree $n - 3$ it may readily be shown that such a reduction is not in general possible.

For suppose that we have a system of p points

$$(\overline{a}_1, \overline{b}_1), \cdots, (\overline{a}_p, \overline{b}_p)$$

lying on an adjoint curve of degree $n - 3$ and that the p points

$$(a_1, b_1), \cdots, (a_p, b_p)$$

do not all lie on such a curve.

Referring now to the expression in (39), which we sought to render finite at ∞ by proper choice of the coefficients γ_λ and of the polynomial $T_{n-1}(z, u)$ and regarding in particular the formulæ in (43), (44), (49), we remark that the constants c are independent of our choice of the p points $(a_1, b_1), \cdots, (a_p, b_p)$. From the form in (49), namely, we see that they depend for their values on the integral

$$\int \{z^{-1}q_1(z, u) + z^{-2}q_2(z, u) + \cdots + z^{-(n-1)}q_{n-1}(z, u) + Q(z, u)\}dz$$

alone, or what is the same thing, on the integral

$$\int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz$$

in (39), independently of what our choice of the system of p points (a_1, b_1), $\cdots$, (a_p, b_p) may happen to be.

From the equations (44) again the quantities $\gamma_{i\kappa}$ depend upon the quantities c alone and are therefore also independent of the choice of our system of points $(a_1, b_1), \cdots, (a_p, b_p)$.

In our system of equations (43), then, the quantities $\gamma_{i\kappa}$ on the left have the same values whatever the p points $(a_1, b_1), \cdots, (a_p, b_p)$ may be and depend solely on the integral in (39).

To prove that we cannot employ the system of p points $(\bar{a}_1, \bar{b}_1), \cdots, (\bar{a}_p, \bar{b}_p)$ for the reduction of the general Abelian integral it will suffice to show in some particular case that the quantities $\gamma_{i\kappa}$ corresponding to an integral of the type of that in (39) are such that the system of equations

$$\gamma_{i\kappa} = \sum_{\lambda=1}^{d+p} \gamma_\lambda \bar{a}_\lambda^i \bar{b}_\lambda^\kappa \tag{71}$$

cannot be satisfied. The d double points for convenience I shall designate indifferently by $(a_{p+1}, b_{p+1}), \cdots, (a_{p+d}, b_{p+d})$ or $(\bar{a}_{p+1}, \bar{b}_{p+1}), \cdots, (\bar{a}_{p+d}, \bar{b}_{p+d})$. Consider now in particular the function

$$\sum_{\lambda=1}^{p+d} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda),$$

where we suppose the coefficients m_λ to be so determined that the function does not become infinite for any of the points (a_λ, b_λ). The derivative of this function with regard to z will not become infinite to an order as high as 1 for any finite point, and for the points at ∞ it will evidently not become infinite to an order higher than $n - 3$ or, what is sufficient for our purpose, it will not become infinite at ∞ to an order higher than $n - 2$. This derivative therefore will plainly be of that type of function which appears under the sign of integration in (39).

On properly determining the coefficients β and the polynomial $T_{n-2}(z, u)$, then, we may write:

$$\sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) = \frac{d}{dz} \left\{ \sum_{\lambda=1}^{p+d} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)} + \sum_{\lambda=1}^{p} m_\lambda \Pi(a_\lambda, b_\lambda) \right\}.$$

The difference

$$(72) \qquad \int \left\{ \sum_e \frac{\beta_e F(a_e, u)}{(z - a_e)(u - b_e)} + T_{n-2}(z, u) \right\} dz + \sum_{\lambda=1}^{d+p} \frac{\gamma_\lambda F(a_\lambda, u)}{(z - a_\lambda)(u - b_\lambda)}$$

is then evidently an expression of the type (39) which does not become infinite for $z = \infty$.

Here the quantities $\gamma_{i\kappa}$ corresponding to the integral in (72) are given by the system of equations:

$$\gamma_{i\kappa} = \sum_{\lambda=1}^{d+p} \gamma_\lambda a_\lambda^i b_\lambda^\kappa$$

and on starting out from a properly chosen set of values for the quantities γ_λ we could arrive at any arbitrarily assigned system of values for the quantities $\gamma_{i\kappa}$, since for the determinant of the coefficients of the quantities γ_λ to the right of these equations we have $|a_\lambda^i b_\lambda^\kappa| \neq 0$. To this assigned system of values for the quantities $\gamma_{i\kappa}$ will correspond a certain set of values γ_λ and therewith a certain integral of the type in (72).

Now for the reduction of this integral the system of p points $(\bar{a}_1, \bar{b}_1), \cdots, (\bar{a}_p, \bar{b}_p)$ will not serve if values have been assigned to the quantities $\gamma_{i\kappa}$ for which the equations (71) cannot be solved. Such values, however, it is always possible to assign to these quantities since for the determinant of the coefficients of the quantities $\bar{\gamma}$ to the right of these equations we have $|\bar{a}_\lambda^i, \bar{b}_\lambda^\kappa| = 0$. It follows then that a system of p points $(\bar{a}_1, \bar{b}_1), \cdots, (\bar{a}_p, \bar{b}_p)$ which lie on an adjoint curve of degree $n - 3$ cannot serve for the reduction of the general Abelian integral.

It will be remembered that in obtaining the reduced form on the right of (70), we assumed that no one of the p points $(a_1, b_1), \cdots, (a_p, b_p)$ was a double point. The only limitation otherwise placed upon our choice of these points was that they should not all be zeros of one and the same adjoint curve of degree $(n - 3)$.

It may however be remarked that in obtaining the reduced form in question, we might also have included under the p elementary integrals of the second kind integrals corresponding to the double points, or more accurately speaking, to certain of the branches through the double points.

To have done this it would have been necessary to introduce some modifications into our reasoning which would have added to the length of the same. The modifications in question however offer no special difficulty and we may in all cases make the general statement that any Abelian integal can be reduced to the sum of a rational function of (z, u), of an integral of the first kind, of an expression linear in p elementary integrals of the second kind, and of an integral of the third kind, where in our choice of the p elementary integrals of the second kind we are subject to this one limitation only, that an adjoint curve of

degree $n - 3$ shall not exist which—over and above the zeros already implied in the fact of adjointness—possesses zeros corresponding to the several points (a_λ, b_λ) (and more precisely in the case of double points, corresponding to the individual branches), for which our p elementary integrals have been respectively constructed.

To cover all cases we should have to modify the notation employed in the formula (70) to designate an elementary integral of the second kind, as the notation $\Pi(a_\lambda, b_\lambda)$ in the case of a double point does not distinguish between the elementary integrals corresponding to the separate branches, either or both of which may happen to be included in our system of p elementary integrals of the second kind.

CHICAGO, *November* 27, 1900.

UEBER FLÄCHEN VON CONSTANTER GAUSSSCHER KRÜMMUNG*

VON

DAVID HILBERT

Ueber Flächen von negativer constanter Krümmung.

Nach BELTRAMI† verwirklicht eine Fläche von negativer constanter Krümmung ein Stück einer LOBATSCHEFSKIJschen (nicht-EUKLIDischen) Ebene, wenn man als Gerade der LOBATSCHEFSKIJschen Ebene die geodätischen Linien der Fläche von constanter Krümmung betrachtet und als Längen und Winkel in der LOBATSCHEFSKIJschen Ebene die wirklichen Längen und Winkel auf der Fläche nimmt. Unter den bisher untersuchten Flächen negativer constanter Krümmung finden wir *keine*, die sich stetig und mit stetiger Aenderung ihrer Tangentialebene in der Umgebung jeder Stelle überall hin ausdehnt; vielmehr besitzen die bekannten Flächen negativer constanter Krümmung singuläre Linien, über die hinaus eine stetige Fortsetzung mit stetiger Aenderung der Tangentialebene nicht möglich ist. Aus diesem Grunde gelingt es mittelst keiner der bisher bekannten Flächen negativer constanter Krümmung, die *ganze* LOBATSCHEFSKIJsche Ebene zu verwirklichen, und es erscheint uns die Frage von principiellem Interesse, *ob die* GANZE LOBATSCHEFSKIJ*sche Ebene überhaupt nicht durch eine analytische Fläche negativer constanter Krümmung auf die* BELTRAMI*sche Weise zur Darstellung gebracht werden kann.*

Um diese Frage zu beantworten, gehen wir von der Annahme einer analytischen Fläche der negativen constanten Krümmung -1 aus, die im Endlichen überall sich regulär verhält und keine singulären Stellen aufweist; wir werden dann zeigen, dass diese Annahme auf einen Widerspruch führt. Eine solche Fläche, wie wir sie annehmen wollen, ist durch folgende Aussagen vollständig charakterisirt:

> Jede im Endlichen gelegene Verdichtungstelle von Punkten der Fläche ist ebenfalls ein Punkt der Fläche.

Bedeutet O irgend einen Punkt der Fläche, so ist es stets möglich, das rechtwinklige Coordinatenkreuz x, y, z so zu legen, dass O der Anfangspunkt des Coordinatensystems wird und die Gleichung der Fläche in der Umgebung dieses Punktes O wie folgt lautet:

* Presented to the Society, October 27, 1900. Received for publication October 9, 1900.

† Giornale di Matematiche, Bd. 6, 1868.

$$z = ax^2 + by^2 + \mathfrak{P}(x, y)\,,$$

wo die Constanten a, b die Relation

$$4ab = -1$$

befriedigen und die Potenzreihe $\mathfrak{P}(x, y)$ nur Glieder dritter oder höherer Dimension in x, y enthält. Offenbar ist dann die z-Axe die Normale der Fläche und die x- und y-Axe geben die Richtungen an, die durch die Hauptkrümmungen der Fläche bestimmt sind.

Die Gleichung

$$ax^2 + by^2 = 0$$

bestimmt die beiden Haupttangenten der Fläche durch den Punkt O in der xy-Ebene; dieselben sind daher stets von einander getrennt und geben die Richtungen an, in denen die beiden Asymptotencurven der Fläche durch den beliebigen Punkt O verlaufen. Jede dieser Asymptotencurven gehört einer einfachen Schaar von Asymptotencurven an, die die ganze Umgebung des Punktes O auf der Fläche regulär und lückenlos überdecken. Verstehen wir daher unter u, v genügend kleine Werthe, so können wir gewiss folgende Construktion ausführen. Wir tragen auf einer der beiden durch O gehenden Asymptotencurven den Parameterwerth u von O als Länge ab, ziehen durch den erhaltenen Endpunkt die andere mögliche Asymptotencurve und tragen auf dieser den Parameterwerth v ab: der nun erhaltene Endpunkt ist ein Punkt der Fläche, der durch die Parameterwerthe u, v eindeutig bestimmt ist. Fassen wir demgemäss die rechtwinkligen Coordinaten x, y, z der Fläche als Functionen von u, v auf, indem wir setzen:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)\,, \tag{1}$$

so sind diese jedenfalls für genügend kleine Werthe von u, v reguläre analytische Functionen von u, v.

Die bekannte Theorie der Flächen von der constanten Krümmung -1 liefert uns ferner die folgenden Thatsachen:

Bedeutet φ den Winkel zwischen den beiden Asymptotencurven durch den Punkt u, v, so erhalten die drei Fundamentalgrössen der Fläche die Werthe:

$$e \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = 1\,,$$

$$f \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial x}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = \cos\varphi\,,$$

$$g \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 1\,;$$

und mithin wird das Quadrat der Ableitung der Bogenlänge einer beliebigen

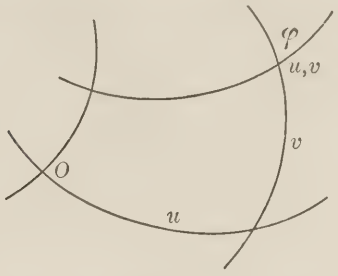


FIG. 1.

Curve auf der Fläche nach einem Parameter t von der Form:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{du}{dt}\right)^2 + 2\cos\varphi\,\frac{du}{dt}\frac{dv}{dt} + \left(\frac{dv}{dt}\right)^2. \tag{2}$$

Der Winkel φ genügt als Function von u, v der partiellen Differentialgleichung

$$\frac{\partial^2\varphi}{\partial u\partial v} = \sin\varphi\,. \tag{3}$$

Die Formel (2) beweist den bekannten Satz :*

In jedem Vierecke, das von vier Asymptotencurven unserer Fläche gebildet wird, sind die gegenüberliegenden Bogen einander gleich.

Die Formel (3) gestattet die Berechnung des Flächeninhaltes eines von Asymptotencurven gebildeten Viereckes mittelst seiner Winkel; DARBOUX † ist auf diesem Wege zu dem folgenden Satze gelangt:

Der Flächeninhalt eines aus Asymptotencurven gebildeten Viereckes auf unserer Fläche ist gleich der Summe der Winkel des Viereckes vermindert um 2π.

Die Formeln (1) liefern eine Parameterdarstellung unserer Fläche, bei welcher die Coordinatenlinien

$$u = \text{const.}, \quad v = \text{const.}$$

die Asymptotencurven sind. Nach den obigen Ausführungen erweisen sich die rechtwinkligen Coordinaten x, y, z gewiss für genügend kleine Werthe von u, v als umkehrbar eindeutige Functionen der Variabeln u, v d. h. die Formeln (1) vermitteln jedenfalls die umkehrbar eindeutige Abbildung eines *Stückes der* uv-Ebene in der Umgebung des Punktes $u = 0$, $v = 0$ auf ein *Stück* unserer Fläche in der Umgebung des Punktes O. Unsere Aufgabe besteht darin, die

* DINI, Annali di Mathematica, Bd. 4, 1870, S. 175.
DARBOUX, Leçons sur la théorie générale des surfaces, Bd. 3, No. 773.
BIANCHI, Lezioni di geometria differenziale, § 67.

† loc. cit., Bd. 3, No. 773.

gesammte Abbildung der uv-Ebene auf unsere Fläche zu untersuchen, welche durch die analytische Fortsetzung der Formeln (1) erhalten wird.

Fassen wir irgend eine Asymptotencurve unserer Fläche ins Auge, so erkennen wir sofort, dass dieselbe im Endlichen keinen singulären Punkt haben und daher auch nirgends aufhören darf; denn bei Annahme einer solchen singulären Stelle könnten wir in dieselbe den Punkt O verlegen und dies gäbe einen Widerspruch mit unseren früheren Ausführungen, wonach durch O stets zwei reguläre Asymptotencurven hindurchlaufen und eine genügend kleine Umgebung des Punktes O auf unserer Fläche durch regulär verlaufende Asymptotencurven lückenlos erfüllt wird.

Aus diesem Umstande entnehmen wir die analytische Thatsache, dass die Functionen x, y, z für alle reellen u, v eindeutig und unbegrenzt fortsetzbar sind. Um dies sicher zu erkennen, tragen wir vom Punkte O aus auf der Asymptotencurve $v = 0$ die Länge u nach der einen oder anderen Richtung hin, je nachdem u positiv oder negativ ist, ab, ziehen durch den erhaltenen Endpunkt die andere Asymptotencurve, tragen dann auf dieser die Länge v nach der einen oder anderen Richtung hin, je nachdem v positiv oder negativ ausfällt, ab und ertheilen endlich dem so erhaltenen Endpunkte, der die rechtwinkligen Coordinaten x, y, z haben möge, die Parameterwerthe u, v. Auf diese Weise ist jedem Punkte der uv-Ebene jedenfalls ein bestimmter Punkt unserer Fläche zugeordnet und die Funktionen x, y, z die diese Zuordnung vermitteln sind eindeutige für alle reellen Variabeln u, v definirte und reguläre analytische Functionen.

Auch zeigt sich sofort, dass umgekehrt jedem Punkte unserer Fläche mindestens ein Werthepaar u, v entspricht. Um dies einzusehen, bezeichnen wir diejenigen Punkte, deren Coordinaten durch Functionswerthe

$$x(u, v), \quad y(u, v), \quad z(u, v)$$

dargestellt werden, mit P, dagegen die Punkte der Fläche, die durch unsere Abbildung (1) nicht betroffen werden, mit Q. Würden nun im Endlichen ein oder mehrere Punkte Q vorhanden sein, so müsste es gewiss mindestens einen Punkt A auf der Fläche geben, in dessen beliebiger Nähe sowohl Punkte P als auch Punkte Q gelegen sind.

Nach den früheren Ausführungen existiren nun für die Umgebung des Punktes A zwei Schaaren von Asymptotencurven, deren jede diese Umgebung einfach und lückenlos überdeckt. Unter diesen Asymptotencurven muss nothwendig mindestens eine solche vorhanden sein, die sowohl einen Punkt P als auch einen Punkt Q enthält. In der That, fassen wir eine der beiden durch A hin durchgehenden Asymptotencurven ins Auge und nehmen wir an dieselbe bestände aus lauter Punkten P (bez. Q), so würden die Asymptotencurven derjenigen Schaar, zu welcher jene erstere Asymptotencurve nicht gehört, mindestens je einen Punkt P (bez. Q) nämlich den Schnittpunkt mit der ersteren Asymptotencurve enthal-

ten. Die sämmtlichen Curven dieser Schaar können aber gewiss nicht aus lauter Punkten P (bez. Q) bestehen, da sonst die ganze Umgebung von A nur Punkte P (bez. Q) enthielte.

Es sei nun l die Länge eines Stückes einer Asymptotencurve, deren Anfangspunkt ein Punkt P und deren Endpunkt ein Punkt Q sein möge. Fassen wir die beiden durch den Anfangspunkt P laufenden Asymptotencurven der Fläche ins Auge, so ist jenes Stück von der Länge l nothwendig die Fortsetzung einer dieser beiden Asymptotencurven, und wenn daher u, v die Coordinaten des Anfangspunktes P sind, so wird der Endpunkt jenes Curvenstückes entweder durch die Parameterwerthe $u \pm l$, v oder u, $v \pm l$ dargestellt — entgegen unserer Annahme derzufolge der Endpunkt Q nicht durch die Formeln (1) darstellbar sein sollte.

Damit ist bewiesen worden, dass durch die Formeln (1) die ganze Fläche zur Darstellung gebracht wird, wenn u, v alle reellen Zahlenwerthe durchläuft.

Endlich ist es für unsere Untersuchung nothwendig einzusehen, dass die Formeln (1) jeden Punkt der Fläche nur durch *ein* Werthepaar u, v darstellen, d. h. dass die gefundene Abbildung (1) unserer Fläche auf die uv-Ebene nicht blos für genügend kleine Gebiete sondern im Ganzen genommen eine *umkehrbar*-eindeutige sein muss.

Wir beweisen zu dem Zwecke der Reihe nach folgende Sätze:

1. *Es giebt auf unserer Fläche keine geschlossene, d. h. in sich zurückkehrende Asymptotencurve.*

Zum Beweise nehmen wir im Gegentheil an, es sei eine solche Asymptotencurve auf unserer Fläche vorhanden. Wir construire durch jeden Punkt

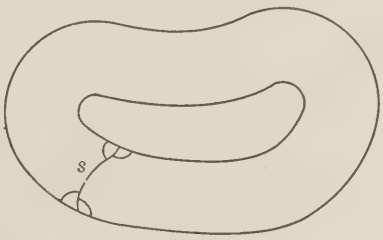


FIG. 2.

derselben die andere Asymptotencurve und tragen auf diesen Curven stets ein Stück s nach derselben Seite hin. Die erhaltenen Endpunkte werden dann entweder eine in sich zurücklaufende Asymptotencurve bilden, oder die Endpunkte des Stückes s beschreiben erst nach zweimaligem Durchlaufen der Grundcurve eine in sich zurückkehrende Asymptotencurve—ein Fall, der eintreten könnte, wenn unsere Fläche eine sogenannte Doppelfläche wäre. Fassen wir nun eine derjenigen Asymptotencurven von der Länge s ins Auge, die uns vorhin zur Construction der neuen geschlossenen Asymptotencurve diente, so bildet dieselbe

doppelt gerechnet zusammen mit den beiden geschlossenen Asymptotencurven ein Asymptotenviereck, dessen Winkelsumme offenbar genau gleich 2π ist. Diese Thatsache aber steht im Widerspruch zu dem vorhin angeführten Satze, wonach der Inhalt eines Asymptotencurvenvierecks stets gleich dem Ueberschuss der Summe seiner Winkel über 2π ist und dieser Ueberschuss daher nothwendig positiv sein muss.

2. *Irgend zwei durch einen Punkt gehende Asymptotencurven schneiden sich in keinem anderen Punkt unserer Fläche.*

Wir denken uns eine Asymptotencurve a nach beiden Richtungen hin ins Unendliche verlängert und dann durch einen Punkt P_0 derselben nach einer Seite die andere Asymptotencurve b gezogen. Nehmen wir dann im Gegensatz zu unserer Behauptung an, dass diese Asymptotencurve b die ursprüngliche a zum ersten Mal im Punkt P_1 schnitte, so sind die folgenden zwei Fälle denkbar:

Erstens: die Asymptotencurve b könnte so verlaufen, dass sie in P_1 von derselben Seite der Asymptotencurve a her eintritt, als sie dieselbe verlassen hat;

Zweitens: die Asymptotencurve b könnte derart verlaufen, dass sie von der anderen Seite der ursprünglichen Asymptotencurve a herkommt und mithin nach Verlassen des Schnittpunktes P_1 nach der nämlichen Seite der Asymptotencurve a gerichtet ist, wie anfänglich, als sie vom Punkte P_0 ausging.

Wir wollen zeigen, dass beide Fälle unmöglich sind. Was den *ersten* Fall betrifft, so bezeichnen wir die Länge der Strecke P_0P_1 auf a mit l und die Mitte dieser Strecke mit Q. Sodann denken wir uns durch jeden Punkt der

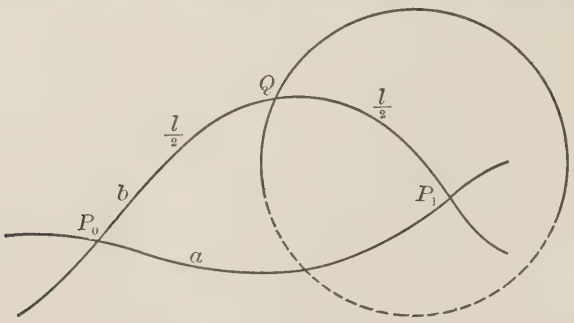


FIG. 3.

Asymptotencurve a die andere Asymptotencurve gezogen und nach der Seite, nach welcher hin die fragliche Strecke P_0P_1 auf b liegt, die Länge $\frac{1}{2}l$ abgetragen. Aus den Punkten P_0 und P_1 der Curve a erhalten wir auf diese Weise den nämlichen Punkt Q als Endpunkt. Die sämmtlichen erhaltenen Endpunkte bilden mithin eine Asymptotencurve, welche durch den Punkt Q geht und zu demselben Punkte Q in der nämlichen Richtung zurückkehrt. Dies ist unmöglich, da es nach 1. auf unserer Fläche keine geschlossene Asymptotencurve giebt.

Damit ist gezeigt, dass der erste Fall nicht stattfinden kann. Aber auch der *zweite* Fall ist unmöglich. Verliefe nämlich die Asymptotencurve in der Weise, dass sie nach Ueberschreitung des Schnittpunktes P_1 die nämliche Richtung aufweist, wie früher in P_0, so könnten wir die Fortsetzung dieses Stückes P_0P_1 der Asymptotencurve b über P_1 hinaus offenbar dadurch erhalten, dass wir von P_0 ausgehend durch jeden Punkt des Stückes P_0P_1 von b die andere Asymptotencurve construiren und auf allen diesen Asymptotencurven nach der betreffenden Seite hin das gleiche Stück P_0P_1 der Asymptotencurve a abtragen. Die erhaltenen Endpunkte bilden die Fortsetzung der Asymptotencurve b von P_1 bis zu einem Punkte P_2 auf a. Aus diesem Stücke P_1P_2 der Asymptotencurve b können wir in gleicher Weise ein neues Stück der Asymptotencurve b construiren, welches über P_2 hinaus geht und bis zu einem Punkte P_3 auf a reicht u. s. f. Auch ist klar, wie wir die Asymptotencurve b nach der anderen Richtung hin über P_0 hinaus durch die entsprechende Construction fortsetzen können und so der Reihe nach zu den Curvenstücken P_0P_{-1}, $P_{-1}P_{-2}, \cdots$ gelangen. Die Asymptotencurve b schneidet also die Asymptotencurve a in den unendlich vielen gleich weit von einander entfernten Punkten:

$$\cdots P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3 \cdots.$$

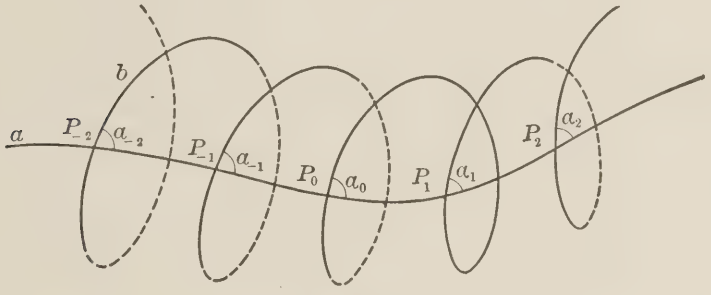


Fig. 4.

Die Winkel, die die Asymptotencurve b mit a in bestimmten Sinne in jenen Schnittpunkten bildet, bezeichnen wir bez. mit

$$\cdots \alpha_{-3},\ \alpha_{-2},\ \alpha_{-1},\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3 \cdots.$$

Wir betrachten nun das Asymptotencurvenviereck $P_0P_1P_2P_1P_0$, welches von den zwei Stücken P_0P_1, P_2P_1 auf a und den zwei Curvenstücken P_1P_2, P_1P_0 auf b gebildet wird. Die vier Winkel dieses Vierecks sind

$$\alpha_0,\ \pi - \alpha_1,\ \alpha_2,\ \pi - \alpha_1,$$

und da nach dem angeführten Satze über das Asymptotencurvenviereck der Inhalt desselben gleich dem Ueberschuss der Summe seiner Winkel über 2π ist und dieser Ueberschuss daher positiv sein muss, so folgt

$$a_0 + \pi - a_1 + a_2 + \pi - a_1 > 2\pi,$$

d. h.

$$a_0 - a_1 > a_1 - a_2. \tag{4}$$

Ebenso folgt allgemein

$$a_k - a_{k+1} > a_{k+1} - a_{k+2}, \qquad (k = 0, \pm 1, \pm 2, \cdots). \tag{5}$$

Wegen der obigen Ungleichung (4) können jedenfalls $a_0 - a_1$ und $a_1 - a_2$ nicht zugleich 0 sein; wir dürfen die Annahme

$$a_0 - a_1 \neq 0$$

treffen. Aus (5) folgen die Ungleichungen:

$$a_{-p} - a_{-p+1} > a_0 - a_1 \qquad (p = 1, 2, 3, \cdots), \tag{6}$$

$$a_0 - a_1 \quad > a_p - a_{p+1}$$

oder

$$a_{p+1} - a_p \quad > a_1 - a_0 \qquad (p = 1, 2, 3, \cdots). \tag{7}$$

Bilden wir die Ungleichungen (6) und (7) für $p = 1, 2, 3, \cdots, n$, so folgt durch Addition derselben leicht

$$a_{-n} > a_0 + n(a_0 - a_1),$$

$$a_{n+1} > a_1 + n(a_1 - a_0).$$

Fällt nun $a_0 - a_1 > 0$ aus, so ist für genügend grosse Werthe von n jedenfalls die erstere dieser beiden Gleichungen unmöglich, da die Winkel a_k sämmtlich kleiner als π sind; fällt dagegen $a_1 - a_0 > 0$ aus, so folgt aus demselben Grunde für genügend grosse Werthe von n die Unmöglichkeit der letzteren Gleichung.

Die Asymptotencurve b darf daher auf keine der beiden angenommenen Arten verlaufen und mithin ist der Beweis für 2. vollständig erbracht.

3. *Eine Asymptotencurve unserer Fläche durchsetzt sich selbst an keiner Stelle, d. h. sie besitzt keinen Doppelpunkt.*

Zum Beweise nehmen wir im Gegentheil an, es existire eine Asymptotencurve mit einem Doppelpunkt; dann verlegen wir den Anfangspunkt der krummlinigen Coordinaten u, v in diesen Doppelpunkt und wählen die beiden Zweige der Curvenschleife zu Coordinatenlinien, nach der Schleife hin den positiven Sinn gerechnet.

Wir ziehen jetzt vom Doppelpunkte (0, 0) oder vom Punkte $(-s, 0)$ beginnend auf der u-Coordinate durch jeden Punkt der Schleife die andere Asymptotencurve und tragen auf dieser nach der positiven Seite hin eine Strecke s ab: wählen wir diese Strecke s genügend klein, so werden die sämmtlichen erhaltenen Endpunkte nach einem oben angeführten Satze über das Asymptotencurvenviereck wiederum eine Asymptotencurve bilden. Diese Asymptotencurve geht vom

Punkte $(0, s)$ bez. $(-s, s)$ aus, durchsetzt sich selbst im Punkte (s, s) bez. $(-s, s)$ und endigt im Punkte $(s, 0)$ bez. $(-s, 0)$. Wir sehen also, dass die eben construirte Asymptotencurve die ursprüngliche Asymptotencurve in zwei

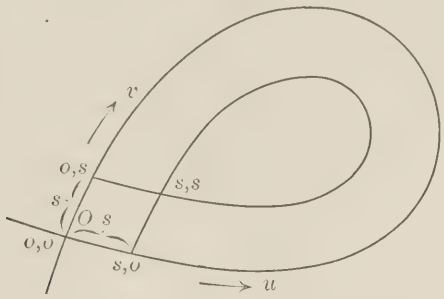


Fig. 5.

verschiedenen Punkten $(0, s)$ und $(s, 0)$ bez. $(0, s)$ und $(-s, 0)$ schneidet; dies ist nach 2. unmöglich.

4. *Wenn wir durch jeden Punkt einer Asymptotencurve a die andere Asymptotencurve ziehen und auf dieser nach der nämlichen Seite hin eine bestimmte Strecke s abtragen, so bilden die erhaltenen Endpunkte eine neue Asymptotencurve b, die die ursprüngliche Asymptotencurve a an keiner Stelle schneidet.*

Denn wäre P ein Schnittpunkt der Asymptotencurve b mit der ursprünglichen a, und tragen wir von P aus auf der Asymptotencurve b die Strecke s nach der betreffenden Seite von a hin ab, so müsste der weitere durch den entstehenden Endpunkt Q hindurchgehende Asymptotencurvenzweig ebenfalls zur Asympto-

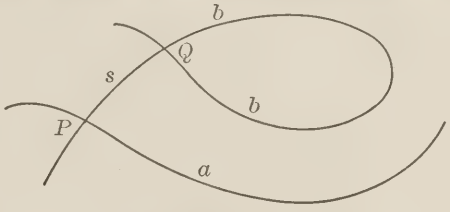


Fig. 6.

tencurve b gehören und mithin wäre Q ein Doppelpunkt der Asymptotencurve b; das Auftreten eines Doppelpunktes ist aber nach 3. unmöglich.

Aus den Sätzen 1. bis 4. können wir sofort diese Schlussfolgerungen ziehen:

Die sämmtlichen Asymptotencurven unserer Fläche zerfallen in zwei Schaaren. Irgend zwei derselben Schaar angehörende Asymptotencurven schneiden sich nicht; dagegen schneiden sich je zwei Asymptotencurven, die verschiedenen Schaaren angehören, stets in einem und nur einem Punkte der Fläche.

Die Coordinatenlinien $u = 0$, $v = 0$ sind zwei Asymptotencurven die verschiedenen Schaaren angehören. Wegen der Bedeutung der Coordinaten u, v als Längen gewisser Coordinatenabschnitte entnehmen wir aus den eben ausge-

sprochenen Thatsachen zugleich, dass zu bestimmt gegebenen Werthen von u, v stets nur ein Punkt unserer Fläche gehört d. h. *die zu untersuchende Abbildung* (1) *unserer Fläche auf die uv-Ebene ist nothwendig eine umkehrbar eindeutige. Insbesondere folgt hieraus, dass unsere Fläche einen einfachen Zusammenhang besitzt und keine Doppelfläche ist.*

Nachdem wir zu dieser wichtigen Einsicht gelangt sind, berechnen wir den gesammten Inhalt unserer Fläche auf zwei Wegen; wir werden dadurch zu einem Widerspruch gelangen.

Der erstere Weg ist der folgende. Wir betrachten auf unserer Fläche dasjenige aus Asymptotencurven gebildete Viereck, dessen Ecken durch die Coordinaten

$$u,\ v; \qquad -u,\ v; \qquad -u,\ -v; \qquad u,\ -v$$

bestimmt sind. Da jeder Winkel dieses Vierecks $< \pi$ sein muss, so ist die Summe der Winkel des Viereckes jedenfalls $< 4\pi$ und der Inhalt des Viereckes d. h. der Ueberschuss der Summe seiner Windel über 2π ist mithin nothwendig $< 2\pi$. Lassen wir nun die Werthe von u, v unbegrenzt wachsen, so kommt jeder bestimmte Punkt der Fläche sicher einmal im Inneren eines Viereckes zu liegen und bleibt dann im Inneren aller weiteren Vierecke, so dass das unbegrenzt wachsende Viereck schliesslich die ganze Oberfläche umfasst. Wir entnehmen daraus, dass der Gesammtinhalt unserer Fläche $\leqq 2\pi$ sein muss.

Andrerseits betrachten wir die geodätischen Linien auf unserer Fläche. Wegen der negativen Krümmung unserer Fläche ist jede geodätische Linie zwischen irgend zweien ihrer Punkte gewiss kürzeste Linie, d. h. von kleinerer Länge als jede andere Linie, die auf der Fläche zwischen den nämlichen zwei Punkten verläuft und sich durch stetige Aenderung in die geodätische Linie überführen lässt. Wir fassen nun irgend zwei vom Punkt O ausgehende geodätische Linien auf unserer Fläche ins Auge und nehmen an, dieselben schnitten sich noch in einem anderen Punkt P der Fläche. Da nach dem oben Bewiesenen unsere Fläche einen einfachen Zusammenhang besitzt, so lässt sich jede dieser beiden geodätischen Linien $O\,P$ in die andere durch stetige Veränderung überführen; es müsste also nach dem eben Ausgeführten jede derselben kürzer sein als die andere, was nicht möglich ist. Unsere Annahme der Existenz eines Schnittpunktes P ist also zu verwerfen. Durch die nämlichen Schlüsse erkennen wir auch, dass eine geodätische Linie unserer Fläche weder sich durchsetzen noch in sich selbst zurücklaufen darf.

Denken wir uns nun auf allen von O ausgehenden geodätischen Linien die gleiche Länge r abgetragen, so bilden die erhaltenen Endpunkte eine geschlossene doppelpunktslose Curve auf unserer Fläche. Das von dieser Curve umspannte Gebiet besitzt nach den bekannten Formeln der LOBATSCHEFSKIJschen Geometrie den Flächeninhalt

$$\pi\left(e^{\frac{r}{2}} - e^{-\frac{r}{2}}\right)^2.$$

Da dieser Ausdruck für unendlich wachsende Werthe von r selbst über alle Grenzen wächst, so entnehmen wir hieraus, dass auch der Gesammtinhalt unserer Fläche unendlich gross sein müsste. Diese Folgerung steht im Widerspruch mit der vorhin bewiesenen Thatsache, wonach jener Inhalt stets $\leqq 2\pi$ ausfallen sollte. Wir sind daher gezwungen, unsere Grundannahme zu verwerfen, d. h. wir erkennen, *dass es eine singularitätenfreie und überall regulär analytische Fläche von constanter negativer Krümmung nicht giebt. Insbesondere ist* daher auch *die zu Anfang aufgeworfene Frage zu verneinen, ob auf die* Beltramische *Weise die* ganze Lobatschefskij*sche Ebene durch eine regulär analytische Fläche im Raume sich verwirklichen lässt.*

Ueber Flächen von positiver constanter Krümmung.

Wir gingen zu Anfang dieser Untersuchung aus von der Frage nach einer Fläche der negativen constanten Krümmung, die überall im Endlichen regulär analytisch verläuft, und gelangten zu dem Resultate, dass es eine solche Fläche nicht giebt. Wir wollen nunmehr mittelst der entsprechenden Methode die gleiche Frage für die positive constante Krümmung behandeln. Offenbar ist die Kugel eine geschlossene singularitätenfreie Fläche der positiven constanten Krümmung, und nach dem von H. Liebmann * geführten Beweise giebt es auch keine andere geschlossene Fläche von derselben Eigenschaft. Diese Thatsache nun wollen wir aus einem Satze herleiten der von einem beliebigen singularitätenfreien Stücke einer Fläche der positiven constanten Krümmung gilt und folgendermassen lautet.

Auf einer analytischen Fläche der positiven constanten Krümmung $+1$ *sei ein singularitätenfreies einfach oder mehrfach zusammenhängendes Gebiet im Endlichen abgegrenzt: denken wir uns dann in jedem Punkte dieses Gebietes sowie in den Randpunkten desselben die beiden Hauptkrümmungsradien der Fläche construirt, so wird das Maximum der grösseren und folglich auch das Minimum der kleineren der beiden Hauptkrümmungsradien gewiss in keinem Punkte angenommen, der im Inneren des Gebietes liegt — es sei denn unsere Fläche ein Stück der Kugel mit dem Radius* 1.

Zum Beweise bedenken wir zunächst, dass wegen unserer Voraussetzung das Product der beiden Hauptkrummungsradien überall $= 1$ und daher der grössere der beiden Hauptkrümmungsradien stets $\geqq 1$ sein muss. Aus diesem Grunde ist das Maximum der grösseren Hauptkrümmungsradien offenbar nur dann $= 1$, wenn beide Hauptkrümmungsradien in jedem Punkte unseres Flächenstückes $= 1$ sind. In diesem besonderen Falle ist jeder Punkt des Flächenstückes ein Nabelpunkt, und man schliesst dann leicht in bekannter Weise, dass das Flächenstück ein Stück der Kugel mit dem Radius 1 sein muss.

* Göttinger Nachrichten, 1899, S. 44.

Nunmehr sei das Maximum der grösseren der beiden Hauptkrümmungsradien unserer Fläche > 1; dann nehmen wir im Gegensatz zu der Behauptung an, es gäbe *im Inneren* des Flächenstückes einen Punkt O, in welchem jenes Maximum stattfinde. Da dieser Punkt O gewiss kein Nabelpunkt sein kann und überdies ein regulärer Punkt unserer Fläche ist, so wird die Umgebung dieses Punktes lückenlos und einfach von jeder der beiden Schaaren von Krümmungslinien der beiden Flächen bedeckt. Benutzen wir diese Krümmungslinien als Coordinatenlinien und den Punkt O selbst als Anfangspunkt des krummlinigen Coordinatensystems, so gelten nach der bekannten Theorie der Flächen positiver constanter Krümmung die folgenden Thatsachen.*

Es bedeute r_1 den grösseren der beiden Hauptkrümmungsradien für den Punkt (u, v) in der Umgebung des Anfangspunktes $O = (0, 0)$; es ist in dieser Umgebung $r_1 > 1$. Man setze

$$\rho = \tfrac{1}{2} \log \frac{r_1 + 1}{r_1 - 1};$$

dann genügt die positive reelle Grösse ρ als Function von u, v der partiellen Differentialgleichung

$$\frac{\partial^2 \rho}{\partial u^2} + \frac{\partial^2 \rho}{\partial v^2} = \frac{e^{-2\rho} - e^{2\rho}}{4}. \tag{8}$$

Da bei abnehmendem r_1 die Function ρ nothwendig wächst, so muss ρ als Function von u, v an der Stelle $u = 0, v = 0$ einen Minimalwerth aufweisen, und demnach hat die Entwickelung von ρ nach Potenzen der Variabeln u, v nothwendig die Gestalt

$$\rho = a + \alpha u^2 + 2\beta uv + \gamma v^2 + \cdots,$$

wo a, α, β, γ Constante bedeuten und dabei die quadratische Form

$$\alpha u^2 + 2\beta uv + \gamma v^2,$$

für reelle u, v niemals negative Werthe annehmen darf. Aus letzterem Umstande folgen für die Constanten α und γ nothwendig die Ungleichungen:

$$\alpha \geqq 0 \text{ und } \gamma \geqq 0. \tag{9}$$

Andrerseits wollen wir die Entwicklung für ρ in die Differentialgleichung (8) einsetzen: für $u = 0, v = 0$ erhalten wir dann

$$2(\alpha + \gamma) = \frac{e^{-2a} - e^{2a}}{4}.$$

Da die Constante a den Werth von ρ im Punkte $O = (0, 0)$ darstellt und mithin

*Darboux, *Leçons sur la theorie générale des surfaces*, Bd. 3, No. 776.
Bianchi, *Lezioni di geometria differenziale*, § 264.

positiv ausfällt, so ist hier der Ausdruck rechter Hand jedenfalls < 0; die letztere Gleichung führt desshalb zu der Ungleichung

$$\alpha + \gamma < 0 ,$$

welche mit den Ungleichungen (9) in Widerspruch steht. Damit ist unsere ursprüngliche Annahme, wonach die Stelle des Maximums im Inneren des Flächenstückes liege als unzutreffend und mithin der oben aufgestellte Satz als richtig erkannt.

Der eben bewiesene Satz lehrt offenbar folgende Thatsache. Wenn wir aus der Kugeloberfläche ein beliebiges Stück ausgeschnitten denken und dann dieses Stück beliebig verbiegen, so findet sich das Maximum aller grösseren vorkommenden Hauptkrümmungsradien stets auf dem Rande des Flächenstückes. Eine geschlossene Fläche besitzt keinen Rand und daraus folgt, wie bereits oben bemerkt, sofort der Satz, *das eine geschlossene analytische singularitätenfreie Fläche mit der positiven constanten Krümmung* 1 *stets die Kugel mit dem Radius* 1 *sein muss.* Dieses Resultat drückt zugleich aus, dass man die Kugel als Ganzes nicht verbiegen kann, ohne den regulär analytischen Charakter der Fläche irgendwo zu stören.

GÖTTINGEN, 1900.

NOTE ON THE FUNCTIONS OF THE FORM

$$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

WHICH IN A GIVEN INTERVAL DIFFER THE LEAST POSSIBLE FROM ZERO*

BY

H. F. BLICHFELDT

TSCHEBYCHEFF has considered the following problem: $f(x)$ is a given function of x and of n arbitrary constants $a_1, a_2, \cdots, a_n$, and is, together with its partial derivatives with respect to $x, a_1, a_2, \cdots, a_n$, continuous in the interval $a \leqq x \leqq b$; to determine the constants $a_1, a_2, \cdots, a_n$ so that the greatest value of $[f(x)]^2$, in the same interval, shall differ as little as possible from zero. †

His solution is as follows:

For any given set of values of the constants $a_1, a_2, \cdots, a_n$, let $x_1, x_2, \cdots, x_\mu$ be all the different values of x for which $[f(x)]^2$, in the interval $a \leqq x \leqq b$, reaches its greatest value L^2. Then must $x_1, x_2, \cdots, x_\mu$ evidently satisfy the two equations

$$(\mathrm{I}) \qquad [f(x)]^2 - L^2 = 0, \quad (x-a)(x-b)f'(x) = 0.$$

If it is now possible to satisfy the μ equations

$$(\mathrm{II}) \qquad \sum_{k=1}^{n} \frac{\partial f(x_i)}{\partial a_k} N_k = f(x_i) = \pm L \qquad (i = 1, 2, \cdots, \mu),$$

by finite, determinate values of the n quantities $N_1, \cdots, N_n$, then it will be possible to give to the constants $a_1, \cdots, a_n$ such small increments, proportional to $N_1, \cdots, N_n$, that the greatest absolute value of $f(x)$, for $a \leqq x \leqq b$, becomes less than L, which is taken positive.

---

* Presented to the Society (Chicago) April 14, 1900, under a slightly different title. Received for publication November 21, 1900.

† *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions*, Pétersbourg Mémoires, series 6, vol. 7. The above statement of TSCHEBYCHEFF'S results is taken from J. BERTRAND, *Calcul Différentiel*, p. 512.

Consequently, the set of constants $a_1, \cdots, a_n$, for which the greatest absolute value of $f(x)$ in the given interval is the least possible, must be such that the μ equations (II) are inconsistent for finite values of the constants $N_1, \cdots, N_n$.

This requires, in many cases, that $\mu > n$, only. In particular, if

$$f(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

and the interval is $-h \leqq x \leqq +h$, the necessary and sufficient condition imposed upon $x_1, \cdots, x_n$ is that the equations (I),

$$[f(x)]^2 - L^2 = 0, \quad (x^2 - h^2) f'(x) = 0,$$

possess $n+1$ different, common solutions. These solutions are therefore $-h$, $+h$, and all the $n-1$ roots of $f'(x) = 0$, which last are obviously double roots of $[f(x)]^2 - L^2 = 0$. Hence, as $f'(x) = nx^{n-1} + \text{etc.}$, we must have identically

$$[f(x)]^2 - L^2 = x^{2n} + \text{etc.} = (x^2 - h^2)\left(\frac{f'(x)}{n}\right)^2,$$

and therefore

$$\frac{f'(x)}{\sqrt{L^2 - [f(x)]^2}} = \frac{n}{\sqrt{h^2 - x^2}},$$

from which we get by integration,

$$f(x) = \frac{h^n}{2^{n-1}} \cos n \cos^{-1}\left(\frac{x}{h}\right) = \frac{1}{2^n}\{[x + \sqrt{x^2 - h^2}]^n + [x - \sqrt{x^2 - h^2}]^n\},$$

the values of L and the constant of integration being readily determined. The result just given is in accord with that of BERTRAND obtained (loc. cit., pp. 514–518) by lengthy considerations involving the theory of continued fractions. *

In general, several sets of constants $a_1, \cdots, a_n$ can be found to satisfy TSCHEBYCHEFF'S conditions. If the function $f(x)$ involves the constants in the following manner:

$$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n,$$

the labor of selecting the required set is much simplified by the following considerations.

Let us by the "maxima" of $[f(x)]^2$ in the interval $a \leqq x \leqq b$ understand those only which are equal to L^2, the greatest value of $[f(x)]^2$ in the given interval. If we classify these maxima as *positive* or *negative* according as the corresponding value of $f(x)$ is $+L$ or $-L$, and plot the curve $y = f(x)$, *we*

* In Liouville's Journal, ser. 2, vol. 19 (1874), pp. 319-347, TSCHEBYCHEFF has solved this problem with the additional condition imposed upon $f(x)$, that it either never decreases or never increases in the given interval.

*shall find at least n alternations of the two kinds of maxima in the given interval.**

The curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ is continuous and can be made to cross the axis of X at $n-1$ given points. If the curve $y = f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$, whose maxima are assumed to have their least possible absolute value, had less than n alternations of the two kinds of maxima, we could construct a curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ which would have positive ordinates whenever $y = f(x)$ possessed negative maxima, and vice versa, at the same time making the largest of its ordinates in the given interval as small as we please. The maxima of the curve

$$y = f(x) + y' \equiv \phi(x) + (a_1 + \beta_1)x^{n-1} + (a_2 + \beta_2)x^{n-2} + \cdots + (a_n + \beta_n)$$

would thus be less in absolute value than those of $y = f(x)$, contrary to hypothesis.

For instance, the nearest approximation to $\sin x$ of the form $a_1 x + a_2$ in the interval $0 \leqq x \leqq h \leqq \pi/2$ must be such that the curve $y = \sin x - a_1 x - a_2$ shall have two positive maxima including one negative, or vice versa; there being just three maxima in the given interval, namely, for $x = 0$, $\cos^{-1} a_1$, h. These maxima, being given by

$$-a_2, \quad +\sqrt{1-a_1^2} - a_1 \cos^{-1} a_1 - a_2, \quad \sin h - a_1 h - a_2,$$

must therefore satisfy the relations:

$$(-a_2) + (+\sqrt{1-a_1^2} - a_1 \cos^{-1} a_1 - a_2) = 0,$$
$$-(-a_2) + (\sin h - a_1 h - a_2) = 0,$$

so that

$$a_1 = \frac{1}{h}\sin h, \quad a_2 = +\tfrac{1}{2}\sqrt{1 - \left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h}\cos^{-1}\left(\frac{\sin h}{h}\right).$$

The approximation is, therefore,

$$\frac{\sin h}{h}x + \tfrac{1}{2}\sqrt{1 - \left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h}\cos^{-1}\left(\frac{\sin h}{h}\right).$$

STANFORD UNIVERSITY, CAL.

* The writer has not access to the original memoirs of TSCHEBYCHEFF, in which this property may have been indicated.

LIBRARY
OF THE
UNIVERSITY of ILLINOIS.

CANONICAL FORMS OF QUATERNARY ABELIAN SUBSTITUTIONS IN AN ARBITRARY GALOIS FIELD*

BY

LEONARD EUGENE DICKSON

§1. *Introduction.*

For application to the problem of the distribution of the substitutions of a given group into complete sets of conjugates within the group, a set of canonical forms for its substitutions should have the property that two substitutions are conjugate within the group if, and only if, they are reducible to the same canonical form according to a definite scheme of reduction. In particular, if the canonical form belongs to a higher field than the initial $GF[p^n]$, the new indices introduced must be conjugate with respect to the initial field.

In the present paper is given a set of canonical forms of quaternary abelian substitutions in the $GF[p^n]$ such that the canonical forms likewise belong to the special abelian group $SA(4, p^n)$, the reduction being effected within the group. From them are derived the ultimate canonical forms, not all belonging to the given abelian group. In the former case, the canonical forms depend on the *coefficients* of the characteristic equation, in the latter case upon its *roots*.

When the given group is the *general* linear homogeneous group on m indices with coefficients in the $GF[p^n]$, a set of ultimate canonical forms is furnished by a theorem due (for the case $n = 1$) to JORDAN.† Likewise for the group of ternary linear homogeneous substitutions of determinant unity in the $GF[p^n]$, a complete set of ultimate canonical forms has been determined. ‡ The problem has also been solved for the corresponding binary group. The corresponding problem for a linear group of special character (i. e., not directly related to the general linear group) has not been previously solved so far as is known to the writer. The simplicity of the canonical substitutions for the quaternary abelian group makes comparatively easy the classification of abelian substitutions into

* Presented to the Society (Chicago) December 28, 1900. Received for publication January 5, 1901.

† *Traité des substitutions*, pp. 114–126. A simple proof by induction of the general theorem has been given by the writer in the American Journal of Mathematics, vol. 22, p. 121, 1900.

‡ DICKSON, American Journal of Mathematics, vol. 22, p. 231, 1900.

sets of conjugates within the abelian group (§ 23). The analogous problem is then solved (§ 25) for the simple quotient-group $A(4, p^n)$ and the results are discussed for the case $p^n = 3$, which leads to a simple group of order 25920 of frequent occurrence in geometrical problems (§§ 26–27). In addition to the checks mentioned in §§ 23, 24 upon the calculations of the paper, it may be stated that the results for the case $p^n = 3$ were previously derived by methods independent of those employed in this paper.*

Frequent use will be made of the theorem † that the equation

$$a\xi^2 + \beta\eta^2 = 1$$

has in the $GF[p^n]$ $(p > 2)$, $p^n - \nu$ solutions (ξ, η), where ν denotes $+1$ or -1 according as $-a\beta$ is a square or a not-square in the field.

§ 2. *Definition of the abelian group.*

The quaternary special abelian group $SA(4, p^n)$ is composed of the linear substitutions

$$(1) \qquad S: \quad \begin{array}{c|cccc} & \xi_1 & \eta_1 & \xi_2 & \eta_2 \\ \hline \xi_1' = & \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \eta_1' = & \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \xi_2' = & \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \eta_2' = & \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{array}$$

with coefficients in the $GF[p^n]$ which satisfy the relations ‡

$$(2) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{11} & \delta_{11} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{vmatrix} = 1, \qquad \begin{vmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 1,$$

$$(3) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \alpha_{21} & \gamma_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \alpha_{22} & \gamma_{22} \end{vmatrix} = 0, \qquad \begin{vmatrix} \beta_{11} & \delta_{11} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \beta_{12} & \delta_{12} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 0,$$

$$(4) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 0, \qquad \begin{vmatrix} \beta_{11} & \delta_{11} \\ \alpha_{21} & \gamma_{21} \end{vmatrix} + \begin{vmatrix} \beta_{12} & \delta_{12} \\ \alpha_{22} & \gamma_{22} \end{vmatrix} = 0,$$

and equivalent relations (2′), (3′), (4′), formed from the columns of (1) as the former were formed from its rows.

* An account of these elementary methods, sufficient for the case $p^n = 3$, was presented January 7, 1901, to the London Mathematical Society.

† Compare American Journal of Mathematics, vol. 21, p. 195, 1899.

‡ For $n = 1$, the abelian group was studied by JORDAN, *Traité des substitutions*, pp. 171–179; for general n, it was investigated by the writer, Quarterly Journal of Mathematics, vol. 29, pp. 169–178, 1897; vol. 31, pp. 383–4, 1899.

Of the simplest substitutions satisfying these relations, the following are frequently employed in this paper, the notations being standard: †

$$
\begin{aligned}
M_i &: \quad \xi_i' = \eta_i, \quad \eta_i' = -\xi_i;\\
L_{i\lambda} &: \quad \xi_i' = \xi_i + \lambda\eta_i;\\
L'_{i\lambda} &: \quad \eta_i' = \eta_i + \lambda\xi_i;\\
T_{i\lambda} &: \quad \xi_i' = \lambda\xi_i, \quad \eta_i' = \lambda^{-1}\eta_i;\\
P_{12} &: \quad \xi_1' = \xi_2, \quad \eta_1' = \eta_2, \quad \xi_2' = \xi_1, \quad \eta_2' = \eta_1;\\
N_{ij\lambda} &: \quad \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j + \lambda\eta_i.
\end{aligned}
$$

The order N of $SA(4, p^n)$ is $p^{4n}(p^{4n}-1)(p^{2n}-1)$.

Since the general substitution (1) may be derived from the generators $L_{i\lambda}$, M_i and $N_{ij\lambda}$, its determinant is unity.

The reciprocal of S, given by (1), is

$$
(5) \qquad S^{-1}: \quad \begin{Vmatrix} \delta_{11} & -\gamma_{11} & \delta_{21} & -\gamma_{21} \\ -\beta_{11} & \alpha_{11} & -\beta_{21} & \alpha_{21} \\ \delta_{12} & -\gamma_{12} & \delta_{22} & -\gamma_{22} \\ -\beta_{12} & \alpha_{12} & -\beta_{22} & \alpha_{22} \end{Vmatrix}.
$$

It follows that the first minors (taken without prefixed sign) of $\alpha_{ij}, \delta_{ij}, \beta_{ij}, \gamma_{ij}$ are respectively $\delta_{ij}, \alpha_{ij}, \gamma_{ij}, \beta_{ij}$.

§ 3. *Characteristic equation of an abelian substitution.*

By definition, the characteristic determinant of S is

$$
\Delta(\kappa) \equiv \begin{vmatrix} \alpha_{11}-\kappa & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11}-\kappa & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22}-\kappa & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22}-\kappa \end{vmatrix}.
$$

The constant term $\Delta(0)$ of $\Delta(\kappa)$ expanded according to powers of κ is unity, being the determinant of the substitution. The coefficient of $-\kappa$ is

$$
\begin{vmatrix} \delta_{11} & \beta_{12} & \delta_{12} \\ \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \delta_{21} & \beta_{22} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \beta_{22} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \gamma_{11} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \delta_{12} \\ \beta_{21} & \delta_{21} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} \end{vmatrix}
$$

$$
= \delta_{11} + \alpha_{11} + \delta_{22} + \alpha_{22}:
$$

† If an index is not altered, it is not written in the formula.

The coefficient of κ^2 is the sum of six determinants of the second order. In particular, $\Delta(\kappa)$ has the form

$$\kappa^4 - (a_{11} + \delta_{11} + a_{22} + \delta_{22})\kappa^3 + (\ )\kappa^2 - (a_{11} + \delta_{11} + a_{22} + \delta_{22})\kappa + 1 .$$

Hence the reciprocal of any root of $\Delta(\kappa) = 0$ is itself a root. *The characteristic equation of a special abelian substitution is a reciprocal equation.**

§ 4. *Substitutions whose characteristic equations have all their roots in the $GF[p^n]$, no root being ± 1.*

Suppose first that all the roots of $\Delta(\kappa) = 0$ belong to the $GF[p^n]$. Designate them by κ, κ^{-1}, λ, λ^{-1} and consider first the case in which no root is ± 1. The root κ leads to a linear function $\omega \equiv a\xi_1 + b\eta_1 + c\xi_2 + d\eta_2$ which S multiplies by κ. But $SA(4, p^n)$ contains a substitution V which replaces ξ_1 by ω. Then $V^{-1}SV \equiv S_1$ replaces ξ_1 by $\kappa\xi_1$. Likewise the root κ^{-1}, which is also a root of the characteristic equation for S_1, leads to a linear function

$$\omega_1 \equiv a_1\xi_1 + b_1\eta_1 + c_1\xi_2 + d_1\eta_2 ,$$

which S_1 multiplies by κ^{-1}.

If $b_1 \neq 0$, the group contains the abelian substitution

$$U \equiv \begin{Vmatrix} b_1^{-1} & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & d_1 \\ -b_1^{-2}d_1 & 0 & b_1^{-1} & 0 \\ c_1 & 0 & 0 & b_1 \end{Vmatrix} .$$

Then $U^{-1}S_1U$, being abelian, takes the form

$$\begin{Vmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \end{Vmatrix} .$$

From the assumption concerning $\Delta(\kappa)$, the equation

$$\begin{vmatrix} \alpha - \kappa & \gamma \\ \beta & \delta - \kappa \end{vmatrix} = 0$$

has as its roots the distinct marks λ, λ^{-1} of the $GF[p^n]$. Hence the given substitution S is conjugate with the canonical form

* The theorem is true for any number $2m$ of indices. For proof, the direct method of the text may be employed; another proof may be based upon the canonical forms of linear substitutions in a Galois field. In a subsequent paper the writer intends to extend the present investigation to the case $2m > 4$.

$$(6) \qquad \xi' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = \lambda\xi_2, \ \eta_2' = \lambda^{-1}\eta_2.$$

We take $\lambda \neq \kappa$ or κ^{-1}, the contrary cases leading to the type (7) below. Here κ has $p^n - 3$ and λ has $p^n - 5$ values if $p > 2$; while, for $p = 2$, κ has $2^n - 2$ and λ has $2^n - 4$ values. But the substitution (6) is transformed by M_1 into a similar one with κ and κ^{-1} interchanged; by M_2 into one with λ and λ^{-1} interchanged; by P_{12} into one with κ and λ, κ^{-1} and λ^{-1} interchanged. The eight resulting combinations give all the substitutions of the type (6) with the distinct roots $\kappa, \kappa^{-1}, \lambda, \lambda^{-1}$. The number of types of canonical forms is therefore

$$\tfrac{1}{8}(p^n - 3)(p^n - 5), \text{ for } p > 2; \quad \tfrac{1}{8}(2^n - 2)(2^n - 4), \text{ for } p = 2.$$

The most general substitution of $SA(4, p^n)$ commutative with a given substitution (6) has the form

$$\xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad \xi_2' = b\xi_2, \quad \eta_2' = b^{-1}\eta_2.$$

Their number being $(p^n - 1)^2$, it follows that each substitution (6) is one of $N \div (p^n - 1)^2 \equiv p^{4n}(p^{2n} + 1)(p^n + 1)^2$ conjugates within $SA(4, p^n)$.

If, however, $b_1 = 0$ in ω_1, we may suppose that $c_1 \neq 0$. For, if $c_1 = 0$, $d_1 \neq 0$, then $M_2^{-1} S_1 M_2$ multiplies $a_1\xi_1 + d_1\xi_2$ by κ^{-1}. With $c_1 \neq 0$, the group contains the abelian substitution

$$V = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a_1c_1^{-1} \\ a_1 & 0 & c_1 & d_1 \\ 0 & 0 & 0 & c_1^{-1} \end{Vmatrix}.$$

Hence $V^{-1}S_1V$ belongs to the group and has the form

$$\begin{Vmatrix} \kappa & 0 & 0 & 0 \\ \beta_{11} & \kappa^{-1} & \beta_{12} & 0 \\ 0 & 0 & \kappa^{-1} & 0 \\ \beta_{21} & 0 & \beta_{22} & \kappa \end{Vmatrix} \qquad (\beta_{21} = \kappa^2\beta_{12}).$$

Transforming by $L_{1\tau}'L_{2\sigma}'$ and taking

$$\beta_{11} + \tau(\kappa - \kappa^{-1}) = 0, \quad \beta_{22} + \sigma(\kappa^{-1} - \kappa) = 0,$$

we find a substitution of the same form, having $\beta_{11} = \beta_{22} = 0$. If $\beta_{12} = 0$, we have the canonical form

$$(7) \qquad \xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = \kappa^{-1}\xi_2, \quad \eta_2' = \kappa\eta_2.$$

If $\beta_{12} \neq 0$, we transform by $T_{2\,\beta_{12}\kappa}$ and obtain the type

$$(8) \qquad \begin{Vmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & \kappa^{-1} & 0 \\ 0 & 0 & \kappa^{-1} & 0 \\ \kappa & 0 & 0 & \kappa \end{Vmatrix}.$$

The number of non-conjugate substitutions (7) with $\kappa^2 \neq 1$ is $\frac{1}{2}(p^n - 3)$ if $p > 2$ and $\frac{1}{2}(2^n - 2)$ if $p = 2$. A substitution commutative with (7) has the form

$$\begin{Vmatrix} \alpha & 0 & 0 & \beta \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ \gamma & 0 & 0 & \delta \end{Vmatrix}.$$

The abelian relations give

$$\alpha a - \beta b = 1, \quad ac - \beta d = 0, \quad -\gamma a + \delta b = 0, \quad -\gamma c + \delta d = 1.$$

Letting $\Delta \equiv ad - bc$, we have, as the solution of these relations,

$$a = d/\Delta, \quad \beta = c/\Delta, \quad \gamma = b/\Delta, \quad \delta = a/\Delta.$$

The number of the commutative substitutions is therefore $(p^{2n} - 1)(p^{2n} - p^n)$, so that each substitution (7) is conjugate within $SA(4, p^n)$ with exactly $p^{3n}(p^{2n} + 1)(p^n + 1)$ substitutions.

The substitution (8) is transformed by P_{12} into a similar one with κ^{-1} in place of κ. Hence there are $\frac{1}{2}(p^n - 3)$ or $\frac{1}{2}(2^n - 2)$ non-conjugate substitutions (8) with $\kappa^2 \neq 1$. A linear substitution commutative with (8) has the form

$$\begin{Vmatrix} a & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & c & 0 \\ b & 0 & 0 & a \end{Vmatrix}.$$

The abelian relations give $ac = 1$, $-bc + ad = 0$, whence

$$c = a^{-1}, \quad d = bc^2.$$

The number of such substitutions is therefore $p^n(p^n - 1)$. Hence each substitution (8) is one of $p^{3n}(p^{4n} - 1)(p^n + 1)$ conjugates.

§ 5. *Roots in the field, two of them being* ± 1.

Suppose next that the roots are $\kappa, \kappa^{-1}, \pm 1, \pm 1$, where κ is a mark $\neq 0$ or ± 1 of the $GF[p^n]$. The canonical form is

$$(9) \qquad \begin{Vmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \beta & \pm 1 \end{Vmatrix}.$$

For $\beta = 0$ there are $p^n - 3$ or $\frac{1}{2}(2^n - 2)$ types each commutative with

$$(10) \qquad \begin{Vmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{Vmatrix} \qquad (ad - bc = 1),$$

giving $(p^n - 1)p^n(p^{2n} - 1)$ substitutions. Hence each type yields a set of $p^{3n}(p^{2n} + 1)(p^n + 1)$ conjugate substitutions of $SA(4, p^n)$.

For $\beta \neq 0$, the substitutions with β a square are transformed into each other by abelian substitutions $T_{2\sigma}$ and likewise those with β a not-square. The two sets are seen to be not conjugate within $SA(4, p^n)$. The number of types is therefore $2(p^n - 3)$ or $\frac{1}{2}(2^n - 2)$ according as $p > 2$ or $p = 2$. A substitution commutative with (9) for $\beta \neq 0$, $\kappa^2 \neq 1$ has the form (10) with $b = 0$ and $a = d$. Each type is therefore commutative with $2p^n(p^n - 1)$ or $2^n(2^n - 1)$ substitutions and thus conjugate with exactly $\frac{1}{2}p^{3n}(p^{4n} - 1)(p^n + 1)$ or $2^{3n}(2^{4n} - 1)(2^n + 1)$ substitutions within $SA(4, p^n)$.

§ 6. *Two roots each* ± 1 *and two roots each* ∓ 1.

If the roots are $\pm 1, \pm 1, \mp 1, \mp 1$, the canonical form is

$$(11) \qquad \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ a & \pm 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ 0 & 0 & \beta & \mp 1 \end{Vmatrix}.$$

One may chose the lower sign, transforming if necessary by P_{12}.

For $a = \beta = 0$, the substitution becomes T_{1-1} and, for $p > 2$, is commutative only with the $[p^n(p^{2n} - 1)]^2$ substitutions

$$(12) \qquad \begin{Vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \end{Vmatrix} \qquad (ad - bc = 1,\ a_1 d_1 - b_1 c_1 = 1).$$

For $a \neq 0$, we may suppose that $a = 1$ or ν, where ν is a particular not-square in the $GF[p^n]$; indeed, a is replaced by $a\sigma^{-2}$ upon transforming (11) by $T_{1\sigma}$. Similarly, we may assume that $\beta = 0$, 1 or ν. The resulting eight types are:

$$L_{1\mu}T_{1-1}, \quad L_{2\mu}T_{1-1}, \quad L_{11}T_{1-1}, \quad L_{2\mu}, \quad L_{1\nu}T_{1-1}L_{2\mu}$$

where $\mu = 1$ or ν. The number of conjugates to each may be determined directly or more simply by the method of §§ 9–10.

§ 7. *Four equal roots each* ± 1.

If the roots be ± 1, ± 1, ± 1, ± 1, the canonical form is either

$$L \equiv \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ a & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \beta & \pm 1 \end{Vmatrix} \quad \text{or} \quad R \equiv \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} \\ a_{21} & 0 & \pm 1 & 0 \\ \beta_{21} & 0 & a_{22} & \pm 1 \end{Vmatrix}$$

according as the linear function ω_1 determined by the second root ± 1 contains η_1 or does not. If $\beta_{12} = \delta_{12} = 0$, R is of the form L. If $\delta_{12} = 0$, $\beta_{12} \neq 0$, the transform of R by $T_{2\beta_{12}}$ is of the form R with $\beta_{12} = 1$, $\delta_{12} = 0$. The abelian relations give $a_{21} = 0$, $\beta_{21} = 1$. Then, for $\beta_{11} \neq 0$, we transform by the abelian substitution

$$\xi_1' = \xi_1 + \beta_{11}^{-1}\xi_2, \quad \eta_2' = \eta_2 - \beta_{11}^{-1}\eta_1,$$

and reach a substitution of the form L. A similar result follows if $\beta_{11} = 0$, $a_{22} \neq 0$, since the transform of R by P_{12} then has $\beta_{11} \neq 0$. Finally, if $\beta_{11} = a_{22} = 0$, we have the substitution

$$\begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 1 & 0 & 0 & \pm 1 \end{Vmatrix}. \tag{13}$$

For $p > 2$, this is transformed into L (with $a = 1$, $\beta = -1$) by

$$\begin{Vmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{Vmatrix}.$$

For $p = 2$, substitution (13) will be seen to furnish a new type. A substitution (1) commutative with (13) must have the form

$$\begin{Vmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \delta_{12} & 0 & \delta_{11} & 0 \\ \beta_{21} & \alpha_{12} & \beta_{22} & \alpha_{11} \end{Vmatrix},$$

subject to the abelian conditions

$$\alpha_{11}\delta_{11} + \alpha_{12}\delta_{12} = 1, \quad \beta_{11}\alpha_{12} + \delta_{11}\beta_{21} + \alpha_{11}\beta_{12} + \delta_{12}\beta_{22} = 0.$$

By the former, α_{11} and α_{12} are not both zero, so that the latter determines one of the β_{ij} in terms of the other three, which may be chosen arbitrarily in the $GF[2^n]$. Hence there are $2^n(2^{2n}-1)2^{3n}$ substitutions of $SA(4, 2^n)$ commutative with (13). A substitution L is conjugate with the identity or L_{11} or $L_{11}L_{21}$. By §10, $L_{11}L_{21}$ is commutative with exactly 2^{4n} substitutions of $SA(4, 2^n)$, so that $L_{11}L_{21}$ and (13) are not conjugate within the group. A different argument is necessary for the case of L_{11} and (13); but the latter are readily shown to be not conjugate under abelian transformation.

It remains to consider R when $\delta_{12} \neq 0$. The transformed of R by $T_{2\,\delta_{12}^{-1}}$ is of the form R with $\delta_{12} = 1$. The latter is transformed by $L'_{2\,\beta_{12}}$ into the substitution

$$R_1 = \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & 0 & 1 \\ -1 & 0 & \pm 1 & 0 \\ \mp a & 0 & a & \pm 1 \end{Vmatrix}.$$

Suppose first that $p = 2$. If $a = 0$, the transform of R_1 by M_2 is of the form R with $\delta_{12} = 0$, a case previously considered. If $a \neq 0$, the transform of R_1 by $T_{1\kappa}T_{2\kappa}$ is of the form R_1 with $\kappa^{-2}a$ in place of a. Choosing $\kappa = a^{\frac{1}{2}}$, we obtain a substitution

$$R_\beta = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ \beta & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{Vmatrix}$$

of period 4. Transforming R_β by the abelian substitution

$$\xi'_1 = \xi_1, \quad \eta'_1 = \eta_1 + \tau\eta_2 + \tau^2\xi_2, \quad \xi'_2 = \xi_2 - \tau\xi_1, \quad \eta'_2 = \eta_2 + \tau\xi_2,$$

we obtain, for $p = 2$, the substitution R_b, where $b \equiv \beta + \tau + \tau^2$. In order

that the $GF[2^n]$ shall contain a mark τ for which $b = 0$, it is necessary and sufficient that

$$B \equiv \beta + \beta^2 + \beta^4 + \cdots + \beta^{2^{n-1}} = 0 .$$

Since β belongs to the field, $B^2 = B$. Inversely, there are 2^{n-1} marks β making $B = 0$ and as many making $B = 1$. The 2^{n-1} substitutions R_β for which $B = 0$ are therefore conjugate within $SA(4, 2^n)$. Likewise the 2^{n-1} substitutions R_β for which $B = 1$ are all conjugate; indeed, R_β is conjugate with R_b and $b \equiv \beta + \tau + \tau^2$ takes 2^{n-1} distinct values when τ runs through the series of 2^n marks, while

$$b + b^2 + b^4 + \cdots + b^{2^{n-1}} \equiv \beta + \beta^2 + \beta^4 + \cdots + \beta^{2^{n-1}} + \tau + \tau^{2^n} \equiv B \pmod 2.$$

That the substitutions R_β for which $B = 1$ are not conjugate with R_0 may be shown by considering the condition $R_\beta S = S R_0$, S being of the general form (1). We find that S must have the form

$$(14) \qquad \begin{Bmatrix} 1 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \beta_{22} \\ \beta_{22} & 0 & 1 & 0 \\ \beta_{21} & 0 & \beta_{22} & 1 \end{Bmatrix} \qquad \begin{pmatrix} \beta_{21} = \beta_{12} + \beta_{22} + \beta \\ \beta_{21} = \beta_{12} + \beta_{22}^2 \end{pmatrix}.$$

The latter conditions require that $\beta + \beta_{22} + \beta_{22}^2 \equiv 0 \pmod 2$.

For $p = 2$, the only substitutions of $SA(4, 2^n)$ commutative with R_β are of the form (14) subject, however, to the conditions

$$\beta_{21} = \beta_{12} + \beta_{22}^2, \quad \beta_{21} = \beta_{12} + \beta_{22} .$$

Hence $\beta_{22} = 0$ or 1, while β_{11} and β_{12} are arbitrary; thus there are $2 \cdot 2^{2n}$ substitutions.

For $p > 2$, R_1 is transformed into a similar substitution R' having $\beta_{11} = 0$ by the following abelian substitution

$$\eta_1' = \eta_1 + \tfrac{1}{2}\beta_{11}\xi_2, \quad \eta_2' = \eta_2 + \tfrac{1}{2}\beta_{11}\xi_1 .$$

For $a = 0$, the transform of R' by M_2 is of the form (13).

For $a \neq 0$, the transform of R' by $T_{1,\pm 1}$ gives the substitutions

$$A_a = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -a & 0 & a & 1 \end{Bmatrix}, \qquad A_{-a} T_{1,-1} T_{2,-1} .$$

Transforming A_a by $T_{1,\kappa} T_{2,\kappa}$ we obtain $A_{a\kappa^{-2}}$. Hence, if $p > 2$, there are only four canonical types, ν being a particular not-square:

$$A_1,\quad A_\nu,\quad A_1T_{1-1}T_{2-1},\quad A_\nu T_{1-1}T_{2-1}. \tag{15}$$

The substitutions A_μ and $A_\mu T_{1-1}T_{2-1}$ are not conjugate, if $p>2$, their characteristic equations having different roots. But every substitution commutative with one is commutative with the other, $T_{1-1}T_{2-1}$ being commutative with every quaternary linear substitution. The period of A_μ is readily seen to be p if $p>3$, 9 if $p=3$, or 4 if $p=2$.* Then $A_\mu T_{1-1}T_{2-1}$ is of period $2p$ if $p>3$, 18 if $p=3$, or 4 if $p=2$.

If S be the general substitution (1), the identity $A_\alpha S = SA_{\alpha'}$ requires that

$$\gamma_{11}=\gamma_{12}=\gamma_{21}=\gamma_{22}=\alpha_{12}=\delta_{21}=0,\quad \alpha_{22}=\alpha_{11},\quad \delta_{22}=\delta_{11},\quad \beta_{22}=a\delta_{12},$$

$$\beta_{21}=-\beta_{12}-a\delta_{12},\quad a\delta_{22}=a'\alpha_{22},\quad -a'\alpha_{11}+a'\alpha_{21}=-\beta_{22}-a\delta_{22}.$$

The second abelian relation (2) then gives $\alpha_{22}\delta_{22}=1$, whence

$$a\delta_{22}^2=a'.$$

Hence A_α and $A_{\alpha'}$ are conjugate within $SA(4,p^n)$ only when a and a' are both squares or both not-squares in the $GF[p^n]$. To determine the substitutions S commutative with A_α, set $a'=a$. Then

$$S=\begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & 0 & \alpha_{11} & 0 \\ -\beta_{12}-a\delta_{12} & 0 & a\delta_{12} & \alpha_{11} \end{bmatrix},$$

subject to the abelian relations

$$\alpha_{11}^2=1,\quad 2\alpha_{11}\beta_{12}+a\alpha_{11}\delta_{12}-a\delta_{12}^2=0.$$

For each of the two values of α_{11}, the second relation determines δ_{12}. Hence there are $2p^{2n}$ substitutions commutative with A_α. Hence each of the substitutions (15) is conjugate with exactly $\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$ substitutions within $SA(4,p^n)$, while no two of the four canonical types are conjugate.

§ 8. *Study of the abelian substitutions of type L.*

Upon transforming L by $T_{1\sigma}T_{2\kappa}$, we obtain a substitution of the form L with a, β replaced by $a\sigma^{-2}$, $\beta\kappa^{-2}$. Hence the substitutions L are conjugate with one of the following:

* For example, by introducing the new indices

$$X=-\xi_1,\quad Y=\xi_2,\quad Z=\xi_2-a^{-1}\eta_2,\quad W=-a^{-1}\eta_1-\xi_2+a^{-1}\eta_2,$$

A_α takes the standard canonical form (not abelian):

$$X'=X,\quad Y'=Y+X,\quad Z'=Z+Y,\quad W'=W+Z.$$

$$L_{1\mu}L_{2\tau}, \quad L_{1\mu}L_{2\tau}T_{1-1}T_{2-1} \qquad (\mu, \tau = 0, 1, \nu).$$

Of these, $L_{11}L_{2\nu}$ is transformed into $L_{1\nu}L_{21}$ by P_{12}, and $L_{11}L_{21}$ is transformed into $L_{1\nu}L_{2\nu}$ by the abelian substitution

$$\begin{Vmatrix} \nu\delta & 0 & \nu\sigma & 0 \\ 0 & \delta & 0 & \sigma \\ -\nu\sigma & 0 & \nu\delta & 0 \\ 0 & -\sigma & 0 & \delta \end{Vmatrix},$$

subject only to the condition $\nu(\delta^2 + \sigma^2) = 1$, which has solutions in every $GF[p^n]$. The identity and $T_{1-1}T_{2-1}$ are conjugate only with themselves. Hence the types L remaining for consideration are:

$$L_{1\mu}, \quad L_{1\mu}T_{1-1}T_{2-1}, \quad L_{1\mu}L_{21}, \quad L_{1\mu}L_{21}T_{1-1}T_{2-1} \qquad (\mu = 1, \nu).$$

The characteristic equations for the second and fourth substitutions have the roots -1 and hence are not conjugate with the first or third.

The substitutions L_{11} and $L_{1\nu}$, ν being a not-square in the $GF[p^n]$, $p > 2$, are not conjugate within the $SA(4, p^n)$. In fact, $SL_{11} = L_{1\nu}S$ gives the conditions:

$$\beta_{11} = 0, \quad \delta_{11} = \nu\alpha_{11}, \quad \beta_{12} = \beta_{21} = \alpha_{21} = \delta_{12} = 0.$$

Thus S does not satisfy the abelian relation (2):

$$1 = \alpha_{11}\delta_{11} - \beta_{11}\gamma_{11} + \alpha_{12}\delta_{12} - \beta_{12}\gamma_{12} \equiv \nu\alpha_{11}^2.$$

The fact that $L_{11}L_{21}$ and $L_{1\nu}L_{21}$ are not conjugate with each other and that neither is conjugate with either L_{11} or $L_{1\nu}$ within $SA(4, p^n)$ follows incidentally from the following determination of the number of abelian substitutions conjugate with each of the four. To determine the number of substitutions of $SA(4, p^n)$ conjugate with $L_{1\mu}L_{2\tau}$, let S denote the general substitution (1) commutative with it. The conditions for the identity

$$SL_{1\mu}L_{2\tau} = L_{1\mu}L_{2\tau}S \qquad (\mu \neq 0)$$

are found at once to be the following:

$$\beta_{11} = 0, \quad \beta_{12} = 0, \quad \beta_{21} = 0, \quad \alpha_{11} = \delta_{11}, \quad \tau\beta_{22} = 0,$$
$$\mu\alpha_{21} = \tau\delta_{21}, \quad \mu\delta_{12} = \tau\alpha_{12}, \quad \tau\alpha_{22} = \tau\delta_{22}.$$

For $\tau = 0$, S has the form

$$(16) \qquad \begin{Vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} \end{Vmatrix}.$$

In particular, an abelian relation gives $\alpha_{11}^2 = 1$. Hence $S = \Gamma$ or $T_{1,-1}\Gamma$, where Γ is the most general substitution of $SA(4, p^n)$ which leaves η_1 fixed. The number of substitutions Γ is $p^{3n}p^n(p^{2n}-1)$, being equal to the number of substitutions of $SA(4, p^n)$ which leave fixed the index ξ_1.* According as $p > 2$ or $p = 2$, $L_{1\mu}$ is one of $\frac{1}{2}(p^{4n}-1)$ or $2^{4n}-1$ conjugate substitutions within $SA(4, p^n)$.

For $\tau = 1$, a substitution S commutative with $L_{1\mu}L_{21}$ has the form

$$(17) \qquad \begin{Vmatrix} \alpha_{11} & \gamma_{11} & \mu\delta_{12} & \gamma_{12} \\ 0 & \alpha_{11} & 0 & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \mu\alpha_{21} & 0 & \alpha_{22} \end{Vmatrix},$$

subject to the abelian relations

$$\alpha_{11}^2 + \mu\delta_{12}^2 = 1, \quad \alpha_{22}^2 + \mu\alpha_{21}^2 = 1, \quad \alpha_{11}^2 + \mu\alpha_{21}^2 = 1,$$

$$\alpha_{11}\alpha_{21} + \delta_{12}\alpha_{22} = 0, \quad \alpha_{11}\gamma_{21} - \gamma_{11}\alpha_{21} + \mu\delta_{12}\gamma_{22} - \gamma_{12}\alpha_{22} = 0.$$

Hence

$$\delta_{12}^2 = \alpha_{21}^2, \quad \alpha_{22} = \pm\alpha_{11}, \quad \alpha_{11}(\alpha_{21} \pm \delta_{12}) = 0.$$

Suppose first that μ is a not-square ν in the $GF[p^n]$, $p > 2$. Then $\alpha_{11} \neq 0$ and $\alpha_{21} = \mp\delta_{12}$, $\alpha_{22} = \pm\alpha_{11}$. For any one of the $p^n + \epsilon$ sets of solutions of $\alpha_{11}^2 + \nu\delta_{12}^2 = 1$, ϵ being ± 1 according as $p^n = 4l \pm 1$, α_{21} and α_{22} are determined except in sign; while γ_{21} is determined in terms of $\gamma_{11}, \gamma_{12}, \gamma_{22}$. Hence there are $2p^{3n}(p^n + \epsilon)$ substitutions of $SA(4, p^n)$ commutative with $L_{1\nu}L_{21}$.

Suppose next that $\mu = 1$. Whether α_{11} be zero or not, we may set $\alpha_{22} = \pm\alpha_{11}$, $\alpha_{21} = \mp\delta_{12}$. For any one of the $p^n - \epsilon$ sets of solutions of $\alpha_{11}^2 + \delta_{12}^2 = 1$ in the $GF[p^n]$, $p > 2$, α_{22} and α_{21} are determined except in sign, and one of the γ_{ij} is determined in terms of the remaining three. Hence there are $2p^{3n}(p^n - \epsilon)$ substitutions of $SA(4, p^n)$, $p > 2$, commutative with $L_{11}L_{21}$. For $p = 2$, we get $\alpha_{22} = \alpha_{11}$, $\delta_{12} = \alpha_{21}$, $\alpha_{11} + \delta_{12} = 1$, so that $SA(4, 2^n)$ contains exactly 2^{4n} substitutions commutative with $L_{11}L_{21}$.

§ 9. *Study of the substitutions* $L_{1\mu}T_{1,-1}$ *and* $L_{1\mu}T_{2,-1}$ ($\mu = 1$ *or* ν).

If ν be a not-square in the $GF[p^n]$, $p > 2$, no two of the substitutions $L_{11}T_{1,-1}$, $L_{1\nu}T_{1,-1}$, $L_{11}T_{2,-1}$, $L_{1\nu}T_{2,-1}$ are conjugate within $SA(4, p^n)$. The first two are not conjugate and the last two are not conjugate since their $(p+1)$-th powers are L_{11} and $L_{1\nu}$ and are not conjugate by § 8. Finally, a relation $SL_{1\mu}T_{1,-1} = L_{1\tau}T_{2,-1}S$ is proved impossible by forming the respective products.

*Quarterly Journal of Mathematics, vol. 29, pp. 171–173, 1897.

Within $SA(4, p^n)$, $p > 2$, each of the four substitutions is one of a complete set of $\frac{1}{2}p^{2n}(p^{4n} - 1)$ conjugate substitutions. In proof, let S be an abelian substitution commutative with $L_{1\mu}T_{1-1}$. Then S is commutative with the p-th and $(p+1)$-th powers of the latter, which are T_{1-1} and $L_{1\mu}$ respectively. Hence S is at the same time of the forms (12) and (16). Hence

$$S = \begin{Vmatrix} \alpha_{11} & \gamma_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{22} & \gamma_{22} \\ 0 & 0 & \beta_{22} & \delta_{22} \end{Vmatrix},$$

subject to the abelian relations

$$\alpha_{11}^2 = 1, \quad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

The number of substitutions S is therefore $2p^n p^n(p^{2n} - 1)$.

A substitution S' commutative with $L_{1\mu}T_{2-1}$ will be commutative with $L_{1\mu}T_{2-1} \cdot T_{1-1}T_{2-1} \equiv L_{1\mu}T_{1-1}$ and vice versa. Hence every S' is an S and vice versa. Hence the final theorem:

Within $SA(4, p^n)$, $p > 2$, the substitutions $L_{11}T_{1-1}$, $L_{11}T_{2-1}$, $L_{1\nu}T_{1-1}$ and $L_{1\nu}T_{2-1}$, where ν is a not-square in the field, are not conjugate and each gives rise to a complete set of $\frac{1}{2}p^{2n}(p^{4n} - 1)$ conjugate substitutions.

§ 10. *Study of the substitutions $L_{1\mu}T_{1-1}L_{2\tau}$ ($\mu, \tau = 1$ or ν).*

The substitutions $L_{11}T_{1-1}L_{21}$ and $L_{1\nu}T_{1-1}L_{21}$ are not conjugate within $SA(4, p^n)$, $p > 2$. Indeed, their $p+1$-th powers, $L_{11}L_{21}$ and $L_{1\nu}L_{21}$ are not conjugate by § 8. Likewise $L_{11}T_{1-1}L_{2\nu}$ and $L_{1\nu}T_{1-1}L_{2\nu}$ are not conjugate within $SA(4, p^n)$. Finally, $L_{1\mu}T_{1-1}L_{21}$ is not conjugate with $L_{1\kappa}T_{1-1}L_{2\nu}$; for, if S transform the former into the latter, it is seen that S must replace ξ_2 by $\nu\delta_{22}\xi_2 + \gamma_{22}\eta_2$ and η_2 by $\delta_{22}\xi_2$, where $\nu\delta_{22}^2 = 1$. Combining the four non-conjugate substitutions into the single type $E \equiv L_{1\mu}T_{1-1}L_{2\tau}$, where $\mu, \tau = 1, \nu$, it will be shown that each E is commutative with exactly $4p^{2n}$ substitutions of $SA(4, p^n)$, $p > 2$, and hence is one of a set of $\frac{1}{4}(p^{4n} - 1)(p^{2n} - 1)p^{2n}$ conjugate substitutions. In proof, let S be commutative with E. Then S must be commutative with $E^p = T_{1-1}$, so that S must be of the form (12). Also S must be commutative with $E^{p+1} = L_{1\mu}L_{2\tau}$ and hence have $\beta_{11} = 0$, $\beta_{22} = 0$, $\alpha_{11} = \delta_{11}$, $\alpha_{22} = \delta_{22}$ (§ 8). Hence S has the form

$$\begin{Vmatrix} \alpha & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha_{22} & \gamma_{22} \\ 0 & 0 & 0 & \alpha_{22} \end{Vmatrix} \qquad (\alpha^2 = \alpha_{22}^2 = 1).$$

Inversely, each of these substitutions is evidently commutative with E.

In view of the number of the substitutions of $SA(4, p^n)$ commutative with the substitutions E and the number commutative with the four of § 9, none of the latter are conjugate with a substitution E.

§ 11. *Canonical form of a binary substitution of irreducible characteristic determinant.*

Theorem. *A binary linear substitution in the* $GF[p^n]$

$$\Sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \qquad (\alpha\delta - \beta\gamma = 1)$$

whose characteristic determinant $D(\kappa) \equiv \kappa^2 - \kappa(\alpha + \delta) + 1$ *is irreducible in the field may be transformed into the canonical form*

$$\Sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & \alpha + \delta \end{pmatrix}$$

by a linear substitution of determinant unity and belonging to the field.

If $\beta\gamma = 0$, then $D(\kappa) = (\kappa - \alpha)(\kappa - \delta)$, contrary to hypothesis. Transforming Σ by $\begin{pmatrix} 1 & 0 \\ \alpha/\gamma & 1 \end{pmatrix}$, we obtain $S \equiv \begin{pmatrix} 0 & \gamma \\ -\gamma^{-1} & \alpha + \delta \end{pmatrix}$. If γ be a square in the field, the transform of S by $T_{1\gamma^{1/2}}$ gives Σ_1. If γ be a not-square, so that $p > 2$, the transform of S by $\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ is a substitution of the form Σ with $c \equiv \gamma + \tau(\alpha + \delta) + \tau^2\gamma^{-1}$ in place of the coefficient γ. Since

$$c\gamma^{-1} = 1 + \tau\gamma^{-1}(\alpha + \delta) + (\tau\gamma^{-1})^2$$

is irreducible in the field, c cannot vanish. Moreover, at most two values of τ give the same value to the expression c. Hence c has at least $\frac{1}{2}(p^n + 1)$ values $\neq 0$, at least one of which is therefore a square in the field. By the earlier case, the substitution is conjugate with Σ_1.

By an analogous proof, Σ may be transformed into

$$\Sigma_1^{-1} \equiv \begin{pmatrix} \alpha + \delta & -1 \\ 1 & 0 \end{pmatrix}.$$

§ 12. *Substitutions whose characteristic determinant is the product of two linear factors and an irreducible quadratic factor.*

Let the characteristic determinant $\Delta(\kappa)$ of the abelian substitution S be the product of two linear factors and an irreducible quadratic factor each belonging to the $GF[p^n]$. Denote the roots of the former by a, b and those of the latter by λ, λ^{p^n}. Then (§ 3), $b = a^{-1}$, $\lambda^{p^n} = \lambda^{-1}$; hence $\lambda^{p^n+1} = 1$, $\lambda^{p^n-1} \neq 1$, the latter excluding only the two roots ± 1 of the former, so that λ may have $p^n - 1$ or 2^n distinct values, according as $p > 2$ or $p = 2$.

If $a^{-1} \neq a$, S may be transformed by a substitution of $SA(4, p^n)$ into a substitution S' which replaces ξ_1 by $a\xi_1$ and η_1 by $a^{-1}\eta_1$. On account of the abelian conditions, S' is seen to affect ξ_2, η_2 according to a substitution of the form

$$\Sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \qquad (\alpha\delta - \beta\gamma = 1).$$

From the invariance of $\Delta(\kappa)$, it follows that λ, λ^{-1} are the roots of

$$\kappa^2 - \kappa(\alpha + \delta) + 1 = 0.$$

In particular, the latter is irreducible in the $GF[p^n]$. By §11, there exists a substitution on ξ_2, η_2 with coefficients in the field and having determinant unity which transforms Σ into

$$\Sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & \alpha + \delta \end{pmatrix}.$$

A substitution S of the group $SA[4, p^n]$ whose characteristic determinant is the product of two distinct linear factors $\kappa - a$, $\kappa - a^{-1}$ and an irreducible quadratic factor $\kappa^2 - A\kappa + 1$ is conjugate within the group with the canonical substitution

$$(18') \qquad \begin{Vmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A \end{Vmatrix}.$$

If $a^{-1} = a$, S may be transformed by a substitution of the group into a substitution S_1 which replaces ξ_1 by $a\xi_1$ and η_1 by $a\eta_1 + b\xi_1$ and therefore ξ_2, η_2 by linear functions of ξ_2, η_2 only. If $b \neq 0$, we transform S_1 by T_{1c} and obtain a substitution of the form S_1 with $c^{-2}b$ in place of b, so that b may be restricted to unity and (for $p > 2$) a particular not-square ν of the field. *The canonical forms within $SA(4, p^n)$ of substitutions whose characteristic determinant is the product of two equal linear factors $\kappa \mp 1$ and an irreducible quadratic factor $\kappa^2 - A\kappa + 1$ are*

$$(19') \qquad \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ b & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A \end{Vmatrix} \qquad (b = 0, 1 \text{ or } \nu).$$

To obtain ultimate canonical forms, we introduce the new indices, conjugate with respect to the $GF[p^n]$,

$$X \equiv -\xi_2 + \lambda\eta_2, \quad X' = -\xi_2 + \lambda^{-1}\eta_2.$$

Then (18′) and (19′) each replace X by

$$-\lambda\xi_2 + (\lambda\lambda - 1)\eta_2 = -\lambda\xi_2 + \lambda^2\eta_2 = \lambda(-\xi_2 + \lambda\eta_2).$$

Hence (18′) and (19′) are reducible to the canonical forms

$$(18) \qquad \xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad X' = \lambda X, \quad Y' = \lambda^{-1}Y \qquad (a \neq \pm 1),$$

$$(19) \qquad \xi_1' = \pm\xi_1, \quad \eta_1' = \pm\eta_1 + b\xi_1, \quad X' = \lambda X, \quad Y' = \lambda^{-1}Y.$$

Two substitutions of $SA(4, p^n)$ reducible in this manner to the same canonical form (18) or (19) are conjugate within the group and inversely.

There are $\frac{1}{4}(p^n - 1)(p^n - 3)$ or $\frac{1}{4}2^n(2^n - 2)$ non-conjugate canonical types (18). Indeed, M_1 transforms (18) into a like substitution with a and a^{-1} interchanged; M_2 transforms (18) into a like substitution with λ and λ^{-1} interchanged. A substitution of $SA(4, p^n)$ commutative with one having the canonical form (18) has simultaneously the canonical form

$$\xi_1' = c\xi_1, \quad \eta_1' = c^{-1}\eta_1, \quad X' = \tau X, \quad Y' = \tau^{-1}Y,$$

where c is any mark $\neq 0$ of the $GF[p^n]$ and τ any root of $\tau^{p^n+1} = 1$, giving $(p^n - 1)(p^n + 1)$ substitutions. Hence each type (18) represents a complete set of $p^{4n}(p^{4n} - 1)$ conjugate substitutions of $SA(4, p^n)$.

For $b = 0$, there are $p^n - 1$ or 2^{n-1} non-conjugate types (19). A substitution of $SA(4, p^n)$ commutative with one having the canonical form (19) with $b = 0$ has simultaneously the canonical form

$$(20) \qquad \xi_1' = r\xi_1 + s\eta_1, \quad \eta_1' = t\xi_1 + u\eta_1, \quad X' = \tau X, \quad Y' = \tau^{-1}Y,$$

where $ru - st = 1$, giving $p^n(p^{2n} - 1)(p^n + 1)$ substitutions. Each type thus represents $p^{3n}(p^{2n} + 1)(p^n - 1)$ conjugate substitutions of $SA(4, p^n)$.

For $b = 1$ or a not-square ν, there are altogether $2(p^n - 1)$ or 2^{n-1} non-conjugate types (19). The commutative substitutions have the canonical form (20) with $s = 0$, $r = u = \pm 1$, giving $2p^n(p^n + 1)$ or $2^n(2^n + 1)$ substitutions. Hence each type represents a set of $\frac{1}{2}p^{3n}(p^{4n} - 1)(p^n - 1)$ or $2^{3n}(2^{4n} - 1)(2^n - 1)$ conjugate substitutions.

§ 13. *Substitutions whose characteristic equations have no root in the* $GF[p^n]$.

THEOREM. *Within* $SA(4, p^n)$ *any substitution* S, *whose characteristic equation* $\Delta(\kappa) = 0$ *has no root in the* $GF[p^n]$, *is conjugate with a substitution replacing* ξ_1 *by* $\gamma\eta_1$.

Let S replace ξ_1 by $\omega \equiv \alpha_{11}\xi_1 + \gamma_{11}\eta_1 + \alpha_{12}\xi_2 + \gamma_{12}\eta_2$. Suppose first that $\gamma_{11} \neq 0$. There exists in $SA(4, p^n)$ a substitution T which replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by ω [compare § 4]. Then $T^{-1}ST$ replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$. The same result will follow if S be conjugate within $SA(4, p^n)$ with a substitution having $\gamma_{11} \neq 0$. In the contrary case $\gamma_{11} = \beta_{11} = \gamma_{22} = \beta_{22} = 0$ in S and all its conjugates; indeed, by transforming S by M_1, P_{12} or $P_{12}M_1$, we may bring the

coefficient β_{11}, γ_{22} or β_{22} in the place of the former γ_{11}. By an earlier transformation, we can make $\beta_{12} = \gamma_{12} = 0$.* The resulting substitution is

$$S' \equiv \begin{Vmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & \delta_{11} & 0 & \delta_{12} \\ a_{21} & \gamma_{21} & a_{22} & 0 \\ \beta_{21} & \delta_{21} & 0 & \delta_{22} \end{Vmatrix}.$$

The transform of S' by $\eta_1' = \eta_1 + \eta_2$, $\xi_2' = \xi_2 - \xi_1$ has β_{21} as the coefficient of ξ_1 in η_1'. Hence $\beta_{21} = 0$. Similarly $\gamma_{21} = 0$. Next L_{11} transforms S' into a substitution with $\delta_{11} - a_{11}$ as the coefficient of η_1 in ξ_1'. Hence $\delta_{11} = a_{11}$ and similarly $\delta_{22} = a_{22}$. The transform of S' by $N_{121}: \xi_1' = \xi_1 + \eta_2$, $\xi_2' = \xi_2 + \eta_1$, has $\delta_{21} - a_{12}$ as the coefficient of η_1 in ξ_1'. Hence $a_{12} = \delta_{21}$ and similarly $a_{21} = \delta_{12}$. Now a_{12} or δ_{12} is not zero, since otherwise the characteristic equation has a root a_{11} or δ_{11} in the field. The transformed of S' by

$$\xi_1' = \xi_1 + \tau\xi_2, \quad \eta_2' = \eta_2 - \tau\eta_1 \qquad (a_{11} + \tau\delta_{12} = 0),$$

has the form

$$\begin{Vmatrix} 0 & 0 & a_{12}' & 0 \\ 0 & 0 & 0 & \delta_{12} \\ \delta_{12} & 0 & 0 & 0 \\ 0 & a_{12}' & 0 & 0 \end{Vmatrix}.$$

It is transformed by $T_{2\,a_{12}'}$ into P_{12} whose characteristic determinant is $(1 - \kappa^2)^2$, contrary to the hypothesis concerning the roots of $\Delta(\kappa) = 0$.

§ 14.

The further discussion of the resulting substitution

$$(21) \qquad \begin{Vmatrix} 0 & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ 0 & \gamma_{21} & a_{22} & \gamma_{22} \\ 0 & \beta_{21} & \beta_{22} & \delta_{22} \end{Vmatrix}$$

is separated into the cases $\beta_{12} = \delta_{12} = 0$, when the substitution has the form (12), and β_{12}, δ_{12} not both zero. In the latter case we may take $\beta_{12} \neq 0$, first transforming by M_2 if $\delta_{12} \neq 0$. Transforming by $T_{2\,\beta_{12}}$, we have a similar substitution with $\beta_{12} = 1$. Then the transform by $L_{2\,\delta_{12}}$ becomes

* Quarterly Journal of Mathematics, vol. 32, pp. 42–63, § 5, 1900.

$$S_1 \equiv \begin{Vmatrix} 0 & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & 1 & 0 \\ 0 & \gamma_{11}\gamma_{22} & \alpha_{22} & \gamma_{22} \\ 0 & \gamma_{11}\delta_{22} & \beta_{22} & \delta_{22} \end{Vmatrix} \qquad (\beta_{11}\gamma_{11} = -1, \quad \alpha_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1).$$

It has the characteristic determinant

$$(22) \quad \kappa^4 - \kappa^3(\delta_{11} + \alpha_{22} + \delta_{22}) + \kappa^2(2 + \delta_{11}\alpha_{22} + \delta_{11}\delta_{22} - \gamma_{11}\gamma_{22}) - \kappa(\delta_{11} + \alpha_{22} + \delta_{22}) + 1 .$$

It is the product of two factors $\kappa^2 - \sigma\kappa + 1$ and $\kappa^2 - \lambda\kappa + 1$ if, and only if,

$$\sigma + \lambda = \alpha_{22} + \delta_{22} + \delta_{11}, \quad \sigma\lambda = (\alpha_{22} + \delta_{22})\delta_{11} - \gamma_{11}\gamma_{22}$$

may be satisfied by marks σ, λ of the $GF[p^n]$. For $p > 2$, the necessary and sufficient condition is that $(\alpha_{22} + \delta_{22} - \delta_{11})^2 + 4\gamma_{11}\gamma_{22}$ be zero or a square in the field, viz., $(\sigma - \lambda)^2$. In particular, if $\gamma_{22} = 0$, we have

$$\Delta(\kappa) \equiv (\kappa^2 - \delta_{11}\kappa + 1)[\kappa^2 - \kappa(\alpha_{22} + \delta_{22}) + 1].$$

§ 15.

We seek the conditions under which new indices

$$X \equiv a\xi_1 + b\eta_1 + c\xi_2 = d\eta_2, \quad Y \equiv a_1\xi_1 + b_1\eta_1 + c_1\xi_2 + d_1\eta_2$$

may be introduced so that S_1 will replace X by Y and Y by $-X + AY$, where A, a, b, c, d, a_1, $\cdots$, are marks of the $GF[p^n]$ satisfying the equation

$$(23) \qquad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} + \begin{vmatrix} c & d \\ c_1 & d_1 \end{vmatrix} = 1 .$$

In order that S_1 shall replace X by Y and Y by $-X + AY$, we must have

$$\begin{cases} a_1 = b\beta_{11}, & b_1 = a\gamma_{11} + b\delta_{11} + c\gamma_{11}\gamma_{22} + d\gamma_{11}\delta_{22}, \\ c_1 = b + c\alpha_{22} + d\beta_{22}, & d_1 = c\gamma_{22} + d\delta_{22}, \end{cases}$$

$$\begin{cases} -a = -a_1 A + b_1\beta_{11}, & -b = a_1\gamma_{11} + b_1(\delta_{11} - A) + c_1\gamma_{11}\gamma_{22} + d_1\gamma_{11}\delta_{22}, \\ -c = b_1 + c_1(\alpha_{22} - A) + d_1\beta_{22}, & -d = c_1\gamma_{22} + d_1(\delta_{22} - A) . \end{cases}$$

On substitution of the values of a_1, b_1, c_1, d_1 into the second set of equations, we have the following equations:

$$(24) \qquad b\beta_{11}(\delta_{11} - A) - c\gamma_{22} - d\delta_{22} = 0 ,$$

$$(25) \quad a(\delta_{11} - A)\gamma_{11} + b(\gamma_{11}\gamma_{22} + \delta_{11}^2 - \delta_{11}A) + c\gamma_{11}\gamma_{22}(\alpha_{22} + \delta_{22} + \delta_{11} - A) + d\gamma_{11}(\delta_{22}^2 + \beta_{22}\gamma_{22} + \delta_{11}\delta_{22} - A\delta_{22}) = 0 ,$$

$$(26) \quad a\gamma_{11} + b(\delta_{11} + \alpha_{22} - A) + c\{\gamma_{11}\gamma_{22} + \alpha_{22}(\alpha_{22} + \delta_{22} - A)\} + d\{\gamma_{11}\delta_{22} + \beta_{22}(\alpha_{22} + \delta_{22} - A)\} = 0 ,$$

$$(27) \qquad b\gamma_{22} + c\gamma_{22}(\alpha_{22} + \delta_{22} - A) + d\delta_{22}(\alpha_{22} + \delta_{22} - A) = 0 .$$

Multiplying (24) by $a_{22}+\delta_{22}-A$ and adding to (27), we find

$$\gamma_{22}+\beta_{11}(\delta_{11}-A)(a_{22}+\delta_{22}-A)=0\,. \tag{28}$$

Indeed, if $b=0$, then $a_1=0$, $a=0$; then (24) and (26) require

$$a_{22}+\delta_{22}-A=0\,,$$

whence (24) and (25) require $d=0$, $\gamma_{22}c=0$. Then $d_1=0$, so that (23) will not hold. If we calculate the determinant of the coefficients of (24), (25), (26), (27), we find γ_{11}^2 times the square of the left member of (28). Hence the latter is the condition that these equations have solutions other than $a=b=c=d=0$.

In order that A determined by (28), viz.,

$$A^2-A(\delta_{11}+a_{22}+\delta_{22})+\delta_{11}(a_{22}+\delta_{12})-\gamma_{11}\gamma_{22}=0\,,$$

shall belong to the $GF[p^n]$ it is necessary and sufficient that $\Delta(\kappa)$ decomposes into quadratic factors $\kappa^2-A\kappa+1$, $\kappa^2-A'\kappa+1$ [see §14].

For the case of $\gamma_{22}=0$, the decomposition is evident:

$$A=\delta_{11}\,,\quad A'=a_{22}+\delta_{22}\,.$$

If these roots be equal, we have

$$a_{22}(a_{22}+\delta_{22}-\delta_{11})=a_{22}^2-\delta_{11}a_{22}+1=0\,,$$

since $a_{22}\delta_{22}=1$, so that $\kappa^2-\delta_{11}\kappa+1$ would be reducible in the field. Hence if $\Delta(\kappa)$ has equal irreducible quadratic factors, then $\gamma_{22}\neq 0$.

In case (28) is satisfied by a mark A, equation (27) may be dropped from consideration. Multiplying (24) by γ_{11} and adding to (26), we find

$$a\gamma_{11}+ba_{22}+ca_{22}(a_{22}+\delta_{22}-A)+d\beta_{22}(a_{22}+\delta_{22}-A)=0\,. \tag{26'}$$

Multiplying (24) by $\gamma_{11}\delta_{11}$ and (27) by $-\gamma_{11}$, and adding the resulting equations to (25), we find

$$d=a(\delta_{11}-A)\,. \tag{25'}$$

Using the relation $1+\beta_{22}\gamma_{22}=a_{22}\delta_{22}$ and (28), we have from (26') and (24)

$$a_{22}\{a\gamma_{11}\delta_{22}+b+c(a_{22}+\delta_{22}-A)\}=0\,,$$

$$(\delta_{11}-A)\{a\gamma_{11}\delta_{22}+b+c(a_{22}+\delta_{22}-A)\}=0\,.$$

If $\delta_{11}-A=0$, then $\gamma_{22}=0$ and $a_{22}\neq 0$. Hence, in every case

$$b=-a\gamma_{11}\delta_{22}-c(a_{22}+\delta_{22}-A)\,. \tag{29}$$

Hence, if (28) be satisfied, equations (24), (25), (26), (27) are equivalent to the two (25') and (29).

It remains to prove that the condition (23) may be satisfied. For $\gamma_{22} = 0$, we may take $A = \delta_{11}$ so that $d = 0$, $d_1 = 0$. Then

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \equiv a^2\gamma_{11} + ab\delta_{11} - \delta^2\beta_{11} = 1 .$$

For $p = 2$, we may take $b = 0$, $a = \gamma_{11}^{\frac{1}{2}}$, every mark of the $GF[2^n]$ being a square. For $p > 2$, the condition may be written

$$(a - \tfrac{1}{2} b\beta_{11}\delta_{11})^2 + b^2\beta_{11}^2(1 - \tfrac{1}{4}\delta_{11}^2) = -\beta_{11} .$$

It has solutions a, b in the $GF[p^n]$, unless $\delta_{11} = \pm 2$. In the latter case $\kappa^2 - \delta_{11}\kappa + 1 = (\kappa \mp 1)^2$, contrary to hypothesis. Then c is determined by (29) since $a_{22} + \delta_{22} - \delta \neq 0$.

For $\gamma_{22} \neq 0$, we may suppose $\delta_{22} = 0$. Indeed the transform of S_1 by $L'_{2\tau}$, $\delta_{22} + \tau\gamma_{22} = 0$, is of the form S_1 with $\delta_{22} = 0$, $\gamma_{22} \neq 0$. Then

$$d = a(\delta_{11} - A), \quad b = -c(a_{22} - A), \quad a_1 = -c\beta_{11}(a_{22} - A),$$

$$b_1 = a\gamma_{11} - cA(a_{22} - A), \quad c_1 = a\beta_{22}(\delta_{11} - A) + cA, \quad d_1 = c\gamma_{22} .$$

Hence the condition (23) becomes

$$a^2\{\gamma_{11} - \beta_{22}(\delta_{11} - A)^2\} - acA(a_{22} + \delta_{11} - 2A) + c^2\{\gamma_{22} - \beta_{11}(a_{22} - A)^2\} = 1 .$$

Since $\beta_{22}\gamma_{22} = -1$, $\gamma_{11}\gamma_{22} = (\delta_{11} - A)(a_{22} - A)$, the condition may be written

$$(a_{22} + \delta_{11} - 2A)\{a^2\gamma_{11}^2 - ac\gamma_{11}A(a_{22} - A) + c^2(a_{22} - A)^2\} = \gamma_{11}(a_{22} - A) .$$

Here $a_{22} - A \neq 0$. Hence the equation is impossible if $2A = a_{22} + \delta_{11}$, whence $4\gamma_{11}\gamma_{22} + (a_{22} - \delta_{11})^2 = 0$. In this case $\Delta(\kappa)$ is the square of a quadratic factor (see § 14).

This case being excluded here the above condition may be satisfied, when $p > 2$, unless $1 - \frac{1}{4}A^2 = 0$ [cf. § 1]. Then $\gamma_{11}\gamma_{22} = (\delta_{11} \mp 2)(a_{22} \mp 2)$, so that

$$4\gamma_{11}\gamma_{22} + (a_{22} - \delta_{11})^2 \equiv (a_{22} + \delta_{11} \mp 4)^2 ,$$

$$\Delta(\kappa) \equiv [\kappa^2 - \kappa(a_{22} + \delta_{11} \mp 2) + 1][\kappa^2 \mp 2\kappa + 1] ,$$

contrary to the hypothesis concerning the roots of $\Delta(\kappa) = 0$. For $p = 2$, the condition is satisfied by the values

$$a = \gamma_{11}^{-\frac{1}{2}}A(a_{22} - A)^{\frac{1}{2}}(a_{22} + \delta_{11})^{-\frac{1}{2}}, \quad c = \gamma_{11}^{\frac{1}{2}}(a_{22} - A)^{-\frac{1}{2}}(a_{22} + \delta_{11})^{-\frac{1}{2}} .$$

As a first conclusion of the preceding investigation it follows that if the characteristic determinant $\Delta(\kappa)$ of a substitution S of $SA(4, p^n)$ decomposes in the $GF[p^n]$ into two distinct irreducible quadratic factors $\kappa^2 - A\kappa + 1$ and $\kappa^2 - A'\kappa + 1$, then S is conjugate within $SA(4, p^n)$ with a substitution of the form (12) and therefore conjugate with

$$(30)\qquad \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & A & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A' \end{Vmatrix}.$$

If $\Delta(\kappa)$ be the square of an irreducible quadratic factor, then S is either conjugate with (21) for $\beta_{12} = \delta_{12} = 0$ and hence conjugate with (30), or else is conjugate with S_1 with $\gamma_{22} \neq 0$. In the latter case we transform by $L'_{2\tau}$ and make $\delta_{22} = 0$, obtaining the substitution

$$S_2 \equiv \begin{Vmatrix} 0 & \gamma_1 & 0 & 0 \\ -\gamma_1^{-1} & \delta & 1 & 0 \\ 0 & \gamma_1\gamma & \alpha & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{Vmatrix}.$$

The characteristic determinant of S_2 is

$$\Delta(\kappa) \equiv \kappa^4 - \kappa^3(\delta + \alpha) + \kappa^2(2 + \delta\alpha - \gamma\gamma_1) - \kappa(\delta + \alpha) + 1.$$

In particular, $\Delta(\kappa)$ is a perfect square * if, and only if,

$$(31)\qquad -4\gamma\gamma_1 = (\alpha - \delta)^2.$$

A second result of the investigation is that S is conjugate with a substitution of the form S_2 within $SA(4, p^n)$ when $\Delta(\kappa)$ is irreducible in the $GF[p^n]$ or is the product of two factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, belonging to and irreducible in the $GF[p^n]$. The latter cases are treated in §17; the case in which (31) holds is considered in the next section.

§16. *Characteristic determinant the square of an irreducible quadratic.*

Consider the following substitution of the form S_2:

$$(32)\qquad \Sigma_2 = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & c & \alpha & c \\ 0 & 0 & -c^{-1} & 0 \end{Vmatrix} \qquad (-4c = \alpha^2).$$

It has the characteristictic determinant (whether $p = 2$ or $p > 2$)

$$\kappa^4 - \kappa^3\alpha + \kappa^2(2 - c) - \kappa\alpha + 1 \equiv [\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1]^2.$$

* If $p > 2$ and (31) is satisfied, $\Delta(\kappa) = \{\kappa^2 - \frac{1}{2}(\alpha + \delta)\kappa + 1\}^2$.

A substitution S_2 satisfying (31) will have the same characteristic determinant as Σ_2 if, and only if, $a = a + \delta$, and for $p = 2$, $c = a^2 - \gamma\gamma_1$. The latter are consequently necessary conditions for the conjugacy of S_2 and Σ_2 under linear transformation. We proceed to prove that, if S_2 satisfy the condition (31) and if $\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1$ be irreducible in the $GF[p^n]$, then $a = a + \delta$, together with $c = a^2 - \gamma\gamma_1$ if $p = 2$, are sufficient conditions for the conjugacy of S_2 and Σ_2 within $SA(4, p^n)$. Assuming these conditions satisfied, we may determine a substitution S of $SA(4, p^n)$ such that $SS_2 = \Sigma_2 S$. We take for S the general substitution (1), the latter relation imposes a set of conditions which reduce to the following upon applying $a = a + \delta$, $-4c = a^2$ and (31):

$$\begin{aligned}
&\gamma_{11} = -\gamma_1\beta_{11}, \quad ca_{12} = \gamma_1\delta_{12}, \quad \gamma_{22} = -\gamma c\beta_{22}, \quad a_{21} = \gamma\delta_{21},\\
&ca_{12} = -a_{11} + \gamma_1\delta_{11}, \qquad c\gamma_1\beta_{12} = -aa_{11} + a\gamma_1\delta_{11} + c\gamma_{11} - \gamma_{12},\\
&a_{21} = \gamma_1^{-1}a_{11} - \delta_{11} - \delta\beta_{11}, \qquad \gamma_1\gamma_{21} = a\gamma_1\delta_{11} + c\gamma_{11} - aa_{11} - \gamma_{12},\\
&\gamma_1\gamma\beta_{21} = \delta a_{11} + \gamma_{12}, \qquad \gamma\gamma_1 c\beta_{22} = aa_{11} - c\gamma_{11} - a\gamma_1\delta_{11},\\
&a_{22} = \delta_{11} + a\beta_{12}, \qquad \gamma\gamma_1\delta_{22} = ca_{11} - a\gamma_{12}.
\end{aligned}$$

It suffices to take $a_{12} = 0$. Setting $\rho \equiv \gamma_1^{-1}\gamma_{12}$, we have S in the form

$$\begin{bmatrix}
\gamma_1\delta_{11} & -\gamma_1\beta_{11} & 0 & \gamma_1\rho\\
\beta_{11} & \delta_{11} & -\beta_{11} - c^{-1}\rho & 0\\
-\delta\beta_{11} & -\delta\delta_{11} - c\beta_{11} - \rho & \delta_{11} - a\beta_{11} - ac^{-1}\rho & -c\beta_{11}\\
\gamma^{-1}\delta\delta_{11} + \gamma^{-1}\rho & -\gamma^{-1}\delta\beta_{11} & \gamma^{-1}\beta_{11} & c\gamma^{-1}\delta_{11} - a\gamma^{-1}\rho
\end{bmatrix}.$$

The abelian relations (2), (3), (4) here reduce to the two:

$$\begin{aligned}
&\delta_{11}^2 + \beta_{11}^2 + \beta_{11}\rho + c^{-1}\rho^2 = \gamma_1^{-1},\\
&-\delta\delta_{11}^2 - \delta\beta_{11}^2 - c\delta_{11}\beta_{11} - 2\delta_{11}\rho + a\beta_{11}\rho + ac^{-1}\rho^2 = 0.
\end{aligned} \tag{33}$$

Multiplying the first by a and the second by -1 and adding, we have

$$(a + \delta)(\delta_{11}^2 + \beta_{11}^2) + c\delta_{11}\beta_{11} + 2\delta_{11}\rho = a\gamma_1^{-1}. \tag{34}$$

If $p > 2$, we may eliminate ρ between (33) and (34) and find

$$\delta_{11}^2\{-\beta_{11}^2(c + 4) + 8a^{-1}a\gamma_1^{-1} - 4\gamma_1^{-1}\} = \{2\beta_{11}^2 - (-c)^{-\frac{1}{2}}a\gamma_1^{-1}\}^2. \tag{35}$$

We prove that there exists a mark β_{11}^2 making the coefficient of δ_{11}^2 a square and the right member not zero. Let

$$\tau \equiv 8a^{-1}a\gamma_1^{-1} - 4\gamma_1^{-1}.$$

For $p > 2$, $\tau \neq 0$; since $2a^{-1}a = 1$ requires $a = \delta$ and therefore $\gamma\gamma_1 = 0$. If τ be square, we may take $\beta_{11} = 0$, when (35) determines $\delta_{11} \neq 0$ and (34) determines ρ. Suppose finally that τ is a not-square. Since $\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1$

is irreducible, $-c-4$ is a not-square. After we divide (35) by τ, the question is the possibility of finding a mark μ^2 such that μ^2+1 is a not-square. There are $\frac{1}{4}(p^n \mp 1)$ such squares $\mu^2 \neq 0$, according as $p^n = 4l \pm 1$.* Hence for $p^n > 5$, there are at least two marks β_{11}^2 making the coefficient of δ_{11}^2 in (35) a square, and hence at least one mark β_{11}^2 making also the right member $\neq 0$. Then δ_{11} is determined different from zero, so that (34) gives ρ.

For $p^n = 5$, the right member of (35) vanishes only if

$$\beta_{11}^2 = \pm 1, \quad (-c)^{-\frac{1}{2}} a\gamma_1^{-1} = \pm 2.$$

Then τ is a not-square only when $\gamma_1^{-1} = +1$ or -1. Since γ_1 is a square, it may be taken to be $+1$ by an earlier transformation of S_2 by $T_{1\omega}T_{2\omega^{-1}}$, ω being suitably chosen. With $\gamma_1 = 1$, then $a^{-1}a = \pm 1$. But $a^{-1}a = 1$ makes τ a square. Hence $a^{-1}a = -1$, so that $\delta = -2a$, $a = -2\delta$. Since $-c-4$ is a not-square, and $-c = \delta^2$, it follows that $\delta^2 = 1$, $a^2 = -1$. Since β_{11}^2 was chosen to make the coefficient of δ_{11}^2 a square, and since τ is now 3, it follows that $\beta_{11}^2 = -1$, and $\delta_{11} = 0$. Then (34) is an identity and (33) becomes $(2\rho - \beta_{11})^2 = 1$ and may be satisfied.

For $p^n = 3$, $c \equiv 0 \pmod 3$, since $-c-4$ is to be a not-square, necessarily -1. Hence the substitution Σ_2 cannot be employed. Since $a+\delta = 0$, (31) gives $\gamma\gamma_1 \equiv -a^2 \equiv -1 \pmod 3$. Hence S_2 takes the form

$$[a, \gamma] \equiv \begin{Vmatrix} 0 & -\gamma^{-1} & 0 & 0 \\ \gamma & -a & 1 & 0 \\ 0 & -1 & a & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{Vmatrix}.$$

But $P_{12}M_1M_2$ transforms $[a, \gamma]$ into $[-a, -\gamma]$. Hence $[a, \gamma]$ is conjugate either with $[1, -1]$ or with $[-1, -1]$. The latter is transformed into the former by the special abelian substitution

$$\begin{Vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{Vmatrix}.$$

For $p = 2$, we have $a = 0$, $a+\delta = 0$, $c = a^2 - \gamma\gamma_1$, so that (34) becomes $c\delta_{11}\beta_{11} = a\gamma_1^{-1}$. The latter together with (33) are to be satisfied by marks β_{11}, δ_{11}, ρ of the $GF[2^n]$. Letting $\rho = c\beta_{11}r^2$ in (33), dividing by $c\beta_{11}^2$ and extracting the square root, we find

$$r^2 + r \equiv c^{-\frac{1}{2}} + \frac{\gamma_1^{-\frac{1}{2}} + \delta_{11}}{c^{\frac{1}{2}}\beta_{11}} \pmod 2$$

$$= c^{-\frac{1}{2}} + g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})$$

*American Journal of Mathematics, vol. 21, p. 196.

upon eliminating β_{11} and denoting the constant $c^{\frac{1}{2}}a^{-1}\gamma_1$ by g. It remains to prove that a mark δ_{11} may be founded for which this equation has a root r belonging to the field. But, by § 7, $r^2 + r = \tau$ is irreducible if, and only if,

$$\tau + \tau^2 + \tau^4 + \cdots + \tau^{2^{n-1}} = 1 .$$

By hypothesis $\kappa^2 + c^{\frac{1}{2}}\kappa + 1 = 0$ and therefore $\mu^2 + \mu + c^{-1} = 0$ is irreducible in the field, so that

$$c^{-1} + (c^{-1})^2 + (c^{-1})^4 + \cdots + (c^{-1})^{2^{n-1}} = 1 .$$

If, for every mark δ_{11}, the equation in r be irreducible, then

$$g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11}) + \{g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})\}^2 + \cdots + \{g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})\}^{2^{n-1}} \equiv 0$$

must be an identity modulo 2 in the variable δ_{11}. This will be the case if, and only if, $g = g^2\gamma_1^{-1}$, whence $g = \gamma_1$. The latter requires $c = a^2$. But $c = a^2 - \gamma\gamma_1$. Hence $\gamma = 0$, which is impossible.

In addition to the determination of the canonical form (32) and for $p^n = 3$, $[1, -1]$, we have derived the theorem:

If two substitutions S_2 have as (common) characteristic determinant the square of an irreducible quadratic, they are conjugate within the group $SA(4, p^n)$.

§ 17.

It remains to consider the cases in which $\Delta(\kappa)$ is irreducible in the $GF[p^n]$ or is the product of two irreducible quadratic factors of the form $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$. In either case the roots of $\Delta(\kappa) = 0$ are σ, σ^{-1}, σ^{p^n}, σ^{-p^n}; in the former case $\sigma^{p^{2n}+1} = 1$, in the latter case $\sigma^{p^{2n}-1} = 1$. We may write $\Delta(\kappa) = 0$ in the form

$$\sigma^2\gamma\gamma_1 = (\sigma^2 - \sigma a + 1)(\sigma^2 - \sigma\delta + 1) . \tag{36}$$

The substitution S_2 given at the end of § 15, multiplies by σ the function

$$X_1 \equiv -\gamma\xi_1 + \sigma\gamma\gamma_1\eta_1 + (\sigma^2 - \sigma\delta + 1)\xi_2 + \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1)\eta_2 .$$

Denote by Y_1, X_2, Y_2 the linear functions derived from X_1 upon replacing σ by σ^{-1}, σ^{p^n}, σ^{-p^n} respectively. If $\Delta(\kappa)$ be irreducible, so that σ belongs to the $GF[p^{4n}]$, the functions X_1, Y_1, X_2, Y_2 are conjugate with respect to the $GF[p^n]$. In the second case, σ belongs to the $GF[p^{2n}]$, so that X_1 and X_2, Y_1 and Y_2 are conjugate with respect to the $GF[p^n]$. Hence the four functions satisfy the requirements as to the conjugacy. In terms of these functions taken as new indices,* the substitution S_2 takes the canonical form

$$X_1' = \sigma X_1 , \quad Y_1' = \sigma^{-1} Y_1 , \quad X_2' = \sigma^{p^n} X_2 , \quad Y_2' = \sigma^{-p^n} Y_2 . \tag{37}$$

* It may be verified by direct calculation that the determinant of the transformation of indices does not vanish; but the result follows from the abelian character of the transformation.

The transformation of indices satisfies those abelian conditions specified by formulæ (3) and (4). In fact,

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{p^n}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2 - \sigma\delta + 1 & \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1) \\ \sigma^{2p^n} - \sigma^{p^n}\delta + 1 & \sigma^{-p^n}\gamma(\sigma^{2p^n} - \sigma^{p^n}\delta + 1) \end{vmatrix}$$

$$= \gamma(\sigma - \sigma^{p^n})\{\gamma\gamma_1 + (\sigma^2 - \sigma\delta + 1)(\sigma^{2p^n} - \sigma^{p^n}\delta + 1)\sigma^{-p^n-1}\}.$$

On elimination of $\gamma\gamma_1$ by (36), the quantity in brackets vanishes if

$$\sigma^{p^n-1}(\sigma^2 - \sigma a + 1) + \sigma^{2p^n} - \sigma^{p^n}\delta + 1 = 0.$$

The latter is derived by multiplying by σ^{p^n} the identity

$$\sigma + \sigma^{-1} + \sigma^{p^n} + \sigma^{-p^n} = a + \delta,$$

which follows from the form of $\Delta(\kappa) = 0$, with the roots $\sigma, \sigma^{-1}, \cdots$. In a similar manner, we find that

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{-p^n}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2 - \sigma\delta + 1 & \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1) \\ \sigma^{-2p^n} - \sigma^{-p^n}\delta + 1 & \sigma^{p^n}\gamma(\sigma^{-2p^n} - \sigma^{-p^n}\delta + 1) \end{vmatrix} = 0.$$

Replacing σ by σ^{-1} in the two identities just established, we obtain two new identities. The four embrace the relations (3) and (4). Consider next the left member of the first abelian relation (2):

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{-1}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2 - \sigma\delta + 1 & \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1) \\ \sigma^{-2} - \sigma^{-1}\delta + 1 & \sigma\gamma(\sigma^{-2} - \sigma^{-1}\delta + 1) \end{vmatrix}$$

$$= \gamma(\sigma - \sigma^{-1})\{\gamma\gamma_1 + \sigma^{-2}(\sigma^2 - \sigma\delta + 1)^2\}.$$

Denoting, for the moment, the quantity in brackets by C, we observe that $C\sigma^2$ may be written

$$2(\sigma^2 - \sigma\delta + 1)\left(\sigma^2 - \frac{a+\delta}{2}\sigma + 1\right)$$

and hence does not vanish. If $\sigma - \sigma^{-1} = 0$, then $\sigma^2 - 1 = 0$, contrary to hypothesis. The left member of the second abelian relation (2) is seen in like manner to be

$$\gamma(\sigma^{p^n} - \sigma^{-p^n})C^{p^n}.$$

If $\sigma^{p^{2n}} = \sigma$, it is only necesary that X_1 and X_2, Y_1 and Y_2 be conjugate with respect to the $GF[p^n]$. When we take μX_1 in place of X_1 and $\mu^{p^n}X_2$ in place of X_2, the canonical form (37) is preserved, as well as the abelian relations (3) and (4) just established. In case μ is the reciprocal of $\gamma(\sigma - \sigma^{-1})C$, the resulting transformation of indices satisfies also the abelian relations (2) and is therefore an abelian substitution in the $GF[p^{2n}]$ on two pairs of conjugate indices. In this case it follows that two substitutions S_2 having the same characteristic equation are conjugate within $SA(4, p^n)$.

If $\Delta(\kappa) = 0$ be irreducible, so that $\sigma^{p^{2n}} = \sigma^{-1}$, the replacement of X_1 by μX_1 requires the replacement of Y_1 by $\mu^{p^{2n}} Y_1$, X_2 by $\mu^{p^n} X_2$, Y_2 by $\mu^{p^{3n}} Y_2$. The above expression $\gamma(\sigma - \sigma^{-1})C$ is then multiplied by $\mu^{p^{2n}+1}$, a mark of the $GF[p^{2n}]$; the product can not be made unity by choice of μ, since C belongs to the $GF[p^{2n}]$, while $\sigma - \sigma^{-1}$ does not. Hence S_2 is not reducible to the canonical form (37) by a special abelian substitution with conjugate indices. Nor is S_2 so reducible by a general abelian substitution; for the product

$$\mu^{p^{2n}+1}\gamma(\sigma - \sigma^{-1})C$$

differs from its (p^n)-th power; indeed, the product equals the negative of its (p^{2n})-th power. By a suitable choice of μ, $\mu^{p^{2n}+1}\gamma C = 1$, so that the left members of (2) become $\sigma - \sigma^{-1}$ and $\sigma^{p^n} - \sigma^{-p^n}$. Hence if T denote the transformation of indices reducing S_2 to the canonical form (37), then T replaces ϕ by ϕ', where

$$\phi \equiv \begin{vmatrix} \xi_1 & \eta_1 \\ \bar{\xi}_1 & \bar{\eta}_1 \end{vmatrix} + \begin{vmatrix} \xi_2 & \eta_2 \\ \bar{\xi}_2 & \bar{\eta}_2 \end{vmatrix}, \quad \phi' \equiv (\sigma - \sigma^{-1}) \begin{vmatrix} \xi_1 & \eta_1 \\ \bar{\xi}_1 & \bar{\eta}_1 \end{vmatrix} + (\sigma^{p^n} - \sigma^{-p^n}) \begin{vmatrix} \xi_2 & \eta_2 \\ \bar{\xi}_2 & \bar{\eta}_2 \end{vmatrix},$$

when T operates cogrediently upon the indices ξ_i, η_i and $\bar{\xi}_i$, $\bar{\eta}_i$. If T' denote the transformation of indices which reduces a second substitution S_2' to the canonical form (37), the product $A \equiv T'T^{-1}$ leaves ϕ absolutely invariant and transforms S_2' into S_2. In view of the conjugacy of the indices X_i, Y_i and the invariance of ϕ under A, the latter may be expressed as a special abelian substitution on ξ_i, η_i with coefficients in the $GF[p^n]$. Hence, *if two substitutions S_2 have the same characteristic determinant and if the latter be irreducible or the product of two irreducible quadratic factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, they are conjugate within the group $SA(4, p^n)$.*

§ 18.

Let S be an abelian substitution whose characteristic determinant is irreducible in the $GF[p^n]$, so that S may be reduced to the canonical form (37), where $\sigma^{p^{2n}+1} = 1$. Of the solutions of the latter, only $\sigma = \pm 1$ satisfy also $\sigma^{p^{2n}-1} = 1$, so that there remain $p^{2n} - 1$ or 2^{2n} suitable values of σ. Replacing σ by σ^{-1}, we obtain from (37) a substitution which is transformed into (37) by M_1M_2; replacing σ by σ^{-p^n}, we obtain the transform of (37) by $P_{12}M_1$. Replacing σ by σ^{p^n}, we obtain the transform of (37) by $P_{12}M_2$. Any new replacement of σ leads to a substitution not conjugate with (37). Hence there are $\frac{1}{4}(p^{2n} - 1)$ or $\frac{1}{4}2^{2n}$ non-conjugate types (37). An abelian substitution S_1 commutative with an abelian substitution S having the canonical form (37) has simultaneously the canonical form

$$X_1' = \rho X_1, \quad Y_1' = \rho^{-1} Y_1, \quad X_2' = \rho^{p^n} X_2, \quad Y_2' = \rho^{-p^n} Y_2 \quad (\rho^{p^{2n}+1} = 1),$$

so that there are $p^{2n} + 1$ such substitutions S_1. Hence S is conjugate with $p^{4n}(p^{2n} - 1)^2$ substitutions within $SA(4, p^n)$.

§ 19.

An abelian substitution S whose characteristic determinant is the product of two irreducible quadratic factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, is reducible to the canonical form (37), where $\sigma^{p^{2n}-1} = 1$, $\sigma^{p^n} \neq \sigma$, $\sigma^{p^n} \neq \sigma^{-1}$. The $(p^n-1)^2$ or $2^{2n} - 2\cdot 2^n$ suitable values of σ give, in sets of four, conjugate substitutions. Hence there are $\frac{1}{4}(p^n-1)^2$ or $\frac{1}{4}2^n(2^n-2)$ non-conjugate types. Each is commutative with $p^{2n}-1$ substitutions and therefore conjugate with exactly $p^{4n}(p^{4n}-1)$ substitutions of $SA(4, p^n)$.

§ 20.

An abelian substitution S whose characteristic determinant is the product of two distinct irreducible factors of the form $\kappa^2 - A\kappa + 1$ is conjugate within $SA(4, p^n)$ with a substitution (30) and therefore is reducible to the canonical form

$$(38) \qquad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = \mu X_2, \quad Y_2' = \mu^{-1} Y_2,$$

where $\lambda^{p^n+1} = 1$, $\mu^{p^n+1} = 1$, $\lambda^{p^n-1} \neq 1$, $\mu^{p^n-1} \neq 1$, $\mu \neq \lambda$, $\mu \neq \lambda^{-1}$. Of the p^n+1 solutions of $\lambda^{p^n+1} = 1$, $\lambda = \pm 1$ are to be excluded; then of the p^n+1 solutions of $\mu^{p^n+1} = 1$, $\mu = \pm 1$ and $\mu = \lambda$, λ^{-1} are to be excluded. Hence there are $(p^n-1)(p^n-3)$ or $2^n(2^n-2)$ pairs of suitable values λ, μ. But λ and λ^{-1} are interchanged upon transforming (38) by M_1; μ and μ^{-1} upon transforming by M_2; λ and μ, λ^{-1} and μ^{-1} upon transforming by P_{12}. Hence eight of the pairs of values λ, μ lead to conjugate types, so that there are exactly $\frac{1}{8}(p^n-1)(p^n-3)$ or $\frac{1}{8}2^n(2^n-2)$ non-conjugate types (38). Each is commutative with exactly $(p^n+1)^2$ substitutions having the canonical form

$$X_1' = \tau^l X_1, \quad Y_1' = \tau^{-l} Y_1, \quad X_2' = \tau^m X_2, \quad Y_2' = \tau^{-m} Y_2,$$

τ being a primitive root of the equation $\tau^{p^n+1} = 1$ and l and m being arbitrary integers. Each type represents a set of $p^{4n}(p^{2n}+1)(p^n-1)^2$ conjugate substitutions of $SA(4, p^n)$.

§ 21.

An abelian substitution S whose characteristic determinant is the square of an irreducible quadratic is either conjugate within $SA(4, p^n)$ with a substitution (30) having $A' = A$ or else with a substitution S_2 satisfying (31). In the former case, S has the canonical form

$$(39) \qquad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = \lambda Y_2, \quad Y_2' = \lambda^{-1} Y_2,$$

where $\lambda^{p^n+1} = 1$, $\lambda \neq \pm 1$. There are $\frac{1}{2}(p^n-1)$ or 2^{n-1} types not conjugate within $SA(4, p^n)$. An abelian substitution S_1, commutative with S has simultaneously the canonical form

$$X_1' = aX_1 + \beta X_2, \qquad X_2' = \gamma X_1 + \delta X_2,$$
$$Y_1' = a^{p^n} Y_1 + \beta^{p^n} Y_2, \quad Y_2' = \gamma^{p^n} Y_1 + \delta^{p^n} Y_2,$$

subject to the abelian conditions *

$$a^{p^n+1} + \beta^{p^n+1} = 1, \quad \gamma^{p^n+1} + \delta^{p^n+1} = 1, \quad a\gamma^{p^n} + \beta\delta^{p^n} = 0.$$

But these are the conditions that $\begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}$ shall be a binary hyperorthogonal substitution, the number of which is $(p^n+1)p^n(p^{2n}-1)$. Hence each canonical form (39) represents a set of $p^{3n}(p^{2n}+1)(p^n-1)$ conjugate substitutions of $SA(4, p^n)$.

§ 22.

A substitution S_2 satisfying (31) has the canonical form

(40) $X_1' = \sigma X_1, \quad Y_1' = \sigma^{-1} Y_1 + \sigma^{-1} X_2, \quad X_2' = \sigma^{-1} X_2, \quad Y_2' = \sigma Y_2 + \sigma X_1,$

σ being a root of $\Delta(\kappa) = 0$. As in § 17, we employ new indices

$$X_1 \equiv -\gamma\xi_1 + \sigma\gamma\gamma_1\eta_1 + (\sigma^2 - \sigma\delta + 1)\xi_2 + \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1)\eta_2,$$

and X_2 obtained from X_1 by replacing σ by $\sigma^{p^n} \equiv \sigma^{-1}$; but for Y_1 and Y_2 we now take the functions

$$Y_1 \equiv \gamma\xi_1 + (\sigma^{-2} - 1)\xi_2 + \gamma(\delta - 2\sigma)\eta_2, \quad Y_2 \equiv \gamma\xi_1 + (\sigma^2 - 1)\xi_2 + \gamma(\delta - 2\sigma^{-1})\eta_2.$$

The determinant of the transformation of indices is seen to be

$$-\gamma^3\gamma_1(\sigma - \sigma^{-1})^4 \neq 0.$$

Upon replacing σ by σ^{-1} in (40), we obtain the transform of (40) by P_{12}. Hence there are $\frac{1}{2}(p^n - 1)$ or 2^{n-1} non-conjugate types (40). Each is commutative only with the substitutions

$$X_1' = aX_1, \quad Y_1' = a^{p^n} Y_1 + d^{p^n} X_2, \quad X_2' = a^{p^n} X_2, \quad Y_2' = aY_2 + dX_1,$$

where $a^{p^n+1} = 1$, $a^{p^n} d = ad^{p^n}$, so that $d = a\kappa$, κ a mark of the $GF[p^n]$, giving $p^n(p^n+1)$ commutative substitutions. Hence each type represents a set of $p^{3n}(p^{4n}-1)(p^n-1)$ conjugate substitutions within $SA(4, p^n)$.

§ 23. *Summary of the preceding results.*

The numerical results obtained in the preceding investigation are collected into the following table. The mark μ denotes 1 or a particular not-square ν when $p > 2$; while $\mu = 1$, if $p = 2$. Also, θ denotes $\frac{1}{2}$ or 1 according as $p > 2$ or $p = 2$. Finally, for $p > 2$, $\epsilon = \pm 1$ according as $p^n = 4l \pm 1$. By the "number of types" is meant the number of non-conjugate types of the speci-

* Indeed, S given by (30) with $A' = A$ may be reduced to its canonical form by an abelian substitution (not necessarily special), so that the canonical form of S_1 satisfies the conditions (2), (3), (4).

fied form within $SA(4, p^n)$. As a check upon the enumeration it was verified that the sum of the products of the number of conjugate substitutions of each type (fourth column) by the number of types (second or third column) gives the order $N \equiv p^{4n}(p^{4n}-1)(p^{2n}-1)$ of the group.

TABLE OF THE NON-CONJUGATE TYPES OF OPERATORS OF THE GROUP $SA(4, p^n)$ OF ORDER N.

Type.	Number of types.		Number of conjugate substitutions of each type.
	$p>2$	$p=2$	
(6)	$\frac{1}{8}(p^n-3)(p^n-5)$	$(2^{n-1}-1)(2^{n-2}-1)$	$p^{4n}(p^{2n}+1)(p^n+1)^2$
(7)	$\frac{1}{2}(p^n-3)$	$2^{n-1}-1$	$p^{3n}(p^{2n}+1)(p^n+1)$
(8)	$\frac{1}{2}(p^n-3)$	$2^{n-1}-1$	$p^{3n}(p^{4n}-1)(p^n+1)$
(9), $\beta=0$	p^n-3	$2^{n-1}-1$	$p^{3n}(p^{2n}+1)(p^n+1)$
(9), $\beta=\mu$	$2(p^n-3)$	$2^{n-1}-1$	$\theta p^{3n}(p^{4n}-1)(p^n+1)$
$T_{1\,\pm1}T_{2\,\pm1}$	2	1	1
$L_{1\,\mu}T_{1\,\pm1}T_{2\,\pm1}$	4	1	$\theta(p^{4n}-1)$
$L_{1\,1}L_{2\,1}T_{1\,\pm1}T_{2\,\pm1}$	2		$\frac{1}{2}p^n(p^{4n}-1)(p^n+\epsilon)$
		1	$(2^{4n}-1)(2^{2n}-1)$
$L_{1\,\nu}L_{2\,1}T_{1\,\pm1}T_{2\,\pm1}$	2		$\frac{1}{2}p^n(p^{4n}-1)(p^n-\epsilon)$
(13)		1	$2^{4n}-1$
R_β, R_0		2	$\frac{1}{2}2^{2n}(2^{4n}-1)(2^{2n}-1)$
$A_\mu T_{1\,\pm1}T_{2\,\pm1}$	4		$\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$
(18)	$\frac{1}{4}(p^n-1)(p^n-3)$	$2^{n-1}(2^{n-1}-1)$	$p^{4n}(p^{4n}-1)$
(19), $b=0$	p^n-1	2^{n-1}	$p^{3n}(p^{2n}+1)(p^n-1)$
(19), $b=\mu$	$2(p^n-1)$	2^{n-1}	$\theta p^{3n}(p^{4n}-1)(p^n-1)$
(37), $\sigma^{p^{2n}+1}=1$	$\frac{1}{4}(p^{2n}-1)$	2^{2n-2}	$p^{4n}(p^{2n}-1)^2$
(37), $\sigma^{p^{2n}-1}=1$	$\frac{1}{4}(p^n-1)^2$	$2^{n-1}(2^{n-1}-1)$	$p^{4n}(p^{4n}-1)$
(38)	$\frac{1}{8}(p^n-1)(p^n-3)$	$2^{n-2}(2^{n-1}-1)$	$p^{4n}(p^{2n}+1)(p^n-1)^2$
(39)	$\frac{1}{2}(p^n-1)$	2^{n-1}	$p^{3n}(p^{2n}+1)(p^n-1)$
(40)	$\frac{1}{2}(p^n-1)$	2^{n-1}	$p^{3n}(p^{4n}-1)(p^n-1)$
$T_{1\,-1}$	1		$p^{2n}(p^{2n}+1)$
$L_{1\,\mu}T_{1\,-1}$ $L_{2\,\mu}T_{1\,-1}$	4		$\frac{1}{2}p^{2n}(p^{4n}-1)$
$L_{1\,1}T_{1\,-1}L_{2\,\mu}$ $L_{1\,\nu}T_{1\,-1}L_{2\,\mu}$	4		$\frac{1}{4}p^{2n}(p^{4n}-1)(p^{2n}-1)$

§ 24. *The group* $SA(4, 2)$ *and the symmetric group on six letters.*

A second check upon the above results is furnished by a consideration of the case $p^n = 2$, when the group $SA(4, 2)$ is holoedrically isomorphic with the symmetric group on six letters.* The isomorphism may be established by the correspondence of generators:

$$(12) \sim M_1, \quad (23) \sim L_{11}, \quad (34) \sim S, \quad (45) \sim L_{21}, \quad (56) \sim M_2,$$

where

$$S \equiv \begin{Vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{Vmatrix}, \quad S' \equiv L_{11}L_{21}SL_{21}L_{11} \equiv \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{Vmatrix} \sim (25).$$

Since M_i transforms L_{i1} into L'_{i1}, $L'_{11} \sim (13)$, $L'_{21} \sim (46)$. Then †

$$R_1 \equiv L_{21}M_2S'M_2L'_{21} \sim (2456), \quad R_0 \equiv M_2L'_{11}S'L_{21} \sim (13)(2456),$$

$$[13] \equiv L'_{11}L'_{21}S' \sim (13)(46)(25).$$

By § 12, there is a single type $L_{21}M_2$ given by (19′) for $b = 0$ and a single type $L'_{11}\,L_{21}\,M_2$ given by (19′) for $b = 1$, The single type (39) may be represented by $L_{11}\,M_1L_{21}M_2$. The single types (40) and (37), $\sigma^{p^{2n}+1} = 1$, may be represented respectively by

$$[40] \equiv \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix} \equiv M_1[13]M_2,$$

$$[37] \equiv \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix} \equiv L'_{21}M_1[13]M_2.$$

For $p^n = 2$ the above table gives the following types of abelian substitutions:

* Proceedings of the London Mathematical Society, vol. 31, p. 40, 1899.

† The abelian substitution numbered (13) is now referred to as [13].

Type.	Number of conjugates.	Corresponding substitution on six letters.
identity	1	identity
L_{11}	15	(23)
$L_{11}L_{21}$	45	(23)(45)
[13]	15	(13)(25)(46)
R_1	90	(2456)
R_0	90	(13)(2456)
$L_{21}M_2$	40	(465)
$L'_{11}L_{21}M_2$	120	(13)(465)
[37]	144	(16523)
$L_{11}M_1L_{21}M_2$	40	(123)(465)
[40]	120	(164523)

The third column gives every type of substitution on six letters and the number of conjugates to each type is given by the second column. In view of the independence of the two determinations of the types of substitutions of $SA(4, 2)$, the check is a complete one.

§ 25. *Operators of the simple group* $A(4, p^n)$, $p > 2$.

For $p = 2$, $n > 1$, the group $SA(4, p^n)$ is simple; for $p > 2$, it has the maximal invariant subgroup composed of the identity and $T \equiv T_{1-1}T_{2-1}$, the quotient-group $A(4, p^n)$ being simple.* In view of the importance of the latter group, we proceed to determine, by means of the earlier results, the distribution of its operators into complete sets of conjugate operators.

In the table (§ 23) of the non-conjugate types of substitutions within $SA(4, p^n)$, $p > 2$, the types are grouped into sets (each set being exhibited in a single line of the table, except the last two sets) such that types S and ST always belong to the same set. If S be not conjugate with ST within $SA(4, p^n)$, the number of conjugates with S within $SA(4, p^n)$ equals the number of conjugates with S within $A(4, p^n)$.† If, however, there exists a substitution V in $SA(4, p^n)$ which transforms S into ST, the number of conjugates with S in $A(4, p^n)$ equals one-half the number of conjugates with S in $SA(4, p^n)$; indeed, if W transforms S_1 into S, then WVW^{-1} will transform S_1 into S_1T.

A type S of the group $SA(4, p^n)$, $p > 2$, will be called special and denoted

*Quarterly Journal of Mathematics, vol. 29, pp. 169–178, 1897; vol. 31, pp. 383–4, 1899.

† In the quotient-group, S and ST become the same operator. It is convenient to denote the latter by S, the context sufficing to avoid confusion.

S_s if S be conjugate with ST within the group. For example, there is a single special type $(7)_s$,

$$\xi_1' = i\xi_1, \quad \eta_1' = -i\eta_1, \quad \xi_2' = -i\xi_2, \quad \eta_2' = i\eta_2 \qquad (i^2 = -1),$$

occurring if and only if -1 be a square in the $GF[p^n]$, viz., if $p^n = 4l + 1$. In general, a type can be special only when the negative of each root of the characteristic equation is also a root.

For special types (6), three cases are to be examined. First, if $-\kappa = \kappa^{-1}$, so that $-\kappa^{-1} = \kappa$, then must $-\lambda = \lambda^{-1}$; hence $\kappa^2 = -1$, $\lambda^2 = -1$, so that $\lambda = \kappa$ or κ^{-1}, contrary to the hypothesis for type (6). Second, if $-\kappa = \lambda$, we have the special type

$$(6)_s \qquad \xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = -\kappa\xi_2, \quad \eta_2' = -\kappa^{-1}\eta_2,$$

which may be transformed into $(6)_s T$ by P_{12}. By the hypotheses for a type (6), $\kappa \neq 0$, $\kappa^2 \neq 1$, $\kappa^2 \neq -1$, the latter having solutions in the $GF[p^n]$ if, and only if, $p^n = 4l + 1$. Hence there are $\frac{1}{4}(p^n - 5)$ non-conjugate special types $(6)_s$ if $p^n = 4l + 1$ and $\frac{1}{4}(p^n - 3)$ such types if $p^n = 4l - 1$. Third, if $-\kappa = \lambda^{-1}$, the resulting special type is transformed into $(6)_s$ by M_2.

There is no special type (18), since $\alpha^2 = -1$, $\lambda^2 = -1$ require that α and λ belong to the same field, contrary to hypothesis.

To show that type (37), $\sigma^{p^{2n}+1} = 1$, is never special, we consider three cases. If $-\sigma = \sigma^{-1}$, then $\sigma^2 = -1$, while $p^{2n} + 1$ is not divisible by 4. If $-\sigma = \sigma^{p^n}$, then $\sigma^{p^{2n}} = -\sigma^{p^n} = \sigma \neq \sigma^{-1}$. Similarly, for $-\sigma = \sigma^{-p^n}$, $\sigma^{p^{2n}} = -\sigma^{-p^n} = \sigma \neq \sigma^{-1}$.

To determine the special types (37), $\sigma^{p^{2n}-1} = 1$, $\sigma^{p^n-1} \neq 1$, $\sigma^{p^n+1} \neq 1$, we examine the three cases. If $-\sigma = \sigma^{-1}$, either $\sigma^{p^n-1} = 1$ or $\sigma^{p^n+1} = 1$, contrary to hypothesis. If $-\sigma = \sigma^{p^n}$, each solution of $\sigma^{p^n-1} = -1$, such that $\sigma^2 \neq -1$ and therefore $\sigma^{p^n+1} \neq 1$, furnishes a special type; there are $p^n - 1$ or $p^n - 3$ such values of σ according as $p^n = 4l \pm 1$. If $-\sigma = \sigma^{-p^n}$, each solution of $\sigma^{p^n+1} = -1$, such that $\sigma^2 \neq -1$ and therefore $\sigma^{p^n-1} \neq 1$, furnishes a special type; there are $p^n \mp 1$ such values of σ according as $p^n = 4l \pm 1$. The two sets of values for σ are wholly distinct since $\sigma^2 \neq 1$. Hence there are $2p^n - 2$ values σ, whatever be the form of p^n, $p > 2$. Hence there are $\frac{1}{2}(p^n - 1)$ special types (37) when $\sigma^{p^{2n}-1} = 1$.

Type (38) is not special for $-\lambda = \lambda^{-1}$, $-\mu = \mu^{-1}$, since then $\mu = \lambda$ or λ^{-1}, but is special for $\mu = -\lambda$, viz.:

$$(38)_s \quad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = -\lambda X_2, \quad Y_2' = -\lambda^{-1} Y_2,$$

since P_{12} transforms it into $(38)_s T$. The number of solutions of $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$, $\lambda^2 \neq -1$, is $p^n - 1$ or $p^n - 3$, according as $p^n = 4l \pm 1$.

Of the remaining types in the table of § 23, it may be determined by inspection what special types, if any, exist. Our results may be combined in the following table:

TABLE OF THE NON-CONJUGATE TYPES*
OF OPERATORS OF THE SIMPLE GROUP $A(4, p^n)$, $p > 2$,
OF ORDER $\frac{1}{2}p^{4n}(p^{4n}-1)(p^{2n}-1)$.

Type.	Number of distinct types.	Number of operators conjugate with each type.
	$p^n = 4l+1$, $p^n = 4l-1$	
$(6)_s$	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$\frac{1}{2}p^{4n}(p^{2n}+1)(p^n+1)^2$
(6)	$\frac{1}{16}(p^n-5)^2$, $\frac{1}{16}(p^n-3)(p^n-7)$	$p^{4n}(p^{2n}+1)(p^n+1)^2$
$(7)_s$	1 , 0	$\frac{1}{2}p^{3n}(p^{2n}+1)(p^n+1)$
(7)	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n+1)$
$(8)_s$	1 , 0	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n+1)$
(8)	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{4n}-1)(p^n+1)$
(9), $\beta=0$	$\frac{1}{2}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n+1)$
(9), $\beta=\mu$	p^n-3	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n+1)$
identity	1	1
$L_{1\mu}$	2	$\frac{1}{2}(p^{4n}-1)$
$L_{11}L_{21}$	1	$\frac{1}{2}p^n(p^{4n}-1)(p^n+\epsilon)$
$L_{1\nu}L_{21}$	1	$\frac{1}{2}p^n(p^{4n}-1)(p^n-\epsilon)$
A_μ	2	$\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$
(18)	$\frac{1}{8}(p^n-1)(p^n-3)$	$p^{4n}(p^{4n}-1)$
(19), $b=0$	$\frac{1}{2}(p^n-1)$	$p^{3n}(p^{2n}+1)(p^n-1)$
(19), $b=\mu$	p^n-1	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n-1)$
(37) , $\sigma^{p^{2n}+1}=1$	$\frac{1}{8}(p^{2n}-1)$	$p^{4n}(p^{2n}-1)^2$
$(37)_s$, $\sigma^{p^{2n}-1}=1$	$\frac{1}{2}(p^n-1)$	$\frac{1}{2}p^{4n}(p^{4n}-1)$
(37) , $\sigma^{p^{2n}-1}=1$	$\frac{1}{8}(p^n-1)(p^n-3)$	$p^{4n}(p^{4n}-1)$
$(38)_s$	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$\frac{1}{2}p^{4n}(p^{2n}+1)(p^n-1)^2$
(38)	$\frac{1}{16}(p^n-1)(p^n-5)$, $\frac{1}{16}(p^n-3)^2$	$p^{4n}(p^{2n}+1)(p^n-1)^2$
$(39)_s$	0 , 1	$\frac{1}{2}p^{3n}(p^{2n}+1)(p^n-1)$
(39)	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n-1)$
$(40)_s$	0 , 1	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n-1)$
(40)	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{4n}-1)(p^n-1)$
T_{1-1}	1	$\frac{1}{2}p^{2n}(p^{2n}+1)$
$L_{1\mu}T_{1-1}$	2	$\frac{1}{2}p^{2n}(p^{4n}-1)$
$L_{11}L_{2\nu}T_{1-1}$	1	$\frac{1}{4}p^{2n}(p^{4n}-1)(p^{2n}-1)$
$L_{1\mu}L_{2\mu}T_{1-1}$	2	$\frac{1}{8}p^{2n}(p^{4n}-1)(p^{2n}-1)$

* A type marked s is special, otherwise a type is not special except for T_{1-1} and $L_{1\mu}L_{2\mu}T_{1-1}$.

In the table μ denotes 1 or a particular not-square ν; ϵ denotes ± 1 according as $p^n = 4l \pm 1$. The total number of non-conjugate types in the simple group $A(4, p^n)$, $p > 2$, is the same function of p^n in the two cases $p^n = 4l \pm 1$, viz.:

$$\tfrac{1}{2}(p^{2n} + 1) + 3p^n + 6.$$

§ 26. *The operators of the simple group* $A(4, 3)$ *of order* 25920.

As proved by JORDAN, $SA(4, 3)$ is the group of the equation for the trisection of the periods of a hyperelliptic function of four periods. Moreover, the group of the equation for the determination of the 27 lines on a general cubic surface of the third order is of the same order 51840 as $SA(4, 3)$. After a certain square root has been adjoined to the realm of rationality, the group reduces to the quotient-group $A(4, 3)$ of order 25920. Hence the above two problems are essentially the same. In view of the importance of the group $A(4, 3)$, it is desirable to know the distribution of its operators into complete sets of conjugates and likewise for its cyclic subgroups. By § 25, there are exactly twenty types of non-conjugate operators in the group. It is desirable to have simple representatives in the group for each type. Type (39) may be represented by $M_1 M_2$; type (19), $b = 0$, by M_2; type (19), $b = 1$, by $M_2 L_{11}$; type (19), $b = -1$, by $M_2 L_{1-1}$; type (37), $\sigma^4 = -1$, by $P_{12} M_1$; type (40) by $P_{12} L_{1-1} T_{1-1}$; type (37), $\sigma^{10} = 1$, by S_2 for $\gamma_1 = 1$, $\gamma = a = \delta = -1$, viz.,

$$K \equiv \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

of characteristic determinant $\kappa^4 - \kappa^3 + \kappa^2 - \kappa + 1$. Of the preceding statements, the only ones requiring further proof are those concerning the representatives $P_{12} M_1$ and $P_{12} L_{1-1} T_{1-1}$; but the former has the characteristic determinant $(\kappa^2 + \kappa - 1)(\kappa^2 - \kappa - 1)$, each factor being irreducible modulo 3, and hence has the canonical form (37), $\sigma^8 = 1$; while the latter is reduced to the canonical form (40) by the abelian transformation

$$X_1 = \eta_1 + \sigma\eta_2, \quad Y_1 = \xi_1 - \sigma\xi_2 - \sigma\eta_2, \quad X_2 = \eta_1 - \sigma\eta_2, \quad Y_2 = \xi_1 + \sigma\xi_2 + \sigma\eta_2,$$

where $\sigma^2 \equiv -1 \pmod 3$. We have therefore, by § 25, the complete list of types of operators of $A(4, 3)$ together with their periods and the number of their conjugates within $A(4, 3)$, as given on the following page.

§ 27. *Cyclic subgroups of the simple group* $A(4, 3)$.

To determine the distinct types of cyclic subgroups of $A(4, 3)$, it is necessary to find what powers of each type of substitution are conjugate with

Type.	Period.	Conjugates.	Type.	Period.	Conjugates.
$L_{1\,1}$	3	40	identity	1	1
$L_{1\,-1}$	3	40	A_1	9	2880
$L_{1\,1}L_{2\,1}$	3	240	A_{-1}	9	2880
$L_{1\,-1}L_{2\,1}$	3	480	M_2	4	540
$T_{1\,-1}$	2	45	$M_2L_{1\,1}$	12	2160
$L_{1\,1}T_{1\,-1}$	6	360	$M_2L_{1\,-1}$	12	2160
$L_{1\,-1}T_{1\,-1}$	6	360	K	5	5184
$L_{1\,1}L_{2\,-1}T_{1\,-1}$	6	1440	$P_{12}M_1$	4	3240
$L_{1\,1}L_{2\,1}T_{1\,-1}$	6	720	M_1M_2	2	270
$L_{1\,-1}L_{2\,-1}T_{1\,-1}$	6	720	$P_{12}L_{1\,-1}T_{1\,-1}$	6	2160

that type. Thus, $L_{1\,1}$ is not conjugate with its square $L_{1\,-1}$; $L_{1\,1}L_{2\,1}$ is conjugate with its square. A_{-1} is transformed into A_1^{-1} by $T_{2\,-1}L'_{2\,1}$. Since $A_1^3 = L'_{1\,-1}$ and $A_1^6 = L'_{1\,1}$ are not conjugate, A_1 is not conjugate with either A_1^2 or A_1^5, so that the latter are conjugate with A_1^{-1}. Hence their squares A_1^4, A_1 and A_1^7 are conjugate. Hence A_1 generates a cyclic group self-conjugate only under a G_{27}. Again, M_2 is transformed into M_2^3 by the abelian substitution (modulo 3) $\xi'_2 = \xi_2 + \eta_2$, $\eta'_2 = \xi_2 - \eta_2$. Hence $M_2L_{1\,1}$ is conjugate with its seventh power $M_2^3L_{1\,1}$, so that the fifth and eleventh powers are conjugate. The latter is $M_2^3L_{1\,-1}$ and is consequently conjugate with $M_2L_{1\,-1}$ and hence (by the above table) not conjugate with $M_2L_{1\,1}$ itself. Hence $M_2L_{1\,1}$ is conjugate only with one other generator of the same cyclic group. The fact that $P_{12}L_{1\,-1}T_{1\,-1}$ is conjugate with its reciprocal within $A(4, 3)$ may be simply verified by observing that the canonical form (40) is transformed into its reciprocal by the abelian substitution $P_{12}T_{2\,-1}$ on the indices X_i, Y_i. In a similar way, $P_{12}M_1$ is shown to be conjugate with its reciprocal. Finally, there being but a single type of substitutions of period 5, K must be conjugate with K^2, K^3, K^4. We have therefore the following complete list of the distinct types of cyclic subgroups of $A(4, 3)$, together with the number of conjugates to each cyclic group.

Type of generator.	Conjugate cyclic groups.	Generator.	Groups.
$L_{1\,1}$	40	A_1	960
$L_{1\,1}L_{2\,1}$	120	M_2	270
$L_{1\,-1}L_{2\,1}$	240	$M_2L_{1\,1}$	1080
$T_{1\,-1}$	45	K	1296
$L_{1\,1}T_{1\,-1}$	360	$P_{12}M_1$	1620
$L_{1\,1}L_{2\,-1}T_{1\,-1}$	720	M_1M_2	270
$L_{1\,1}L_{2\,1}T_{1\,-1}$	720	$P_{12}L_{1-1}T_{1\,-1}$	1080

THE UNIVERSITY OF CHICAGO, *January* 3, 1901.

CERTAIN CASES IN WHICH THE VANISHING OF THE WRONSKIAN IS A SUFFICIENT CONDITION FOR LINEAR DEPENDENCE*

BY

MAXIME BÔCHER

PEANO in Mathesis, vol. 9 (1889), p. 75 and p. 110 seems to have been the first to point out that the identical vanishing of the Wronskian of n functions of a single variable is not in all cases a sufficient condition for the linear dependence of these functions.† At the same time he indicated a case in which it *is* a sufficient condition,‡ and suggested the importance of finding other cases of the same sort. Without at first knowing of PEANO'S work, I was recently led to this same question, and found a case not included in PEANO'S in which the identical vanishing of the Wronskian is a sufficient condition.§ It is my purpose in the present paper to consider these cases and others of a similar nature.

By far the most important case in which the identical vanishing of the Wronskian is a sufficient condition for linear dependence is that in which the functions in question are at every point of a certain region analytic functions, whether of a real or complex variable is, of course, immaterial. This case requires no further treatment here.

We shall therefore be concerned exclusively with the case in which the independent variable x is real. This variable we will suppose to be confined to an interval I which may be finite or infinite, and if limited in one or both directions may or may not contain the end points. In some of the proofs we shall use a subinterval $a \leqq x \leqq b$ of I;‖ this subinterval we call I'.

Whether the functions are real or complex is immaterial.

We use the symbol $\equiv$ to denote an identity, i. e., an equality which holds at every point of the interval we are considering.

---

* Presented to the Society December 28, 1900. Received for publication December 28, 1900.

† It is of course a necessary condition provided the functions have finite derivatives of the first $n-1$ orders at every point of the region in question.

‡ See § 4 of the present paper.

§ See Bulletin of the American Mathematical Society, December, 1900, p. 120, and Annals of Mathematics, second series, vol. 2, p. 93.

‖ We suppose here that a and b are finite quantities.

§ 1. *The Fundamental Theorem.*

We consider first the special case of two functions.

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be functions of x which at every point of I have finite first derivatives, while u_1 does not vanish in I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0 , \tag{1}$$

u_1 and u_2 are linearly dependent throughout I, and in particular

$$u_2 \equiv c u_1 . \tag{2}$$

For dividing (1) by u_1^2 we have:

$$\frac{d}{dx}\left(\frac{u_2}{u_1}\right) \equiv 0 .$$

Therefore:

$$\frac{u_2}{u_1} \equiv c .$$

We pass now to the general case which includes the case just considered.

THEOREM II. *Let $u_1(x), u_2(x), \cdots, u_n(x)$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish in I; then if the Wronskian W of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent throughout I, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1} .$$

In the Wronskian:

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} ,$$

we denote by $W_1, W_2, \cdots, W_n$ the minors corresponding to the elements of the last row. We have then:

$$W_1 u_1^{(i)} + W_2 u_2^{(i)} + \cdots + W_n u_n^{(i)} \equiv 0 \quad (i = 0, 1, \cdots, n-1).$$

Differentiating each of the first $n-1$ of these identities and subtracting from it the one next following we get:

$$W_1' u_1^{(i)} + W_2' u_2^{(i)} + \cdots + W_n' u_n^{(i)} \equiv 0 \quad (i = 0, 1, \cdots, n-2).$$

Let us add these identities together after having multiplied the i-th of them

$(i = 1, 2, \cdots, n-1)$ by the first minor of W_n corresponding to $u_1^{(i-1)}$. This gives:

$$W_1' W_n - W_n' W_1 \equiv 0 .$$

Now since by hypothesis W_n does not vanish in I we have by theorem I:

$$W_1 \equiv - c_1 W_n .$$

In the same way:

$$W_2 \equiv - c_2 W_n ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$W_{n-1} \equiv - c_{n-1} W_n .$$

Therefore the identity

$$W_1 u_1 + W_2 u_2 + \cdots + W_n u_n \equiv 0 ,$$

can be written:

$$W_n(- c_1 u_1 - c_2 u_2 \cdots c_{n-1} u_{n-1} + u_n) \equiv 0 ,$$

and, since W_n does not vanish, our theorem follows at once.*

§ 2. *A Generalization for the Case of Two Functions.*

THEOREM III.† *Let u_1 and u_2 be functions of x which at every point of I have finite derivatives of the first k orders $(k \geqq 1)$, while u_1, u_1', u_1'', $\cdots$, $u_1^{(k)}$ do not all vanish at any one point of I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0 ,$$

u_1 and u_2 are linearly dependent, and in particular:

$$u_2 \equiv c u_1 .$$

This theorem will evidently be established if we can prove it for every finite and perfect subinterval I' of I. We will therefore in our proof consider only the interval I'.

There cannot be more than a finite number of points in I' where $u_1 = 0$. For if there were these points would have at least one limiting point x_0 in I', and since u_1 is continuous it would vanish at x_0. By ROLLE'S theorem there would also be an infinite number of points where $u_1' = 0$ and these points would have x_0 as limiting point, and owing to the continuity of u_1' we should have $u_1'(x_0) = 0$. Proceeding in the same way we see that u_1'', u_1''', $\cdots$, $u_1^{(k-1)}$ would all vanish at x_0. That $u_1^{(k)}$ would also vanish at x_0 must be shown in a slightly

* This proof is merely a slight modification of the one given by FROBENIUS, Crelle, vol. 76 (1873), p. 238. Cf. also HEFFTER, *Lineare Differentialgleichungen*, p. 233.

† The special case $k = 1$ of this theorem was given by PEANO, l. c. Cf. also Annals of Mathematics, second series, vol. 2, p. 92.

different manner since we do not know that $u_1^{(k)}$ is continuous. This follows at once, however, from the fact that $u_1^{(k-1)}$ would vanish in every neighborhood of x_0. We thus see that if u_1 vanished at an infinite number of points in I' there would be a point x_0 where $u_1, u_1', \cdots, u_1^{(k)}$ all vanish, and this is contrary to hypothesis.

The points at which $u_1 = 0$ therefore divide the interval I' into a finite number of pieces throughout each of which theorem I tells that u_2 is a constant multiple of u_1, and owing to the continuity of u_1 and u_2 this relation must also hold at the extremities of the piece in question. It remains to show that this constant is the same for all the pieces. It will evidently be sufficient to consider two adjacent pieces separated by the point p. Suppose that in the piece to the left of p we have

$$u_2 = c_1 u_1,$$

and in the piece to the right,

$$u_2 = c_2 u_1.$$

Since the derivatives of u_1 and u_2 at p may be found either by differentiating to the right or to the left we have:

$$u_2^{(i)}(p) = c_1 u_1^{(i)}(p), \qquad (i = 1, 2, \cdots, k).$$

$$u_2^{(i)}(p) = c_2 u_1^{(i)}(p),$$

Therefore

$$(c_1 - c_2) u_1^{(i)}(p) = 0 \qquad (i = 1, 2, \cdots, k).$$

Now, since $u_1(p) = 0$, there must be at least one of the derivatives $u_1', u_1'', \cdots, u_1^{(k)}$ which does not vanish at p. Therefore

$$c_1 = c_2,$$

and our theorem is proved.

§ 3. *Two Extensions to the case of n Functions.*

THEOREM IV.* *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n - 2 + k$ orders ($k \geqq 1$), while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ and its first k derivatives do not all vanish at any one point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ is identically zero $u_1, u_2, \cdots, u_n$ are linearly dependent, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

The proof of this theorem is, in the main, the same as that of theorem II. We will therefore only point out the two points of difference.

1. We must use theorem III instead of theorem I to establish the relation:

* The special case $k = 1$ of this theorem was given by the writer, l. c.

$$W_1 \equiv - c_1 W_n .$$

2. From the identity:

$$W_n(- c_1 u_1 - c_2 u_2 \cdots c_{n-1} u_{n-1} + u_n) \equiv 0 ,$$

we can now infer only that *at the points where* $W_n \neq 0$,

$$u_n = c_1 u_1 + \cdots + c_{n-1} u_{n-1} .$$

In order to prove that this equation also holds at the points where $W_n = 0$, we notice first that these points in any finite and perfect subinterval I' of I are finite in number as otherwise there would be (cf. the proof of theorem III) a point of I' where W_n, W_n', $\cdots$, $W_n^{(k)}$ all vanish. All the points where W_n vanishes are therefore isolated, and since the equation

$$u_n = c_1 u_1 + \cdots + c_{n-1} u_{n-1}$$

holds everywhere except at these points it must on account of the continuity of the u's hold at these points also. Thus our theorem is proved.

A little reflection on the results so far obtained will suggest the question whether the theorem of the last section might not be extended to the case of n functions by requiring, not as we have just done, that W_n, W_n', $\cdots$, $W_n^{(k)}$, do not all vanish at any point of I, but that u_1 and a certain number of its derivatives shall not all vanish at any point of I. The following example shows, however, not only that the theorem thus suggested is not true, but that even when *no one of the u's vanishes at any point of* I the identical vanishing of the Wronskian is not necessarily a sufficient condition for linear dependence when we have more than two functions.

EXAMPLE. Consider the three functions:

$$u_1 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x \neq 0), \\ 1 & (x = 0), \end{cases} \qquad u_2 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x > 0), \\ 1 & (x = 0), \\ 1 - e^{-\frac{1}{x^2}} & (x < 0), \end{cases} \qquad u_3 = 1 .$$

These three functions are obviously linearly independent in any interval including both positive and negative values of x. Moreover no one of them vanishes for any real value of x. Yet the Wronskian of u_1, u_2, u_3 is identically zero.

The following theorems V and VI, which run somewhat along the lines just indicated, are, however, true:

THEOREM V. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* $n - 1$ *orders, while no function (other than zero) of the form:*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(*the g's being constants*) *vanishes together with its first* $n-1$ *derivatives at any point of* I; *then if the Wronskian of* $u_1, u_2, \cdots, u_n$ *vanishes at any point* p *of* I *these functions are linearly dependent.*

From the fact that the Wronskian vanishes at p follows the existence of n constants $c_1, c_2 \cdots, c_n$ not all zero and such that

$$c_1 u_1^{(i)}(p) + c_2 u_2^{(i)}(p) + \cdots + c_n u_n^{(i)}(p) = 0 \quad (i = 0, 1, \cdots, n-1),$$

i. e., the function $c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ vanishes together with its first $n-1$ derivatives at the point p, and must therefore be identically zero. Thus our theorem is proved.

THEOREM VI. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* k *orders* $(k > n-1)$, *while no function* (*other than zero*) *of the form*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n,$$

(*the g's being constants*) *vanishes together with its first* k *derivatives at any point of* I; *then if the Wronskian of* $u_1, u_2, \cdots, u_n$ *vanishes identically these functions are linearly dependent.*

We prove this theorem first on the supposition that the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish identically.* In this case there exists a point p of I where the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. Since this last named Wronskian is continuous it is different from zero throughout the neighborhood of p. We see then by applying II that there exist n constants $c_1, c_2, \cdots, c_n$ not all zero and such that the function

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

is zero throughout the neighborhood of p. Accordingly this function vanishes together with its first k derivatives at p, and therefore vanishes identically. Thus our theorem is proved in this special case.

In order to prove the theorem in general we first notice that if $u_1 \equiv 0$ the u's are surely linearly dependent. If u_1 is not identically zero, consider in succession the Wronskians of u_1, u_2, of u_1, u_2, u_3, of u_1, u_2, u_3, u_4, etc. Suppose the first of these which vanishes identically is the Wronskian of $u_1, u_2, \cdots, u_m$ $(m \leqq n-1)$. Then since the Wronskian of $u_1, u_2, \cdots, u_{m-1}$ does not vanish identically, the special case of our theorem which we have already proved shows that $u_1, u_2, \cdots, u_m$ are linearly dependent. Accordingly $u_1, u_2, \cdots, u_n$ are linearly dependent, and our theorem is proved.

Theorems V and VI admit of immediate application to the theory of linear differential equations, as the following theorem shows.

* The proof of this part of the theorem has been modified since the paper was presented to the Society by making it depend on II instead of on the lemmas of § 5.

THEOREM VII. *Let $p_1, p_2, \cdots, p_n$ be functions of x which at every point of I are continuous, and let $y_1, y_2, \cdots, y_k (k \leqq n)$ be functions of x which at every point of I satisfy the differential equation:*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0;$$

then the identical vanishing of the Wronskian of $y_1, y_2, \cdots, y_k$ (or in the case $k = n$ the vanishing of this Wronskian at a single point of I) is a sufficient condition for the linear dependence of $y_1, y_2, \cdots, y_k$.

This theorem follows at once from theorems V and VI when we recall the fact that a solution of the above written differential equation which vanishes together with its first $n-1$ derivatives at a point of I is necessarily identically zero.

§ 4. *Discussion of Peano's Theorems.* *

One of PEANO'S results, as has already been stated, is the special case $k = 1$ of theorem III. Apart from this PEANO'S results cover no case which is not also covered by the fundamental theorem of § 1. I propose to show this in the present section.

For this purpose we first establish the following:

LEMMA. *Let u_1 and u_2 be functions of x which at every point of I have finite first derivatives, while*

$$u_1 u_2' - u_2 u_1' \equiv 0;$$

if a point p exists in I at which $u_2 = 0$, while in every neighborhood of p lie points where $u_2 \neq 0$, then $u_1(p) = 0$.

For if $u_1(p) \neq 0$ we could, on account of the continuity of u_1, mark off a neighborhood of p throughout which u_1 does not vanish, and throughout which therefore by theorem I

$$u_2 = c u_1.$$

Since at p $u_1 \neq 0$ and $u_2 = 0$ we must have $c = 0$, but this would make u_2 vanish throughout the neighborhood of p, and this is contrary to hypothesis.

PEANO deduces the following theorem in the case of two functions. This theorem includes as a special case the theorem to which theorem III reduces when $k = 1$, and appears at first sight to go beyond it.

PEANO'S FIRST THEOREM. *Let u_1 and u_2 be functions of x which at every point of I have finite first derivatives, while u_1, u_2, u_1', u_2' do not all vanish at any point of I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0,$$

u_1 and u_2 are linearly dependent.

* See, besides the notes in Mathesis referred to at the beginning of this article, a paper by PEANO: Rendiconti della Accademia dei Lincei, ser. 5, vol. 6, 1° sem. (1897), p. 413.

The truth of this theorem will be established, and at the same time it will be proved that it covers no case which is not also covered by the special case $k=1$ of theorem III, if we can show that either there is no point of I where u_1 and u_1' both vanish, or there is no point of I where u_2 and u_2' both vanish. Assume then that there is a point where u_2 and u_2' both vanish. Here we distinguish between two cases:

(*a*) $u_2 \equiv 0$. Here $u_2 = u_2' = 0$ at every point of I, and therefore there can be no point in I where $u_1 = u_1' = 0$.

(*b*) u_2 is not identically zero. Then there exists a point p in I at which $u_2 = u_2' = 0$, but in whose every neighborhood lie points where $u_2 \neq 0$. Therefore by the above lemma $u_1(p) = 0$. We must therefore have $u_1'(p) \neq 0$. Accordingly there exists an ϵ such that throughout the interval $p < x < p + \epsilon$, and also throughout the interval $p > x > p - \epsilon$, u_1 does not vanish. Let us choose that one of these intervals in which lie points where $u_2 \neq 0$. By theorem I we have at every point of this interval, and therefore on account of the continuity of u_1 and u_2 also at p,

$$u_2 = cu_1,$$

where $c \neq 0$ as otherwise u_2 would vanish at every point of this interval. From this last equation we infer that

$$u_2'(p) = cu_1'(p).$$

Therefore since $u_2'(p) = 0$ and $c \neq 0$ we get $u_1'(p) = 0$. We are thus led to a contradiction, and therefore the case (*b*) cannot occur.

PEANO'S SECOND THEOREM. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskians of these functions taken $n-1$ at a time do not all vanish at any point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent.*

We will establish this theorem, and at the same time show that it covers no case which is not also covered by the fundamental theorem II, by proving that there must be one of the Wronskians $W_1, W_2, \cdots, W_n$ (to use the notation employed in the proof of theorem II) which does not vanish at any point of the interval I. Suppose each of these W's vanished in I. They cannot all vanish identically. Suppose that W_n is one of those which does not vanish identically. Then there exists a point p at which $W_n = 0$ but in whose every neighborhood lie points where $W_n \neq 0$.

Now by the reasoning used in the proof of theorem II we see that:

$$W_i' W_n - W_i W_n' \equiv 0 \qquad (i = 1, 2, \cdots, n-1).$$

Therefore, by our lemma, W_i vanishes at p $(i = 1, 2, \cdots, n-1)$ and this is contrary to hypothesis since W_n also vanishes at p.

§ 5. *A Theorem concerning Wronskians.*

I have now completed what I have to say on the subject of linear dependence. There remains however a theorem concerning Wronskians which I have found useful in the course of my work, although in the form which I have finally given to this paper no use has been made of it.

Before stating this theorem we will first establish two lemmas which we shall use in its proof.

Consider a matrix M of $n+m$ rows and n columns. Denote by D_i the n-rowed determinant obtained from M by striking out all of its $m+1$ last rows except the $(n-1+i)$-th row. Denote by M' the matrix obtained from M by striking out its last $m+1$ rows. Denote by Δ_i the $(n-1)$-rowed determinant obtained from M' by striking out its i-th column.

LEMMA I. *If* $D_1 = D_2 = \cdots = D_{m+1} = 0$, *and if* $\Delta_1, \Delta_2, \cdots, \Delta_n$ *are not all zero, then all the n-rowed determinants of* M *are zero.*

For denoting the element of M which stands in the i-th row and j-th column by a_{ij}, we have:

$$a_{i1}\Delta_1 - a_{i2}\Delta_2 + \cdots + (-1)^{n-1}a_{in}\Delta_n = 0 \qquad (i = 1, 2, \cdots, n+m),$$

and these form a set of $n+m$ homogeneous linear equations satisfied by the n Δ's which by hypothesis are not all zero.

LEMMA II. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* k *orders* $(k \geqq n)$, *while their Wronskian vanishes identically; then, except at points where the Wronskian of* $u_1, u_2, \cdots, u_{n-1}$ *is zero, all the n-rowed determinants of the matrix:*

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{vmatrix}$$

are zero.

We first prove this lemma in the case $k = n$. Here the determinant obtained from the above matrix by striking out the next to the last row is simply the derivative of the Wronskian of $u_1, u_2, \cdots, u_n$, and therefore also vanishes identically. The truth of our lemma thus follows at once from lemma I.

In order to prove the lemma in the general case we use the method of mathematical induction, and assume that the lemma has been proved when $k = k_1 - 1$. We wish to prove that the lemma also holds when $k = k_1$. Let us denote by M the above matrix when k has the value k_1, and by N the matrix obtained from M by striking out its last row; and let p be any point of I where the

Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. If then we can prove that the determinant:

$$D = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(k_1)} & u_2^{(k_1)} & \cdots & u_n^{(k_1)} \end{vmatrix},$$

vanishes at p, it will follow at once from lemma I that all the n-rowed determinants of M vanish at p, since this is true of all the n-rowed determinants of N. In order to prove that D vanishes at p let us consider the $(k_1 - n + 1)$-th derivative of the Wronskian of $u_1, u_2, \cdots, u_n$. This derivative will of course vanish identically. If we compute its value we find that it consists of the sum of a number of n-rowed determinants of which D is one while the others are all determinants of the matrix N, and therefore vanish at p. Thus we see that D vanishes at p, and our lemma is proved.

THEOREM VIII. *Let $u_1, u_2, \cdots, u_{n+1}$ be functions of x which at every point of I have continuous derivatives of the first n orders; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically the Wronskian of $u_1, u_2, \cdots, u_{n+1}$ will vanish identically.*

Denote by M the matrix obtained from the Wronskian

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_{n+1} \\ u_1' & u_2' & \cdots & u_{n+1}' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n)} & u_2^{(n)} & \cdots & u_{n+1}^{(n)} \end{vmatrix}$$

by striking out the last column. Then lemma II tells us that all the n-rowed determinants of M vanish except at the points where the Wronskians $\Delta_1, \Delta_2, \cdots, \Delta_n$ of the functions $u_1, u_2, \cdots, u_n$ taken $n-1$ at a time all vanish. Accordingly $W = 0$ except at these points. Let p be any such point of I. Our theorem will be proved if we can show that W vanishes at p.

We must distinguish two cases:

(*a*) $\Delta_1, \Delta_2, \cdots, \Delta_n$ do not all vanish identically throughout the neighborhood of p. There are therefore points in every neighborhood of p where the Δ's are not all zero, and where therefore $W = 0$; accordingly W must also vanish at p since it is a continuous function of x.*

(*b*) The Δ's all vanish identically throughout the neighborhood of p. Before

* This is the only point in the proof where use is made of the assumption that the nth derivatives of the u's are continuous. Would not the theorem still be true without this assumption?

proving in general that $W = 0$ for points of class (b) we will prove it in the simple case $n = 2$. Here we have two Δ's: $\Delta_1 = u_2$, $\Delta_2 = u_1$. Since these vanish identically in the neighborhood of p all the elements of the first two columns of W vanish at p, and therefore W vanishes at p.

We will now complete our proof by the method of mathematical induction by assuming that the theorem has been proved when we have less than $n + 1$ functions. Since each of the Δ's is the Wronskian of $n - 1$ of the functions u_1, u_2, $\cdots$, u_n it follows that throughout the neighborhood of p the Wronskian of any n of the $n + 1$ functions u_1, u_2, $\cdots$, u_{n+1} must vanish. Accordingly W also vanishes at p, as we see by expanding it according to the elements of its last row.

RAPALLO, ITALY, *December* 9, 1900.

AN ELEMENTARY PROOF OF A THEOREM OF STURM*

BY

MAXIME BÔCHER

We shall have to deal with two real solutions y_1 and y_2 of the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are throughout an interval $a \leqq x \leqq b$ real and continuous functions of the real variable x.† One of the most important of STURM'S results (Liouville's Journal, vol. 1 (1836), p. 106) is that, if y_1 and y_2 are linearly independent, between two successive roots of one lies one and only one root of the other.

The following generalization is (implicitly at least) contained in STURM'S paper, and from it what I have called STURM'S theorems of comparison for a single equation ‡ follow at once. It is my object in the present note to prove this theorem by a simple and elementary method which makes use only of a single property of y_1 and y_2, namely that a necessary and sufficient condition for their linear dependence is that $y_1y_2' - y_2y_1'$ should vanish at some point of the interval ab.§

The theorem in question may be stated as follows, and when it is so stated the method of proof is at once suggested:

Suppose that y_1 vanishes neither at a nor at b, and that y_2, if it does not vanish at a, satisfies the relation:

$$\frac{y_2'(a)}{y_2(a)} > \frac{y_1'(a)}{y_1(a)},$$

and, if it does not vanish at b, satisfies the relation:

---

* Presented to the Society February 23, 1901. Received for publication January 14, 1901.

† We may, if we wish, allow p and q to have a finite number of discontinuities in this interval, provided that in the neighborhood of each discontinuity p and q remain finite, or, if they become infinite, do so in such a way that $|p|$ and $|q|$ can be integrated up to these points. See Bulletin of the American Mathematical Society, March, 1899, p. 276.

‡ Bulletin of the American Mathematical Society, May, 1898, pp. 366–367.

§ In a slightly different form I applied this method four years ago to the simple case stated above. See Bulletin of the American Mathematical Society, March, 1897, p. 210.

$$\frac{y_2'(b)}{y_2(b)} < \frac{y_1'(b)}{y_1(b)},$$

*and finally that y_2 does not vanish in the interval $a < x < b$; then y_1 vanishes once and only once in this interval.**

We first prove that y_1 must vanish at least once. For if it did not the function:

$$f(x) = \frac{y_2'(x)}{y_2(x)} - \frac{y_1'(x)}{y_1(x)}$$

would be continuous throughout the interval $a < x < b$. Moreover if $y_2(a) \neq 0$ we have $f(a) > 0$, if $y_2(a) = 0$ we have $f(a) = +\infty$. In any case therefore $f(x)$ is positive in the neighborhood of a. In the same way we see that $f(x)$ is negative in the neighborhood of b. Accordingly $f(x)$ must vanish at some point between a and b. Therefore at this point:

$$y_1 y_2' - y_2 y_1' = 0,$$

and hence y_1 and y_2 are linearly dependent. This however is clearly not the case.

If now y_1 vanished more than once in the interval $a < x < b$, let $x = a$ and $x = \beta$ be two successive points in this interval where $y_1 = 0$. Then, interchanging y_1 and y_2, and applying the part of the theorem we have proved to the interval $a < x < \beta$, we see that y_2 vanishes at least once between a and β, and this is contrary to hypothesis.

RAPALLO, ITALY, *December* 31, 1900.

* If $y_2(a) = y_2(b) = 0$ this theorem reduces to the simple theorem stated above.

ON THE DETERMINATION OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY ALGEBRAIC CURVES*

BY

HENRY FREEMAN STECKER

Introduction.

BELTRAMI has shown † that surfaces of constant curvature can be built upon the plane in such a manner that the geodetic lines shall go over into straight lines, and that this result is true for no other surface. He considered this as the simplest case of building one surface upon another in such a manner that the geodetic lines of one surface shall go over into the geodetic lines of the other. The general question was later solved by DINI. ‡ It is an immediate consequence of BELTRAMI'S memoirs that surfaces of constant curvature are the only surfaces that can be built conformally upon the plane in such a manner that the geodetic lines shall go over into straight lines or arcs of circles. § This latter fact suggests a generalization of BELTRAMI'S problem different from the one which he had in mind, i. e., to so build a surface conformally upon the plane in such a manner that the geodetic lines shall go over into algebraic curves. It is proposed to consider that question.

§ 1.

We consider a doubly infinite system of algebraic curves in the plane:

$$F_3(x, y) + AF_2(x, y) + BF_1(x, y) = 0, \tag{1}$$

where A and B are the parameters of the system.

We think of the surface as given by

$$x_i = \phi_i(\mu, \nu) \qquad (i=1, 2, 3),$$

* Presented to the Society February 23, 1901. Received for publication November 20, 1900.

† Annali di Matematica, vol. 7, 1866.

‡ Annali di Matematica, ser. 2, vol. 3, 1870.

§ For an independent proof of this proposition see F. BUSSE, *Inaugural Dissertation*, Berlin, 1896.

where μ and ν are rectangular surface coördinates, i. e., such that the systems of curves $\mu = \text{const.}$, $\nu = \text{const.}$, intersect orthogonally.

Let the relation between the surface and the plane be given by

$$x = \psi_1(\mu, \nu), \quad y = \psi_2(\mu, \nu).$$

Then we write $f_i(\mu, \nu)$ for $F_i(\psi_1, \psi_2)$ and equation (1) becomes

$$f_3(\mu, \nu) + Af_2(\mu, \nu) + Bf_1(\mu, \nu) = 0, \tag{2}$$

and from this we find:

$$f'_3(\mu, \nu) + Af'_2(\mu, \nu) + Bf'_1(\mu, \nu) = 0, \tag{3}$$

$$f''_3(\mu, \nu) + Af''_2(\mu, \nu) + Bf''_1(\mu, \nu) = 0, \tag{4}$$

where

$$f'_i(\mu, \nu) = \frac{\partial f_i(\mu, \nu)}{\partial \mu} d\mu + \frac{\partial f_i(\mu, \nu)}{\partial \nu} d\nu,$$

$$f''_i(\mu, \nu) = \frac{\partial^2 f_i(\mu, \nu)}{\partial \mu^2} d\mu^2 + 2 \frac{\partial^2 f_i(\mu, \nu)}{\partial \mu \partial \nu} d\mu d\nu + \frac{\partial^2 f_i(\mu, \nu)}{\partial \nu^2} d\nu^2 + \frac{\partial f_i(\mu, \nu)}{\partial \mu} d^2\mu + \frac{\partial f_i(\mu, \nu)}{\partial \nu} d^2\nu.$$

From (2), (3) and (4) we obtain

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix} = 0. \tag{5}$$

We write

$$n_{i1} = \frac{\partial f_i}{\partial \mu}, \quad n_{i2} = \frac{\partial f_i}{\partial \nu}; \quad m_{i1} = \frac{\partial^2 f_i}{\partial \mu^2}, \quad m_{i2} = \frac{\partial^2 f_i}{\partial \nu^2}, \quad m_{i3} = \frac{\partial^2 f_i}{\partial \mu \partial \nu}.$$

Making use of these abbreviations, substituting for f'_i and f''_i their values, and expanding in terms of the first row, we have from equation (5):

$$f_1 \begin{vmatrix} n_{21}d\mu + n_{22}d\nu & m_{21}d\mu^2 + m_{22}d\nu^2 + 2m_{23}d\mu d\nu + n_{21}d^2\mu + n_{22}d^2\nu \\ n_{31}d\mu + n_{32}d\nu & m_{31}d\mu^2 + m_{32}d\nu^2 + 2m_{33}d\mu d\nu + n_{31}d^2\mu + n_{32}d^2\nu \end{vmatrix}$$

$$-f_2 \begin{vmatrix} n_{11}d\mu + n_{12}d\nu & m_{11}d\mu^2 + m_{12}d\nu^2 + 2m_{13}d\mu d\nu + n_{11}d^2\mu + n_{12}d^2\nu \\ n_{31}d\mu + n_{32}d\nu & m_{31}d\mu^2 + m_{32}d\nu^2 + 2m_{33}d\mu d\nu + n_{31}d^2\mu + n_{32}d^2\nu \end{vmatrix}$$

$$+f_3 \begin{vmatrix} n_{11}d\mu + n_{12}d\nu & m_{11}d\mu^2 + m_{12}d\nu^2 + 2m_{13}d\mu d\nu + n_{11}d^2\mu + n_{12}d^2\nu \\ n_{21}d\mu + n_{22}d\nu & m_{21}d\mu^2 + m_{22}d\nu^2 + 2m_{23}d\mu d\nu + n_{21}d^2\mu + n_{22}d^2\nu \end{vmatrix} = 0,$$

or

$$f_1\theta(23) - f_2\theta(13) + f_3\theta(12) = 0,$$

where

$$\theta(23) = \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} d\mu^3 + \begin{vmatrix} n_{21}m_{22} \\ n_{31}m_{32} \end{vmatrix} d\mu d\nu^2 + 2 \begin{vmatrix} n_{21}m_{23} \\ n_{31}m_{33} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{21}n_{22} \\ n_{31}n_{32} \end{vmatrix} d\mu d^2\nu$$

$$+ \begin{vmatrix} n_{22}m_{21} \\ n_{32}m_{31} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{22}m_{22} \\ n_{32}m_{32} \end{vmatrix} d\nu^3 + 2 \begin{vmatrix} n_{22}m_{23} \\ n_{32}m_{33} \end{vmatrix} d\mu d\nu^2 + \begin{vmatrix} n_{22}n_{21} \\ n_{32}n_{31} \end{vmatrix} d\nu d^2\mu ,$$

with corresponding expressions for $\theta(13)$, $\theta(12)$. Adding like terms, we have for terms in $d\mu^3$:

$$\left\{ f_1(\mu, \nu) \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} - f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{31}m_{31} \end{vmatrix} + f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{21}m_{21} \end{vmatrix} \right\} d\mu^3 ,$$

or

$$\begin{vmatrix} f_1(\mu, \nu) & n_{11} & m_{11} \\ f_2(\mu, \nu) & n_{21} & m_{21} \\ f_3(\mu, \nu) & n_{31} & m_{31} \end{vmatrix} d\mu^3 . \tag{6}$$

And we easily see that we have like determinants for the coefficients of the other terms. Indeed if we write $[11]$ for the coefficient of $d\mu^3$, our differential equation is:

$$[11]d\mu^3 + [22]d\nu^3 + \{2[13] + [21]\}d\mu^2 d\nu + \{2[23] + [12]\}d\mu d\nu^2 + \Delta(d\mu d^2\nu - d\nu d^2\mu) = 0 , \tag{7}$$

where Δ is the determinant $|f_1 n_{21} n_{32}|$.

For shortness we write equation (7) in the form:

$$a_1 d\mu^3 + a_4 d\nu^3 + a_2 d\mu^2 d\nu + a_3 d\mu d\nu^2 + a_5(d\mu d^2\nu - d\nu d^2\mu) = 0 . \tag{8}$$

§ 2.

The geodetic lines of our surface are given by the differential equation:

$$\tfrac{1}{2} G \frac{\partial G}{\partial \mu} d\nu^3 - \tfrac{1}{2} E \frac{\partial E}{\partial \nu} d\mu^3 + \left[E \frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} \right] d\mu^2 d\nu$$

$$+ \left[G \frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} \right] d\mu d\nu^2 + EG(d\mu d^2\nu - d\nu d^2\mu) = 0 ; \tag{9}$$

we are to compare this with equation (8). This comparison gives the following system of partial differential equations:

$$E \frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} = \lambda a_2 , \quad G \frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} = -\lambda a_3 ,$$

$$\tfrac{1}{2} G \frac{\partial G}{\partial \mu} = \lambda a_4 , \quad \tfrac{1}{2} E \frac{\partial E}{\partial \nu} = -\lambda a_1 , \quad EG = \lambda a_5 . \tag{10}$$

From this system we eliminate λ by means of the last equation and obtain:

$$E\frac{\partial G}{\partial \mu} - \tfrac{1}{2}G\frac{\partial E}{\partial \mu} = EG\frac{a_2}{a_5}, \quad G\frac{\partial E}{\partial \nu} - \tfrac{1}{2}E\frac{\partial G}{\partial \nu} = -EG\frac{a_3}{a_5},$$

(11)

$$\tfrac{1}{2}G\frac{\partial G}{\partial \mu} = EG\frac{a_4}{a_5}, \quad \tfrac{1}{2}E\frac{\partial E}{\partial \nu} = -EG\frac{a_1}{a_5}.$$

From these we have:

(12)

1. $$\frac{\partial \log G}{\partial \mu} - \tfrac{1}{2}\frac{\partial \log E}{\partial \mu} = \frac{a_2}{a_5},$$

2. $$\frac{\partial \log E}{\partial \nu} - \tfrac{1}{2}\frac{\partial \log G}{\partial \nu} = -\frac{a_3}{a_5},$$

3. $$\frac{\partial \log E}{\partial \nu} = -2\frac{a_1}{a_5}\frac{G}{E},$$

4. $$\frac{\partial \log G}{\partial \mu} = 2\frac{a_4}{a_5}\frac{E}{G}.$$

Integrating 1 and 2 we have:

(13)

1. $$\log G - \tfrac{1}{2}\log E = \log F_1 + \log V_1,$$

2. $$\log E - \tfrac{1}{2}\log G = -\log F_2 + \log U_1,$$

where $F_1 = e^{\int \frac{a_2}{a_5} d\mu}$ and $F_2 = e^{\int \frac{a_3}{a_5} d\mu}$; U_1 is a function of μ only and V_1 of ν only. Then (13) gives

(14)

1. $$\frac{G}{E^{\frac{1}{2}}} = F_1 V_1,$$

2. $$\frac{E}{G^{\frac{1}{2}}} = F_2^{-1} U_1.$$

Squaring each of these equations and multiplying it by the other we find

(15)

1. $$E = U_1^{\frac{4}{3}} V_1^{\frac{2}{3}} F_1^{\frac{2}{3}} F_2^{-\frac{4}{3}},$$

2. $$G = U_1^{\frac{2}{3}} V_1^{\frac{4}{3}} F_1^{\frac{4}{3}} F_2^{-\frac{2}{3}}.$$

From these last two equations we have the following values:

1. $$\frac{E}{G} = U_1^{\frac{2}{3}} V_1^{-\frac{2}{3}} F_1^{-\frac{2}{3}} F^{-}$$

(16)

2. $$\frac{\partial \log G}{\partial \mu} = \tfrac{2}{3}\frac{1}{U_1}\frac{dU_1}{d\mu} + \tfrac{4}{3}\frac{a_2}{a_5} - \tfrac{2}{3}\frac{\partial \log F_2}{\partial \mu},$$

3. $$\frac{\partial \log E}{\partial \nu} = \tfrac{2}{3}\frac{1}{V_1}\frac{dV_1}{\partial \nu} - \tfrac{4}{3}\frac{a_3}{a_3} + \tfrac{2}{3}\frac{\partial \log F_1}{\partial \nu}.$$

We substitute these values in equations (3) and (4) of the system (12) and obtain:

$$\frac{dU_1}{d\mu} + \left[2\frac{a_2}{a_5} - \frac{\partial \log F_2}{\partial \mu} \right] U_1 = 3\frac{a_4}{a_5} V_1^{-\frac{2}{3}} F_1^{-\frac{2}{3}} F_2^{-\frac{2}{3}} U_1^{\frac{5}{3}}, \tag{17}$$

$$\frac{dV_1}{d\nu} + \left[-2\cdot\frac{a_3}{a_5} + \frac{\partial \log F_1}{\partial \nu} \right] V_1 = -3\frac{a_1}{a_5} U_1^{-\frac{2}{3}} F_1^{\frac{2}{3}} F_2^{\frac{2}{3}} V_1^{\frac{5}{3}}. \tag{18}$$

The integrals of these are:

$$1.\quad \frac{V_1^{\frac{2}{3}}}{U_1^{\frac{2}{3}}} = 2F_1^{\frac{4}{3}} F_2^{-\frac{2}{3}} (F_3 + V_2), \tag{19}$$

$$2.\quad \frac{U_1^{\frac{2}{3}}}{V_1^{\frac{2}{3}}} = 2F_1^{\frac{2}{3}} F_2^{-\frac{4}{3}} (F_4 + U_2),$$

where

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^2 d\nu$$

and U_2 and V_2 are functions of μ only and of ν only respectively.

These equations are *sufficient to say whether or not a solution exists and to find such solutions if they do exist.* It is proposed to illustrate this by an example (pp. 156–159) before going farther with the general theory.

EXAMPLE. We take $f_3 = f_3(\nu)$, a function of ν only; also

$$f_2(\mu, \nu) = \mu^n \nu^m, \quad f(\mu, \nu) = \mu^n \nu^\beta.$$

Then after some calculation we find the following values:

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= \mu^{2n-2}\nu^{\beta+m-1} n(n+1)(\beta - m) f_3(\nu), \\
a_3 &= \mu^{2n-1}\nu^{\beta+m-2} \left[n(\beta - m)(m + \beta - 1) f_3(\nu) - 2n(\beta - m)\nu \frac{df_3(\nu)}{d\nu} \right], \\
a_4 &= \mu^{2n}\nu^{\beta+m-3} (\beta - m) \left[\beta m f_3(\nu) - (\beta + m - 1)\nu \frac{df_3(\nu)}{d\nu} + \nu^2 \frac{d^2 f_3(\nu)}{d\nu^2} \right], \\
a_5 &= \mu^{2n-1}\nu^{\beta+m-1} n(\beta - m) f_3(\nu).
\end{aligned} \tag{20}$$

From these we find:

$$\begin{aligned}
\frac{a_2}{a_5} &= \frac{n+1}{\mu}, \\
\frac{a_3}{a_5} &= \frac{m + \beta - 1}{\nu} - 2\frac{1}{f_3(\nu)} \frac{df_3(\nu)}{d\nu}, \\
\frac{a_1}{a_5} &= 0, \\
\frac{a_4}{a_5} &= \frac{\mu}{n} \left[\frac{\beta m}{\nu^2} - (\beta + m - 1)\frac{1}{\nu}\frac{1}{f_3(\nu)}\frac{df_3(\nu)}{d\nu} + \frac{1}{f_3(\nu)}\frac{d^2 f_3(\nu)}{d\nu^2} \right] = \frac{\mu}{n}\theta(\nu).
\end{aligned} \tag{21}$$

Then we have:

$$\int \frac{a_2}{a_5}\, d\mu = \int (n+1)\frac{d\mu}{\mu} = \log \mu^{n+1},$$

$$\int \frac{a_3}{a_5}\, d\nu = \int (m+\beta-1)\frac{d\nu}{\nu} - 2\int d\log f_3(\nu) d\nu = \log \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}.$$

Therefore

$$F_1 = \mu^{n+1}, \quad F_2 = \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}, \quad F_4 = 0,$$

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\theta(\nu)\mu^{-2(n+1)+1} d\mu = \frac{\theta(\nu)}{2n^2}\mu^{-2n}.$$

Introducing these values into (19) we have:

(22)

1. $$V_1^{\frac{2}{3}} U_1^{-\frac{2}{3}} = 2\mu^{\frac{4}{3}(n+1)}\nu^{-\frac{2}{3}(m+\beta-1)} f_3^{\frac{4}{3}}(\nu)\left[\frac{\theta(\nu)\mu^{-2n} + 2n^2 V_2}{2n^2}\right]$$

2. $$V_1^{-\frac{2}{3}} U_1^{\frac{2}{3}} = 2\mu^{\frac{2}{3}(n+1)}\nu^{-\frac{4}{3}(m+\beta-1)} f_3^{\frac{8}{3}}(\nu) U_2.$$

This requires either

(a) $$V_2 = K\theta(\nu) \qquad (K = \text{constant}),$$

or

(b) $$\theta(\nu) = 0.$$

Case (a). If $V_2 = K\theta(\nu)$, then we find:

$$U_1^{\frac{2}{3}} = \lambda\mu^{-\frac{4}{3}(n+1)}[\mu^{-2n} + 2n^2K]^{-1},$$

$$U_1^{\frac{2}{3}} = a\mu^{\frac{2}{3}(n+1)} U_2,$$

which require that

$$U_2 = \frac{\lambda}{a}\mu^{-2(n+1)}[\mu^{-2n} + 2n^2K]^{-1};$$

therefore

$$U_1 = \frac{\lambda^{\frac{3}{2}}\mu^{n-2}}{[2n^2K\mu^{2n} + 1]^{\frac{3}{2}}}.$$

And for V_1 we find:

$$V_1^{\frac{2}{3}} = \lambda'\nu^{-\frac{2}{3}(m+\beta-1)} f_3^{\frac{4}{3}}(\nu)\theta(\nu),$$

$$V_1^{\frac{2}{3}} = a'\nu^{\frac{4}{3}(m+\beta-1)} f_3^{-\frac{8}{3}}(\nu).$$

These require that:

$$\theta(\nu) = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)};$$

Taking $a' = \lambda'$, as we may do, and restoring the value of $\theta(\nu)$ we have the following differential equation to determine $f_3(\nu)$:

$$(23)\qquad \frac{d^2f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \beta m\nu^{-2}f_3(\nu) = \nu^{2m+\beta-1}f_3^{-3}(\nu).$$

Before solving this differential equation we consider case (b). If $\theta(\nu) = 0$ we find:

$$U_1^{\frac{2}{3}} = \lambda\mu^{-\frac{4}{3}(n+1)},$$

$$U_1^{\frac{2}{3}} = a\mu^{\frac{2}{3}(n+1)}U_2,$$

giving

$$U_2 = \frac{\lambda}{\mu}\mu^{-2(n+1)};$$

also

$$V_1^{\frac{2}{3}} = \lambda'\nu^{-\frac{2}{3}(m+\beta-1)}f_3^{\frac{4}{3}}(\nu)\,V_2,$$

$$V_1^{\frac{2}{3}} = a'\nu^{\frac{4}{3}(m+\beta-1)}f_1^{-\frac{6}{3}}(\nu),$$

giving

$$V_2 = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)}.$$

And $f_3(\nu)$ is given by

$$(24)\qquad \frac{d^2f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \frac{\beta m}{\nu^2}f_3(\nu) = 0.$$

It remains then to see if $f_3(\nu)$ can be determined algebraically.

Since equation (24) is (23) with its right hand number put equal to zero we can consider them together.

We put

$$\nu = e^{\vartheta},\quad f_3(\nu) = e^{\frac{(m+\beta)\vartheta}{2}}z,\quad \theta = \frac{dz}{d\vartheta}.$$

After some reduction equations (23) and (24) reduce to the forms:

$$(25)\qquad \theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4}z = z^{-3},$$

$$(26)\qquad \theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4}z = 0.$$

We consider the latter and simpler case first.

It gives:

$$\theta = \tfrac{1}{2}\sqrt{(m-\beta)^2z^2 + 4c}.$$

Hence

$$\vartheta = \int\frac{2dz}{\sqrt{(m-\beta)^2z^2 + 4c}}.$$

Therefore

$$\vartheta = \log\left[\frac{2(m-\beta)^2z + 2(m-\beta)\sqrt{(m-\beta)^2z^2 + 4c_1}}{2(m-\beta)^2z - 2(m-\beta)\sqrt{(m-\beta)^2z^2 + 4c_1}}\right]^{\frac{1}{m-\beta}} + \log c_2.$$

Remembering that $\vartheta = \log \nu$ and that $z = f_3(\nu)/\nu^{\frac{m+\beta}{2}}$, we find after some reduction:

$$f_3^2(\nu) = \frac{4c_1\nu^{m+\beta}}{(m-\beta)^2\{(\nu^{m-\beta}-c_2)^2-1\}};$$

and for the equation of the curve:

$$(m-\beta)^2\{(\nu^{m-\beta}-c_3)^2-1\}(A\mu^n\nu^m + B\mu^n\nu^\beta)^2 - 4c_1\nu^{m+\beta} = 0.$$

Hence

$$V_1 = K_1[\nu^{2(m-\beta)-1} - 2c_2\nu^{m+\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

$$U_1 = K_2\mu^{-2(n+1)} \qquad (K_1 \text{ and } K_2 \text{ constant});$$

these give:

$$E = p\mu^{-2(n+1)},$$

$$G = q[\nu^{2(m-\beta)-1} - 2c_2\nu^{m-\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

where p and q are constant.

In regard to the other integral:

$$\vartheta = \int \frac{2zdz}{2cz + \sqrt{(m-\beta)^2z^4 - 4}},$$

it appears that in order to keep the exponents rational and the coefficients real it is necessary to take $c = 0$.

Then we find finally:

$$f_3^2(\nu) = \frac{4K^2\nu^{2\beta} + \nu^{2m}}{2K(m-\beta)},$$

where K is constant, and the curve is given by the equation:

$$4K^2\nu^{2\beta} + \nu^{2m} = 2K(m-\beta)(A\mu^n\nu^m + B\mu^n\nu^\beta)^2.$$

To revert to the general theory, we may say that if a solution exists, the right hand member of equation 1 of (19) must be of the form

$$\frac{\Theta(\nu)}{\Theta(\mu)},$$

and then

$$V_1^{\frac{3}{3}} = K\Theta(\nu), \quad U_1^{\frac{3}{3}} = K\Theta(\mu).$$

The functions $\psi_1(\mu, \nu)$ and $\psi_2(\mu, \nu)$ are given by the equations:

$$\frac{F_1(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_1(\mu, \nu)}{f_3(\mu, \nu)}, \qquad \frac{F_2(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_2(\mu, \nu)}{f_3(\mu, \nu)}.$$

Since we desire a conformal representation we shall take

$$\psi_1 = \mu, \quad \psi_2 = \nu.$$

But it is plain that by such a choice certain conditions are imposed by equations (19) and we proceed to consider those conditions.

§ 3.

Calculation of V_2 and U_2.

If we represent the right hand members (19) by R and S respectively, then we must have:

1. $$RS = 1\,,$$

(27) 2. $$\frac{\partial^2 \log R}{\partial\mu\partial\nu} = 0\,,$$

3. $$\frac{\partial^2 \log S}{\partial\mu\partial\nu} = 0\,.$$

From the second of these we find:

$$(28)\qquad \tfrac{4}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} - \tfrac{2}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial^2 \log (F_3 + V_2)}{\partial\mu\partial\nu} = 0\,.$$

Considering the last term of this equation we find

$$\frac{\partial^2 \log (F_3 + V_2)}{\partial\mu\partial\nu} = \frac{\partial}{\partial\nu}\frac{\partial F_3/\partial\nu}{F_3 + V_2} = \frac{\partial}{\partial\nu}\left\{-4\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2)\right\}$$

$$= -4F_2^{-2}(F_4 + U_2)\frac{\partial}{\partial\nu}\frac{a_4}{a_5} + 8\frac{a_3}{a_5}\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2) - 4\frac{a_1}{a_5}\frac{a_4}{a_5}\,,$$

since

$$\frac{\partial F_3}{\partial\mu} = -\frac{a_4}{a_5}F_1^{-2} \quad \text{and} \quad 4F_1^2F_2^{-2}(F_3 + V_2)(F_4 + U_2) = 1\,.$$

Whence from (28) we find:

$$(29)\qquad 2F_2^{-2}(F_4 + U_2) = \frac{\tfrac{2}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \tfrac{4}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_3}{a_5}\frac{a_4}{a_5} - 2\frac{\partial}{\partial\nu}\frac{a_4}{a_5}}\,;$$

and from 3 of (27) we find in like manner:

$$(30)\qquad 2F_1^2(F_3 + V_2) = \frac{\tfrac{4}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \tfrac{2}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_1}{a_5}\frac{a_2}{a_5} + 2\frac{\partial}{\partial\mu}\frac{a_1}{a_5}}\,.$$

If we represent the right hand members of (29) and (30) by Δ_1 and Δ_2 respectively, then equations (19) become:

$$(31)\qquad \begin{aligned} \frac{V_1^{\frac{2}{3}}}{U_1^{\frac{2}{3}}} &= F_1^{-\frac{2}{3}} F_2^{-\frac{2}{3}} \Delta_1 , \\ \frac{U_1^{\frac{2}{3}}}{V_1^{\frac{2}{3}}} &= F_1^{\frac{2}{3}} F_2^{\frac{2}{3}} \Delta_2 . \end{aligned}$$

§ 4.

Partial differential equations satisfied by $f_i(\mu, \nu)$.

As a first condition we have

$$\Delta_1 \Delta_2 = 1 .$$

Differentiating (29) with respect to ν, and using (19) we have

$$2 \frac{a_1}{a_5} F_2^2 = F_2^2 \frac{\partial \Delta_1}{\partial \nu} + 2 F_2^2 \frac{a_3}{a_5} \Delta_1 ,$$

or

$$\frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 = 2 \frac{a_1}{a_5} ;$$

and in like manner from (30) we find

$$\frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 = - 2 \frac{a_4}{a_5} .$$

Hence we have the system:

$$(32)\qquad \begin{aligned} \Delta_1 \Delta_2 &= 1 , \\ \frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 &= 2 \frac{a_1}{a_5} , \\ \frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 &= - 2 \frac{a_4}{a_5} . \end{aligned}$$

These are the necessary, and, as is easily seen, sufficient conditions to impose upon $f_i(\mu, \nu)$ in order that a solution may exist for the rectangular coördinate system (μ, ν).

§ 5.

We consider next a somewhat general solution of our problem. We have written

$$\int \frac{a_2}{a_5} d\mu = \log F_1 , \quad \int \frac{a_3}{a_5} d\nu = \log F_2 .$$

We consider one case where these integrals actually yield logarithms, i. e., $\partial a_5/\partial \mu = \lambda a_2$ and $\partial a_5/\partial \nu = a a_3$. To simplify the expressions for a_i we divide equation

(2) through by $f_2(\mu, \nu)$ and then consider $f_3' = 1$, $f_2' = f_2/f_3$, $f_3' = f_1/f_3$. This amounts to putting $f_3 = 1$ in our formulas. We find:

$$a_1 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_4 = \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}, \quad a_5 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial f_3}{\partial \nu} \end{vmatrix},$$

$$a_2 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_3 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}.$$

The case to be considered is found by taking $\partial^2 f_2/\partial\mu\partial\nu = 0$ and $\partial^2 f_3/\partial\mu\partial\nu = 0$, that is, $f_2 = \phi_1(\mu) + \phi_2(\nu)$ and $f_3 = \psi_1(\mu) + \psi_2(\nu)$. Then we shall have:

$$\begin{aligned} a_5 &= \phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu), \\ a_1 &= \phi_1'(\mu)\,\psi_1''(\mu) - \psi_1'(\mu)\,\phi_1''(\mu), \\ a_4 &= \phi_2'(\nu)\,\psi_2''(\nu) - \psi_2'(\nu)\,\phi_2''(\nu), \\ a_2 &= \phi_2'(\nu)\,\psi_1''(\mu) - \psi_2'(\nu)\,\phi_1''(\mu), \\ a_3 &= \phi_1'(\mu)\,\psi_2''(\nu) - \psi_1'(\mu)\,\phi_2''(\nu). \end{aligned}$$

Hence

$$\int \frac{a_2}{a_5}\,d\mu = -\log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\}, \tag{40}$$

$$\int \frac{a_3}{a_5}\,d\nu = \log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\}, \tag{41}$$

Therefore

$$F_1 = 1 \div \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = \frac{1}{a_5}, \qquad F_2 = \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = a_5.$$

Hence

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2}\,d\mu = -a_4 \int a_5\,d\mu = -\begin{vmatrix} \phi_1(\mu) & \phi_2'(\nu) \\ \psi_1(\mu) & \psi_2'(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix},$$

$$F_4 = \int \frac{a_1}{a_5} F_2^2\,d\nu = a_1 \int a_5\,d\nu = \begin{vmatrix} \phi_1'(\mu) & \phi_2(\nu) \\ \psi_1'(\mu) & \psi_2(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_1'(\mu) & \phi_1''(\mu) \\ \psi_1'(\mu) & \psi_1''(\mu) \end{vmatrix}.$$

In order to separate the variables it is necessary that either

$$\text{(a)} \quad \phi_1(\mu) = \psi_1(\mu) \quad \text{or} \quad \text{(b)} \quad \phi_2(\nu) = \psi_2(\nu),$$

and since the equations and expressions are symmetrical in μ and ν it will only be necessary to consider case (a) and afterwards interchange μ and ν in the result.

First of all we have:

$$F_4 = 0\,,\quad F_1 = \frac{1}{\phi'(\mu)}\,\frac{1}{\psi_2'(\nu) - \phi_2'(\nu)}\,,\quad F_2 = \phi'(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\},$$

$$F_3 = -\,\phi(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\}\begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}.$$

Then from (19) we have:

$$\frac{\frac{1}{2}\,\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}{\dfrac{V_2}{\{\psi_2'(\nu) - \phi_2'(\nu)\}} - \phi(\mu)\begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}} = \frac{2\,U_2}{\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}\,. \tag{42}$$

As V_1 must vanish and $U_2 = \phi_1'^4(\mu)/4\phi(\mu)$, we find

$$U_1 = \frac{1}{\sqrt{8}}\,\frac{\phi'^3(\mu)}{\phi^{\frac{3}{2}}(\mu)}\,.$$

Also

$$\begin{vmatrix} \phi_2''(\nu) & \phi_2'(\nu) \\ \psi_2''(\nu) & \psi_2'(\nu) \end{vmatrix} = \{\psi_2'(\nu) - \phi_2'(\nu)\}^3,$$

or, if we write x for $\psi_2'(\nu)$ and y for $\phi_2'(\nu)$,

$$x\frac{dy}{d\nu} - y\frac{dx}{d\nu} = (x - y)^3.$$

Put $x - y = q$ and our differential equation is:

$$q\frac{dx}{d\nu} - x\frac{dq}{d\nu} = q^3,$$

or

$$\frac{d}{d\nu}\frac{x}{q} = q\,.$$

Therefore

$$\frac{x}{q} = \int q\,d\nu = \psi_2(\nu) - \phi_2(\nu) + c\,;$$

hence

$$\frac{\psi_2'(\nu)}{\psi_2'(\nu) - \phi_2'(\nu)} = \psi_2(\nu) - \phi_2(\nu) + c_1\,,$$

and integrating this we have:

$$\psi_2(\nu) = \tfrac{1}{2}\,[\psi_2(\nu) - \phi_2(\nu)]^2 + c_1[\psi_2(\nu) + \phi_2(\nu)] + c_2\,.$$

Remembering that $\phi_2(\nu) = f_1(\nu)/f_3(\nu)$ and $\phi(\mu) = f_1(\mu)/f_3(\mu)$ we have, after some reduction:

$$B^2[K_2 f_1(\nu) + K_3 f_3(\nu)] f_3(\nu) f_3^2(\mu) = \{(K_1 B + 1) f_3(\nu) f_3(\mu) + (A + B)[f_1(\mu) f_3(\nu) + f_3(\mu) f_1(\nu)]\}^2, \tag{43}$$

for the equation of our curve. Here the f_i of μ and of ν are *any algebraic functions whatever*. As mentioned above we find a solution for case (b) by interchanging μ and ν in (43).

§ 6.

Conditions for the equality of E and G.

The work thus far is for $F = 0$, i. e., μ and ν are rectangular surface coördinates. It remains to consider the case $E = G$.

From our system of partial differential equations (10) we find that a_1 must be equal to a_3 and a_2 to a_4. It remains to consider the system of equations (32) which were the necessary and sufficient conditions to impose on the $f_i(\mu, \nu)$. First we consider the values of F_3 and F_4. We recall that

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^{-2} d\nu,$$

$$\log F_1 = \int \frac{a_2}{a_5} d\nu, \quad \log F_2 = \int \frac{a_3}{a_5} d\nu.$$

Hence also

$$\frac{\partial F_1}{\partial \mu} = \frac{a_2}{a_5} F_1, \quad \frac{\partial F_2}{\partial \nu} = \frac{a_3}{a_5} F_2.$$

We have then

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\frac{a_2}{a_5} F_1^{-2} d\mu = \int -F_1^{-3} \frac{\partial F_1}{\partial \mu} d\nu = \tfrac{1}{2} F_1^{-2}.$$

Also

$$F_4 = \int \frac{a_3}{a_5} F_2^2 d\nu = \int F_2 \frac{\partial F_2}{\partial \nu} d\nu = \tfrac{1}{2} F_2^2.$$

Hence the right hand members of (29) and (30) become respectively,

$$2F_2^{-2}(\tfrac{1}{2} F_2^2 + U_2) = 1 + 2F_2^{-2} U_2 = \Delta_1 = \frac{\frac{2}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} - \frac{4}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} - 2\frac{\partial}{\partial \nu}\frac{a_2}{a_5}},$$

$$2F_1^2(\tfrac{1}{2}F_1^{-2} + V_2) = 1 + 2F_1^2 V_2 = \Delta_2 = \frac{\frac{4}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \frac{2}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} + 2\frac{\partial}{\partial\mu}\frac{a_3}{a_5}}.$$

We consider the first of equations (3), i. e., $\Delta_1\Delta_2 = 1$.

We write

$$\Delta_1 = \frac{\frac{2}{3}a - \frac{4}{3}b + c}{c - 2b} = 1 + \frac{\frac{2}{3}(a+b)}{c-2b},$$

and also

$$\Delta_2 = \frac{\frac{4}{3}a - \frac{2}{3}b + c}{c + 2a} = 1 - \frac{\frac{2}{3}(a+b)}{c+2a}.$$

Therefore

$$\Delta_1\Delta_2 = 1 + \frac{\frac{2}{3}(a+b)}{c-2b} - \frac{\frac{2}{3}(a+b)}{c+2a} - \frac{\frac{4}{9}(a+b)^2}{(c+2a)(c-2b)}$$

$$= 1 + \frac{\frac{2}{3}(a+b)}{(c+2a)(c-2b)}\left[c + 2a - c + 2b - \tfrac{2}{3}(a+b)\right]$$

$$= 1 + \frac{\frac{8}{9}(a+b)}{(c+2a)(c-2b)}.$$

Hence we must have $a + b = 0$ and therefore

$$\frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} = 0.$$

Hence $\Delta_1 = 1$ and $\Delta_2 = 1$. Then the other two equations of (32) are satisfied and we may conclude that

$$(44)\qquad \begin{aligned} a_1 &= a_3, \\ a_2 &= a_4, \\ \frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} &= 0, \end{aligned}$$

are the necessary and sufficient conditions that $E = G$.

The next question will be the study of these surfaces, their existence and properties. In particular I have already finished part of the work for the curve $y^2 = ax^3 + bx^2 + cx + d$, i. e., for surfaces whose geodetic lines go into cubics upon the plane by a conformal transformation. The straight line and circle lead to the surfaces of constant curvature with their interesting properties. Here is a much wider and more interesting field.

GÖTTINGEN, *November* 3, 1900.

ON THE EXISTENCE OF A MINIMUM OF THE INTEGRAL

$$\int_{x_0}^{x_1} F(x, y, y')\, dx$$

WHEN x_0 AND x_1 ARE CONJUGATE POINTS, AND THE GEODESICS ON AN ELLIPSOID OF REVOLUTION: A REVISION OF A THEOREM OF KNESER'S*

BY

W. F. OSGOOD

In a paper entitled *Zur Variationsrechnung* KNESER enunciates the theorem † that the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

ceases to be a minimum, not only when the interval (x_0, x_1) contains a point x' conjugate ‡ to x_0, but when x_1 coincides with x'. This theorem is true in general, but not in all cases, and it is the object of this paper to correct and extend the theorem and to give a complete solution of the problem that KNESER proposed to himself. The geodesics on an ellipsoid of revolution afford an example of a case in which the theorem as enunciated by KNESER is not correct.

The methods and results of this paper admit of extension to multiple integrals.

The contents are as follows: §§ 1, 2 are introductory; § 3 gives an account of KNESER'S memoir and obtains his results in a simple manner; §§ 4–6 contain the new material of the paper.

§ 1. *Necessary Conditions.*

A necessary condition that the integral I be made a minimum by the function y is that y satisfy LAGRANGE'S equation

* Presented to the Society December 28, 1900. Received for publication January 30, 1901.

† Mathematische Annalen, vol. 50 (1897), p. 50. Cf. also § 6 of this paper. This theorem had however been stated and proved for the general case twenty years earlier by ERDMANN: Zeitschrift für Mathematik und Physik, vol. 22 (1877), p. 324, who showed that, when $\delta y = \varepsilon \phi_\gamma(x, \gamma_0)$ (in the notation of the present paper) the second variation of the integral I vanishes while

$$\delta^3 I = -\varepsilon^3 (F_{y'y'})_{x=x'} \phi_\gamma'(x', \gamma_0)\, \phi_{\gamma\gamma}(x', \gamma_0),$$

and this is not in general zero. For this reference I am indebted to Professor BOLZA.

‡ The meaning of the word *conjugate* will appear below, § 3.

$$F_y - \frac{dF_{y'}}{dx} = 0\,, \qquad \left(F_y = \frac{\partial F}{\partial y}\,,\ \text{etc.}\right) \tag{1}$$

or

$$F_y - F_{xy'} - y' F_{yy'} - y'' F_{y'y'} = 0\,,$$

and furthermore fulfill the boundary conditions

$$y(x_0) = y_0\,, \quad y(x_1) = y_1\,,$$

where y_0, y_1 are fixed quantities.* Here the function $F(x, y, p)$, regarded as a function of the three independent variables (x, y, p), is to be continuous, together with its partial derivatives of the first and second orders, throughout a region R:

$$A_1 \leqq x \leqq B_1\,, \quad A_2 < y < B_2\,, \quad A_3 < p < B_3\,,$$

and $A_1 \leqq x_0 < x_1 \leqq B_1$. For simplicity we will assume $A_3 = -\infty$, $B_3 = +\infty$, and require that $F_{y'y'}(x, y, p)$ shall vanish at no point of R. The function $F_{y'y'}(x, y, p)$ will, then, not change sign in R.

The functions y admitted to consideration are single valued and continuous throughout the interval $x_0 \leqq x \leqq x_1$ and assume the prescribed boundary values, and they have a first derivative which, at least within the interval $x_0 < x < x_1$, is finite and continuous.

We will denote with KNESER† any solution of (1), or the curve that represents such a function, as an *extremal*.

§ 2. *Sufficient Conditions.*‡

WEIERSTRASS'S sufficient condition that such a function y, represented by a curve C, actually makes I a minimum consists (a) in the existence of a field of extremals surrounding C, and (b) in the relation $F_{y'y'}(x, y, p) > 0$, where (x, y) is any point of C and p may have any value. The notion of the field was first published by SCHWARZ.§ That which is essential may be embraced in the following requirements:

* We confine ourselves for simplicity to the case of integrals with fixed limits; but the method here set forth applies equally well to integrals with variable limits. In fact, one of the first examples that I formed was the following. Reversing the usual order of the processes, I started with the envelope, which I assumed as the semicubical parabola $y^2 = x^3$, and took then as extremals the right lines that envelop it. I then constructed a problem of the calculus of variations, of which these elements afford the solution and which, since the above family of right lines is cut orthogonally by a certain parabola P $(y^2 = ax + \beta)$, may be formulated as follows: *To find the shortest line that can be drawn from the point* $x = 0$, $y = 0$ *to the parabola* P.

† *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, p. 24.

‡ For an exposition of WEIERSTRASS'S sufficient condition for a minimum, together with KNESER'S and HILBERT'S proofs of its validity, cf. an article by the writer, Annals of Mathematics, ser. 2, vol. 2, no 3 (1901), p. 105.

§ Festschrift on the occasion of WEIERSTRASS'S seventieth birthday: *Ueber ein die Flächen kleinsten Inhalts betreffendes Problem der Variationsrechnung*, Acta soc. sci. Fennicae, vol. 15 (1885), p. 315 = Werke, vol. 1, p. 223. Cf. also KNESER, *Variationsrechnung*, chap. 3.

(1) Through each point (x, y) of the neighborhood S of the extremal C there passes one and only one extremal

$$y = \phi(x, \gamma),$$

γ being a parameter. The function ϕ, regarded as a function of the two independent variables (x, γ), is together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi'_\gamma$ continuous at each point (x, γ) corresponding to an interior point (x, y) of S; and $\phi'(x, \gamma)$ remains finite throughout that part of the neighborhoods of (x_0, y_0), (x_1, y_1) that is bounded by any two curves of the class admitted to consideration in § 1. The equation of C is $y = \phi(x, \gamma_0)$.

(2) The function $\phi_\gamma(x, \gamma) \neq 0$, when $x_0 < x < x_1$.

By the "neighborhood" of the extremal C is meant the interior of a strip S of the xy-plane bounded by the lines $x = x_0$, $x = x_1$, $y = \phi(x, \gamma_0) + \epsilon$, $y = \phi(x, \gamma_0) - \epsilon$, where ϵ is a positive constant, together with the points (x_0, y_0), (x_1, y_1).

The definition of the field here given is broader than the one employed by KNESER in his treatise or the one that I used in the paper above referred to. Both KNESER's and HILBERT's proofs for WEIERSTRASS's sufficient condition, given in the latter article, can be extended without difficulty to the present case. Thus, in the case of HILBERT's proof, it is sufficient to show that the value of the integral

$$J = \int_{x_0}^{x_1} \{F(x, Y, y') + (Y' - y') F_{y'}(x, Y, y')\} dx,$$

where x, Y are the coördinates of a point of an arbitrary curve $\bar{C}$ lying in the field, and where

$$y' = \phi'(x, \gamma), \quad Y = \phi(x, \gamma),$$

is independent of $\bar{C}$, i. e., is the same for any two such curves, $\bar{C}_1$, $\bar{C}_2$, and this can be proved as follows. Let the values of J for $\bar{C}_1$ and $\bar{C}_2$ be denoted respectively by J_1 and J_2, and suppose

$$J_1 - J_2 = h > 0.$$

Let (x', y') and (x'', y'') be two points of $\bar{C}_1$ chosen respectively arbitrarily near to the points (x_0, y_0) and (x_1, y_1); and let C' be a curve connecting (x', y')

and (x'', y'') and coinciding throughout the greater part of its extent with $\bar{C}_2$. More precisely, let curves c' and c'' be drawn respectively from (x', y') and

(x'', y'') tangent to C_2 and meeting C_2 at points that lie near to (x', y') and (x'', y'') respectively, and let c' and c'' be so chosen that the curve $\bar{C}'$, consisting of c', c'' and the part of $\bar{C}_2$ lying between the points of tangency of c' and c'', will be a curve of the class admitted to consideration, § 1. Denote the values of J taken along $\bar{C}_1$ from (x', y') to (x'', y'') and along $\bar{C}'$ respectively by J_1' and J'. Then

$$J_1' - J' = 0 .$$

For, all of the conditions of the lemma on which HILBERT'S proof rests are fulfilled for the interval $x' \leqq x \leqq x''$. Now it is clear that, by a proper choice of the points (x', y'), (x'', y'') and the curves c', c'', the integrals J_1' and J' can be made to differ respectively from J_1 and J_2 by quantities each numerically less than $\frac{1}{2}h$. For, though the function $y' = \phi'(x, \gamma)$ may become infinite in the neighborhood of the point (x_0, y_0), still it is possible to cut out from S an angle bounded by the lines $y - y_0 = \pm \lambda(x - x_0)$, where λ is an arbitrarily chosen constant, within which $\phi'(x, \gamma)$ remains finite and from which the curves C_1, C_2, C' never emerge; and since a similar remark applies to the neighborhood of the point (x_1, y_1), this is sufficient for the proof. Hence the supposition that $h > 0$ leads to a contradiction, and

$$J_1 = J_2 .$$

§ 3. *Kneser's Investigation.**

KNESER'S results may be briefly obtained as follows. Let there exist an extremal C connecting the points (x_0, y_0) and (X_1, Y_1). Consider the integral I formed for an arbitrary value of x_1 between x_0 and X_1, the point (x_1, y_1) lying on C. Further assume

(1) That through the point (x_0, y_0) and each point (x, y) of that part of the neighborhood of (x_0, y_0) for which $x > x_0$ (i. e., $x_0 < x < x_0 + \mu$, $|y - y_0| < \mu$), one and only one extremal $y = \phi(x, \gamma)$ passes and that $y = \phi(x, \gamma_0)$ coincides with C. The function $\phi(x, \gamma)$, regarded as a function of the two independent variables (x, γ), shall, together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi_\gamma'$, be continuous at each of the points (x, γ) corresponding to points (x, y) of the above neighborhood; and furthermore at the points (x, γ) of the region

$$x_0 + \mu \leqq x \leqq X_1, \qquad \gamma_0 - \kappa < \gamma < \gamma_0 + \kappa :$$

and $\phi'(x, \gamma)$ shall remain finite throughout that part of the neighborhood of (x_0, y_0) bounded by any two curves of the class considered in § 1.

(2) $\phi_\gamma(x, \gamma) \neq 0$ when $x_0 < x < x_0 + \mu$, $|y - y_0| < \mu$.

$\phi_\gamma(x, \gamma_0) \neq 0$ when $x_0 + \mu \leqq x < x' < X_1$; $\phi_\gamma(x', \gamma_0) = 0$.

* Mathematische Annalen, loc. cit.

Since all the extremals $y = \phi(x, \gamma)$ go through the point (x_0, y_0) it follows that $\phi_\gamma(x_0, \gamma) \equiv 0$.

When $x_1 < x'$, a field exists about the extremal C, as is readily proved by the ordinary methods of analysis.* We will assume once for all that $F_{y'y'}(x, y, p)$ is positive, and hence the integral I is a minimum.

When $x_1 = x'$, x' is called the *conjugate point* of x_0, and it is the study of this case to which KNESER'S paper is devoted. It is desirable to distinguish three cases. We will assume that $\phi_{\gamma\gamma}(x, \gamma)$ exists and is continuous throughout the neighborhood of the point (x', γ_0).

Case I. $\phi_{\gamma\gamma}(x', \gamma_0) \neq 0$. This is the general case.

Case II. $\phi_{\gamma\gamma}(x', \gamma) \equiv 0$, $|\gamma - \gamma_0| < \delta$.

Case III. $\phi_{\gamma\gamma}(x', \gamma_0) = 0$, $\phi_{\gamma\gamma}(x', \gamma) \not\equiv 0$.

Cases I and II are considered by KNESER and his results are as follows. In Case I the extremals $y = \phi(x, \gamma)$ have an envelope † which passes through the point x', $y' = \phi(x', \gamma_0)$ and this point is an ordinary point of the envelope. The proof may be based on DINI'S existence theorem for implicit functions of real variables,‡ according to which the equation

$$0 = \phi_\gamma(x, \gamma)$$

can be solved for γ in the neighborhood of the point (x', γ_0). The function

$$\gamma = \psi(x)$$

thus defined is single valued in the neighborhood of the point x' and it has a continuous derivative throughout this region, given by the formula

$$\frac{d\gamma}{dx} = -\frac{\phi'_\gamma(x, \gamma)}{\phi_{\gamma\gamma}(x, \gamma)}.$$

Moreover, the pairs of values $\big(x, \gamma = \psi(x)\big)$ are the only pairs of values (x, γ) in the neighborhood of the point (x', γ_0) which cause $\phi_\gamma(x, \gamma)$ to vanish.

The equation of the envelope is

$$y = \phi(x, \gamma) \qquad \text{where} \qquad \gamma = \psi(x).$$

Its slope is given by the formula

$$\frac{dy}{dx} = \phi'(x, \gamma) + \phi_\gamma(x, \gamma)\frac{-\phi'_\gamma(x, \gamma)}{\phi_{\gamma\gamma}(x, \gamma)} = \phi'(x, \gamma).$$

---

* The details of the proof are given in a similar case in the writer's paper, Annals of Mathematics, loc. cit., § 2.

† That this *in general* is the case is essentially JACOBI'S criterion. The introduction of sufficient restrictions so that one is able to prove that, in the problem thus restricted, an envelope does exist which has an ordinary point at (x', γ_0), is characteristic of the progress of analysis since Jacobi's time.

‡ Cf. PEANO-GENOCCHI, *Calcolo differenziale*, etc., § 110, or JORDAN, *Cours d'Analyse*, vol. 1, 2d ed., p. 80. KNESER uses the method of power series.

Thus in Case I an envelope in the proper sense of the term exists, and it *enters the interval* (x_0, x'). This fact is the essential characteristic for Case I. It can now be shown that, when $x_1 = x'$, I ceases to be a minimum. Let $P = (x'', y'')$ be an arbitrary point of the envelope lying in the neighborhood of

$B = B' = (x', y')$ and let $x'' < x'$. Then the value of I taken from A to B along C is the same as its value taken first along the extremal AP and then along the envelope from P to B. The proof of this theorem is readily deduced from the analysis by which WEIERSTRASS'S sufficient condition is established.* Thus the function $f(s)$ (loc. cit., § 3) formed for the curve $\overline{C}$ consisting of APB is easily shown to be constant throughout the entire interval $0 \leqq s \leqq l$. If we employ HILBERT'S method (loc. cit., § 5), the proof is still simpler. Let $\overline{C}'$ be a curve connecting A with B' and lying within the region bounded by C and the curve APB'. Then it follows, as in §2 of this article, that HILBERT'S integral J has the same value for $\overline{C}'$ as for C. When $\overline{C}'$ goes over continuously into APB', the integral J goes over continuously into this integral taken along APB', and the proof is complete.

Thus far it has only been shown that the integral I loses the minimum property, inasmuch as there are curves other than C lying in the neighborhood of C for which I has the same value as for C. By making use of the relation †

$$\phi'_\gamma(x', \gamma_0) \neq 0,$$

we can show that the envelope PB' is not an extremal, and hence that there exist curves in the neighborhood of C for which I has a smaller value than for C. In fact, for the envelope,

$$\frac{dy}{dx} = \phi'(x, \gamma), \qquad \frac{d^2y}{dx^2} = \phi''(x, \gamma) - \frac{\{\phi'_\gamma(x, \gamma)\}^2}{\phi_{\gamma\gamma}(x, \gamma)} \neq \phi''(x, \gamma),$$

and so the envelope does not satisfy LAGRANGE'S equation (1).

In Case II no envelope in the proper sense of the term exists. All the extremals $y = \phi(x, \gamma)$ for which $\gamma_0 - \delta < \gamma < \gamma_0 + \delta$ go through the point (x', y'). The value of I taken along each of these extremals is the same, as can be shown by reasoning similar to that employed in Case I, and hence I ceases to be a

* Cf. Annals of Mathematics, loc. cit., §§ 3, 5.

† Cf. KNESER'S memoir, p. 40, formula (32).

minimum. If the function $\phi(x, \gamma)$ satisfies all of the conditions requisite for a field about the arc AB' of C with the one exception that through the point (x', y') more than one extremal passes, there will be no curve $\overline{C}$ in the neighborhood of C for which I has a smaller value than for C. The great circles of a sphere, regarded as shortest lines on the sphere, are typical for this case, A and B' being two points situated diametrically opposite each other.

§ 4. *The Revision of Kneser's Theorem.*

In Case III an envelope may exist which has a singular point (e. g., cusp) at B' and *does not enter the interval* (x_0, x'). This case KNESER appears to have overlooked. * All of the conditions laid down for a field in § 2 may be fulfilled, so that the integral I will still be a minimum, as is shown by the example of the following section. On the other hand, there are cases in which the conditions of Case III are fulfilled, but in which the envelope enters the region (x_0, x') and the reasoning employed in Case I to show that I ceases to be a minimum can be repeated.

Let us, then, examine the possibilities which Case III presents with some detail. Denote the set of points (x, y) lying in the neighborhood of B', for which

$$\phi_\gamma(x, \gamma) = 0$$

by (E). The point B' is a member of (E). We will distinguish two cases.

(a) The set (E) contains a point (x, y) for which $x < x'$.

(b) The set (E) contains no such point.

In case (a) the integral I will cease to be a minimum whenever the points of (E) — or some of these points — for which $x \leqq x'$ lie on a curve E which has a continuously turning tangent and goes through B'; which, furthermore, at each point of E in the neighborhood of B', is tangent to the extremal $y = \phi(x, \gamma)$ that passes through that point; and when lastly $\phi_\gamma(x, \gamma) \neq 0$ at all points on the segment of each line parallel to the y-axis that lies between E and C. For, the proof given in Case I applies here.

A sufficient condition for the existence of case (a) is that in the neighborhood of the point $\gamma = \gamma_0$, the derivatives $\phi_{\gamma^k}(x', \gamma)$, $k = 1, 2, \cdots, n$, exist; that $\phi_{\gamma^k}(x', \gamma_0)$ vanishes when $k < n$, but does not vanish when $k = n$; and that n is even. Case (a) may, however, exist when n is odd.

For, let $\phi_{\gamma^n}(x', \gamma_0) > 0$; n even. Then the function $y = \phi(x', \gamma)$ is a minimum when $\gamma = \gamma_0$, and hence in the neighborhood of the point $\gamma = \gamma_0$, $\phi(x', \gamma) > \phi(x', \gamma_0)$ both for values of γ larger and for values of γ smaller than γ_0. Now at any point $(\bar{x}, \bar{y})$ of C intermediate between A and B', $\phi_\gamma(\bar{x}, \gamma_0) \neq 0$, and if $\phi_\gamma(\bar{x}, \gamma_0) > 0$, the curves $y = \phi(x, \gamma)$ for which

* Cf., however, below, § 6.

$\gamma_0 - \eta < \gamma < \gamma_0$ all lie below C at the point $x = \bar{x}$ and hence cut C between $\bar{x}$ and x'. The equation

$$\phi(x, \gamma) - \phi(x, \gamma_0) = (\gamma - \gamma_0)\phi'_\gamma\big(x, \gamma_0 + \theta(\gamma - \gamma_0)\big) = 0$$

has a solution $x = X$, $\bar{x} < X < x'$, and $\big(X, \gamma_0 + \Theta(\gamma - \gamma_0)\big)$ is a point of (E).

Thus for example if $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of (x', γ_0) and if

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \quad (k = 1, 2, \cdots, \quad n - 1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

n being even, we shall certainly have case (a). But more than this: with the present restrictions on the function $\phi(x, \gamma)$ we can infer that all the conditions named above are fulfilled which suffice that I cease to be a minimum. In fact, the equation

$$0 = \phi_\gamma(x, \gamma)$$

now defines a set of points (E) that may be grouped together so as to constitute a finite number of curves passing through B' and not intersecting one another elsewhere in the neighborhood of B'. Through that part of the neighborhood of B' for which $x < x'$ at least one of these curves will pass, and if we denote by E that one of these curves that lies next to C (on either side), then E will satisfy all of the above requirements. For, it is possible to write $\phi_\gamma(x, \gamma)$ in the form

$$\phi_\gamma(x, \gamma) = \bar{\phi}\,\Phi,$$

where Φ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and does not vanish there; and where $\bar{\phi}$ is of the form:

$$\bar{\phi} = \gamma^n + A_1\gamma^{n-1} + \cdots + A_n,$$

the coefficients A_i being analytic functions of x in the neighborhood of the point $x = x'$.* The irreducible factors † of this polynomial, set equal to zero, yield the curves which constitute the set (E). In particular, let $\bar{\bar{\phi}}$ be the factor which yields the curve lying next to C. This curve satisfies all the conditions imposed on E. For, since $\bar{\bar{\phi}}$ is irreducible, $\bar{\bar{\phi}}_\gamma$ does not vanish simultaneously with $\bar{\bar{\phi}}$ in the neighborhood of (x', γ_0) when $x < x'$, and hence

$$\frac{d\gamma}{dx} = -\frac{\bar{\bar{\phi}}_x}{\bar{\bar{\phi}}_\gamma}$$

exists and is finite at all such points. The slope of the curve is given by the equation

* This is a fundamental theorem of WEIERSTRASS'S; Werke, vol. 2, p. 135.

† "Irreducible" in the sense in which I have used this term in reporting WEIERSTRASS'S theorem, Encyclopädie der Mathematischen Wissenschaften, II., B. 1.

$$\frac{dy}{dx} = \phi' + \phi_\gamma \frac{d\gamma}{dx} = \phi',$$

where $x < x'$, and it is readily shown from the continuity of $\phi'(x, \gamma)$ that at B' the slope of E is given by the same formula.

If all of the foregoing conditions hold except the one that n be *even*, and if now n is odd, it would appear that we may still have case (a) and hence the existence of an envelope E with the above properties. For, the function

$$\phi(x, \gamma) = (x + 1)(x - \gamma)^3, \quad x_0 = -1, \quad x' = 0, \quad \gamma_0 = 0,$$

for example, satisfies all the conditions imposed on the function γ except, possibly, that of being an extremal. It seems probable that there are families of extremals for which this example is typical.

In case (b) a field will exist and I will continue to be a minimum provided that through B' and each point of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes. *

This will always be the case when it is the case for the points of the line $x = x'$ that lie in the neighborhood of B'. For then γ will be a single valued function of y along this line, continuous, monotonic and never constant, since the inverse function, $y = \phi(x', \gamma)$ is continuous and single valued.

Consider the region bounded by the quadrilateral whose sides are

$$y = \phi(x, \gamma_0 + \epsilon), \quad y = \phi(x, \gamma_0 - \epsilon), \quad x = x', x = x' - \epsilon > x_0,$$

where ϵ is so chosen (1) that the points $y' = \phi(x', \gamma_0 \pm \epsilon)$ are points of the above neighborhood of y' and (2) that $\phi_\gamma(x, \gamma)$ does not vanish within the quadrilateral. Let $(\bar{x}, \bar{y})$ be any point lying within the quadrilateral. Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits one and only one solution $\gamma_0 - \epsilon < \gamma < \gamma_0 + \epsilon$. For

$$\phi(\bar{x}, \gamma_0 - \epsilon) < \bar{y} < \phi(\bar{x}, \gamma_0 + \epsilon)$$

and $\phi(\bar{x}, \gamma)$ is a continuous function of γ in the interval $\gamma_0 - \epsilon \leqq \gamma \leqq \gamma_0 + \epsilon$. Hence $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ for at least one value of γ in the interval. Since $\phi_\gamma(\bar{x}, \gamma)$ does not vanish in the interval, it follows from ROLLE's theorem that the equation $\phi(\bar{x}, \gamma) = \bar{y}$ has only one root in the interval. Thus the sufficiency of the above condition is established.

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and we have case (b), then the integral will always be a minimum. For, only a finite number of the derivatives $\phi_{\gamma^k}(x', \gamma_0)$ can vanish. The first one

* More precisely: provided that, if $(\bar{x}, \bar{y})$ is the point B' or any point of its neighborhood, for which $x < x'$, the equation $\bar{y} = \phi(\bar{x}, \gamma)$ has one and only one root γ lying in the neighborhood of γ_0.

that does not vanish cannot be of even order, otherwise we should have case (a). The order being odd, it is readily seen that the equation $y = \phi(x', \gamma)$ admits, for each value y in the neighborhood of y', one and only one root γ in the neighborhood of γ_0.

The foregoing results may be summarized in the following theorems, which may be regarded as a correction and extension of KNESER'S theorem.

THEOREM A. *The integral*

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

ceases to be a minimum when x_1 is a conjugate point, $x_1 = x'$, of x_0 and the conditions of Case I *or* II *are fulfilled.*

This is KNESER'S theorem accurately enunciated.

THEOREM B. *In Case* III, *the integral ceases to be a minimum when we have case (a), i. e., when some of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie in the interval (x_0, x'), and when, besides, such points, or a part of them, can be so grouped as to form the arc of an envelope E entering the region (x_0, x') and, together with the extremal C and the line $x = x' - \epsilon$, bounding a region within which $\phi_\gamma(x, \gamma) \neq 0$.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and if we have case (a), then the further conditions of this theorem will be fulfilled. Here, a sufficient, but apparently not a necessary, condition that we have case (a) is that

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \quad (k = 1, 2, \cdots, n-1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

where n' is even.

THEOREM C. *In Case* III, *the integral continues to be a minimum when we have case (b), i. e., when none of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie within the interval (x_0, x'); and when, besides, through B' and each point (x, y) of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and if we have case (b), then the further conditions of this theorem will be fulfilled.

These theorems exhaust completely the case that $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), the case to which KNESER restricts his investigation.

§ 5. *The Geodesics on an Ellipsoid of Revolution.*

The geodesics on an oblate ellipsoid of revolution furnish an example of an integral I which does not always cease to be a minimum when x_1 is a conjugate point of x_0.

The form of the envelope of the geodesics drawn from a fixed point on an ellipsoid of revolution was given by JACOBI* and the envelope was investigated analytically by VON BRAUNMÜHL.†

Let the equation of a surface of revolution be written:

$$y = f(r), \quad \text{where} \quad r^2 = x^2 + z^2,$$

and let us take as the coördinates of a point on the surface r and the angle ω that the plane of the meridian through the point makes with a fixed plane through the axis of revolution. The length of the arc of a curve joining two points (r_0, ω_0) and (r_1, ω_1) is given by the formula

$$\int_{\omega_0}^{\omega_1} \sqrt{r^2 + \left(1 + f'(r)^2\right) r'^2}\, d\omega .$$

This integral is to be made a minimum. Since the integral does not contain the independent variable explicitly, LAGRANGE'S equation (1) admits a first integral and the differential equation of the extremals (geodesics) can be written in the form:‡

$$r^2(r^2 - \nu^2)\, d\omega^2 = \nu^2(1 + f'(r)^2)\, dr^2, \tag{2}$$

where ν is the constant of integration.

The surface here to be considered is the oblate ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = ca, \quad c < 1, \tag{3}$$

and the point A through which all the extremals pass is to be a point of the equator, for which ω shall vanish. The extremal C is to be the equator, $r = a$. We will choose ω as the independent variable and replace it by x to conform to the earlier notation, and y as dependent variable. The function $F_{y'y'}$ is seen to be positive for all values $-b < y < b$ and $-\infty < y' < \infty$; it does not contain x explicitly. Equation (2) becomes the following:

$$(b^2 - y^2)^2 \{c^2(a^2 - \nu^2) - y^2\}\, dx^2 = c^2\nu^2 \{c^2 b^2 + (1 - c^2) y^2\}\, dy^2 .$$

To integrate this equation set

$$\gamma = \pm c\sqrt{a^2 - \nu^2}, \qquad y = \gamma \sin\theta .$$

**Vorlesungen über Dynamik*, 6th lecture, Werke, Supplementband, p. 47, where the form of the envelope is indicated by a figure.

†Mathematische Annalen, vol. 14 (1879), p. 557. Cf. also the models made by VON BRAUNMÜHL in the Brill (now Schilling) collection, Katalog mathematischer Modelle, ser. 5, XVIII, a, b = Specialkatalog, 104, 106 = Deutsche Mathematiker-Vereinigung, Katalog mathematische Modelle, München, 1892, 2. Teil, II. Abteilung, 214.

In a second paper by the same author (Mathematische Annalen, vol. 20 (1882), p. 557) the geodesics on an ellipsoid with three distinct axes are treated and a number of bibliographical references are given.

‡KNESER, *Lehrbuch der Variationsrechnung*, p. 24.

Then x is given as a function of θ by a quadrature and we have the formulas:

$$(4)\qquad \begin{cases} y = \gamma \sin\theta\,, \\ x = \sqrt{b^2-\gamma^2}\displaystyle\int_0^\theta \frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta\,, \end{cases}$$

where all the radicals are taken positive and $-b<\gamma<b$. For any one of these values of γ, x is a single valued continuous function of θ that increases when θ increases, and hence the inverse function, $\theta(x)$, is also single valued and increases as x increases. Equations (4) thus define y as a single valued function of (x,γ) for all values of (x,γ) in the domain:

$$-\infty<x<\infty\,,\quad -b<\gamma<b\,,$$

and it follows from DINI'S theorem (cf. p. 170) that this function, $y=\phi(x,\gamma)$ has continuous partial derivatives at each point of this region.

The course of the geodetics on the ellipsoid is shown in VON BRAUNMÜHL'S first paper (loc. cit.), Figs. 1–3, and on the models. It is suggested geometrically by Fig. 3 and the corresponding model that either cusp of the envelope lying on the equator is a point x' conjugate to A for the portion of the equator C included between A and x'; that this portion of C is embedded in a family of extremals having an envelope passing through x'; and that this envelope does not enter the region Ax'. Thus it would appear that this portion of C is surrounded by a field and hence that for it the integral is a minimum. It remains to supply the corresponding analytic proofs.

In the case of the prolate ellipsoid, Fig. 1 of VON BRAUNMÜHL'S paper, it is evidently the cusps on the meridian through A that yield the desired example.

For a given value of γ lying between $-b$ and b, the curve $y=\phi(x,\gamma)$ is an extremal and its course on the ellipsoid is easily traced. Since θ always increases with x, the curve lies within the zone of the ellipsoid bounded by the planes $y=\pm\gamma$. It is a periodic curve and the value of x corresponding to a quarter period is given by the formula

$$\lambda = \sqrt{b^2-\gamma^2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta\,.$$

For this value of x, y is a maximum, $=\gamma$. The curve crosses the equator, $y=0$, when $\theta=\pi$, $x=2\lambda$. Let γ converge toward 0; then 2λ approaches πc as its limit and, as will appear in the course of the subsequent work,

$$x' = \pi c$$

is the conjugate point to A $(x=0)$ on the equator, C.

For all values of x that are positive and less than πc, $\phi_\gamma(x,\gamma)\neq 0$ when $\gamma=\gamma_0=0$. In act, x is then equal to $c\theta$, so that $0<\theta<\pi$; and since

$$\phi_\gamma(x, \gamma) = \frac{\partial y}{\partial \gamma} = \sin\theta + \gamma\cos\theta\frac{\partial\theta}{\partial\gamma}; \tag{5}$$

it follows that $\phi_\gamma(x, \gamma_0) = \sin\theta > 0$.

At the point $x = \pi c$, $\gamma = 0$, the function $\phi_\gamma(x', \gamma_0)$ vanishes; and the same is true of the function

$$\phi_{\gamma\gamma}(x, \gamma) = \frac{\partial^2 y}{\partial\gamma^2} = 2\cos\theta\frac{\partial\theta}{\partial\gamma} - \gamma\sin\theta\left(\frac{\partial\theta}{\partial\gamma}\right)^2 + \gamma\cos\theta\frac{\partial^2\theta}{\partial\gamma^2}.$$

For, $\partial\theta/\partial\gamma$ is given by the formula:

$$\begin{aligned} 0 = {} & \sqrt{b^2 - \gamma^2}\,\frac{\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}}{b^2 - \gamma^2\sin^2\theta}\frac{\partial\theta}{\partial\gamma} \\ & - \frac{\gamma}{\sqrt{b^2 - \gamma^2}}\int_0^\theta \frac{\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}}{b^2 - \gamma^2\sin^2\theta}\,d\theta \\ & + \sqrt{b^2 - \gamma^2}\int_0^\theta \left\{ \frac{(1-c^2)\gamma\sin^2\theta}{(b^2 - \gamma^2\sin^2\theta)\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}} \right. \\ & \left. + \frac{2\gamma\sin^2\theta\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}}{(b^2 - \gamma^2\sin^2\theta)^2} \right\} d\theta \end{aligned} \tag{6}$$

and hence $\partial\theta/\partial\gamma$ vanishes when $\gamma = 0$. The function $\phi_{\gamma\gamma}(x', \gamma)$ does not, however, vanish identically at the point $\gamma = 0$. In fact, $\phi_{\gamma^3}(x', \gamma_0) \neq 0$. For

$$\phi_{\gamma^3}(x, \gamma) = 2\cos\theta\frac{\partial^2\theta}{\partial\gamma^2} + \rho,$$

where ρ is composed of terms that vanish at the point (x', γ_0), and it is readily shown by direct computation that $\partial^2\theta/\partial\gamma^2$ does not vanish at this point.

We proceed now to show that the further conditions of Theorem C, § 4, are fulfilled, and hence that the integral taken along C from A to B' is a minimum. It remains to establish two things: (α) that through each point (x, y) of the neighborhood of A, for which $x > 0$, one and only one extremal $\phi(x, \gamma)$ passes, and that at each of these points $\phi_\gamma(x, \gamma) \neq 0$; and that, furthermore, $\phi'(x, \gamma)$ remains finite in that part of this neighborhood lying between any two curves, y_1 and y_2, of the class considered in § 1; (β) that at the point B' we have case (b) (§ 4), for $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of B', and hence the existence of case (b) suffices.

The proof of (β) is direct. At all points of the region

$$x' - \epsilon < x < x', \quad -\kappa < \gamma < \kappa,$$

where ϵ, κ are sufficiently small positive constants, $\phi_\gamma(x, \gamma) > 0$. For, an approximate evaluation of x by (4) in terms of γ near 0 and θ near π shows that

when x and γ lie in the above region, θ is less than π. Similarly it appears from (6) that for such values of θ and γ, $\gamma\,\partial\theta/\partial\gamma \leqq 0$. Hence the first term in the right hand member of (5) is always positive and the second is never negative.

Finally, to establish (a), we show (a) that through each point (x, y) of the neighborhood of A for which $x > 0$ at least one extremal $y = \phi(x, \gamma)$ passes; (b) that for all such points $0 < \theta < \delta$, where δ can be made arbitrarily small by choosing the above neighborhood suitably; (c) that $\gamma\,\partial\theta/\partial\gamma \geqq 0$ when $0 < \theta < \delta$, whence it follows by the aid of (5) that $\phi_\gamma(x, \gamma) > 0$; (d) that $\phi'(x, \gamma)$ remains finite between any two extremals $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$.

ad (a). Let ϵ, η be two positive quantities, the first arbitrarily small, the second, any quantity between 0 and 1, e. g., $\eta = \frac{1}{2}$. Then through each of the points

$$y = Y = \eta b \sin\epsilon, \quad 0 < x \leqq \epsilon',$$

where ϵ' denotes the value of x corresponding to the values $\gamma = \eta b$, $\theta = \epsilon$, there passes at least one extremal $\phi(x, \gamma)$. For, the equation

$$\gamma \sin\theta = \eta b \sin\epsilon$$

defines θ as a single valued continuous function of γ that decreases from the value ϵ to the value $0 < \sin^{-1}(\eta \sin\epsilon) < \epsilon$ when γ increases from the value ηb to b. Substitute this function $\theta(\gamma)$ in the formula for x, (4). Then x becomes a continuous function of γ in the interval $\eta b \leqq \gamma \leqq b$, assuming at the extremities of the interval respectively the values ϵ' and 0, and hence assuming each intermediate value at least once.

Thus the equation

$$Y = \phi(\bar{x}, \gamma),$$

where $\bar{x}$ is chosen arbitrarily between 0 and ϵ', admits at least one solution $\gamma = \Gamma$, where $\eta b < \Gamma < b$. Let $(\bar{x}, \bar{y})$ be any point of the region

$$0 < \bar{x} < \epsilon', \quad 0 \leqq \bar{y} < Y.$$

Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits at least one solution, $\gamma = \bar{\gamma}$, where

$$0 \leqq \bar{\gamma} < b.$$

For

$$\phi(\bar{x}, 0) \leqq \bar{y} < \phi(\bar{x}, \Gamma),$$

and the continuous function $\phi(\bar{x}, \gamma)$ must assume the value $\bar{y}$ at least once when γ passes from 0 to Γ. To show that $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ only once,

it is sufficient to show that $\phi_\gamma(\bar{x}, \gamma) > 0$, when $0 < \gamma < b$ and this relation follows from (b) and (c).

ad (b). At any point $(\bar{x}, \bar{y})$ at which $\gamma \geqq \eta b$,

$$\theta \leqq \epsilon, \quad \text{since} \quad \eta b \sin\theta \leqq \gamma \sin\theta = \bar{y} < Y = \eta b \sin\epsilon.$$

On the other hand, when $\gamma < \eta b$, it appears from (4) that

$$\bar{x} > \sqrt{b^2 - \eta^2 b^2} \int_0^\theta \frac{cb}{b^2 - \eta^2 b^2}\, d\theta = \frac{c}{\sqrt{1-\eta^2}}\,\theta,$$

or

$$\theta < c^{-1}\sqrt{1-\eta^2}\,\bar{x} < c^{-1}\sqrt{1-\eta^2}\,\epsilon'.$$

Hence δ may be taken equal to the larger one of the quantities ϵ, $c^{-1}\sqrt{1-\eta^2}\,\epsilon'$.

ad (c). From (6) it is clear that for all values of $-b < \gamma < b$ and for all values of $0 < \theta < \delta$, where δ is chosen sufficiently small, the sign of $\gamma\,\partial\theta/\partial\gamma$ is dominated by the sign of the first integral, and hence is positive.

From (5) it now follows that, when ϵ is chosen sufficiently small, $\phi_\gamma(x, \gamma) > 0$ at all points (x, y) of the neighborhood of A for which $x > 0$, $y \geqq 0$. Similar reasoning applies to the region $x > 0$, $y < 0$.

ad (d). The value of $\phi'(x, \gamma)$ is given by the formula

$$\phi'(x, \gamma) = \frac{dy}{dx} = \frac{\gamma}{\sqrt{b^2 - \gamma^2}} \frac{\cos\theta\,(b^2 - \gamma^2 \sin^2\theta)}{\sqrt{c^2 b^2 + (1-c^2)\gamma^2 \sin^2\theta}},$$

from which it follows that $|\phi'(x, \gamma)| < b^2 c^{-1}(b^2 - \gamma^2)^{-\frac{1}{2}}$. On the other hand, two extremals, $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$ can always be so chosen as to include between them that part of the neighborhood in question which lies between the curves y_1 and y_2. Hence the finiteness of $\phi'(x, \gamma)$ follows without difficulty.

§6. *The Integral* $I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$.*

In the foregoing, the value of the integral taken along C has been compared

* This section was added 7 March. At the time that I sent the MS. to the TRANSACTIONS, I had not observed that KNESER, in his treatise, § 25, notes the case in which the theorem cited in § 1 of this paper fails as an "exceptional case" (p. 95, l. 4). He recognizes that it is conceivable that the envelope may have a cusp, without however showing that this is ever actually the case. Near the close of the section he says: "Eine Ausnahme findet nur statt, wenn die stets vorhandene Enveloppe der Curven des Feldes im Brennpunkte einen Rückkehrpunkt von besonderer Art besitzt." In what the specialization of the cusp consists is not explained, and, in fact, it is not even clear whether the "Ausnahme" is considered to be an exception to the *proof* or an exception to the *theorem*.

with its value taken along only such curves of the neighborhood of C as are cut by a parallel to the y-axis in at most one point. In the case of the geodesic AB' on the ellipsoid it is readily seen that it continues to be a minimum even when compared with any continuous curve having a continuously turning tangent, which can be drawn from A to B' in the neighborhood of C. (The neighborhood of C shall now include the whole neighborhood of the points A and B'.) For, to begin with, the integral being written in the Weierstrassian form with a parameter t:

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt, \tag{7}$$

it is easily seen that the field may be taken as the neighborhood of C with the exception of that part of this region that is cut out by the envelope. The Weierstrassian proof for the existence of a minimum will apply to any curve connecting A and B' and lying in this field.

Next, consider the family of geodesics through A that lie in the neighborhood of C, and lay off on each of these a length measured from A equal to the length of the geodesic AB'. The locus of the points thus determined is readily seen to be a curve Γ passing through B' and not tangent to C.

Now consider *any* curve lying in the neighborhood of C and connecting A and B'. If this curve meets Γ only at the point B', it lies wholly in the field, and hence is longer than the geodesic AB'. If it meets Γ before it reaches B', then this part of it is at least as long as AB', and hence in this case, too, its total length is greater than that of the arc AB' of the geodesic C.*

It is possible to generalize from this example as follows. Let A, B' be two conjugate points on the extremal C of the general integral of the form (7). If the extremals of the field are given by the formulas†

$$x = \xi(t, a), \qquad y = \eta(t, a),$$

then, to the former function $\phi_\gamma(x, \gamma)$ corresponds now the determinant

$$\Delta = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.$$

We assume that Δ vanishes only at A and at points in the neighborhood of B'. If, furthermore, $\xi(t, a)$, $\eta(t, a)$ are both analytic in the neighborhood of B', the equation

$$\Delta = 0$$

will define one or more curves going through B' and tangent to C there.

* It may furthermore be shown that the arc AB' is shorter than any other curve that can be drawn on the ellipsoid connecting A and B'. For, the curve Γ is a closed curve that does not cut itself, and the field may be so chosen as to include the whole interior of this curve.

† KNESER, *Variationsrechnung*, chap. 3.

Divide the neighborhood S of C into two pieces, S_1 (containing A) and S_2, by a curve through B' cutting C normally. Then to cases (a) and (b) of § 4 correspond here the cases (a) that there are interior points of S_1 at which $\Delta = 0$ (we will also tabulate here the case that $\Delta = 0$ at B', but nowhere else in the neighborhood of B'); (b) that there are no such points, but that there are points within S_2 in the neighborhood of B' at which $\Delta = 0$. In case (a) the integral I will cease to be a minimum. In case (b) the integral will certainly continue to be a minimum when compared with the value that it attains when taken along any curve of the neighborhood of C that meets the locus $\Delta = 0$ only at B', for such a curve will lie in the field; and it will continue to be a minimum even when a curve of the kind just excluded is considered, provided that the integrand

$$F(x_1, y_1, \cos\phi, \sin\phi) > 0,$$

where the coördinates of B' are written as (x_1, y_1) and ϕ has any value. For, the locus of points so taken on the extremals which lie in the neighborhood of C that the value of I along each of these extremals is the same as the value of I along C from A to B' is a curve Γ passing through B' and not tangent to C, and the points at which $\Delta = 0$ lie on a curve or curves tangent to C at B'. The remainder of the proof follows the same lines as the proof of the example above considered.

ON THE GEOMETRY OF PLANES IN A PARABOLIC SPACE OF FOUR DIMENSIONS*

BY

IRVING STRINGHAM

Literature.

Of the literature of the geometry of hyperspace that has accumulated in recent years the following papers are cited as having points of contact with the ideas here set forth:

CLIFFORD: *Preliminary Sketch of Biquaternions* in Proceedings of the London Mathematical Society, vol. 4 (1873), pp. 381–395. CLIFFORD'S theory of parallels in elliptic space is identical with the theory of isoclinal systems of planes in four-dimensional space; namely, planes that pass through a fixed point and make equal dihedral angles with any transversal plane through the same point. (See §§ 30–32 of this paper.)

CHARLES S. PEIRCE: Reprint of the *Linear Associative Algebra* of BENJAMIN PEIRCE in the American Journal of Mathematics, vol. 4 (1881). In the foot-note of page 132 attention is called to the fact that in four-dimensional space two planes may be so related to one another that every straight line in the one is perpendicular to every straight line in the other. (See § 28 (3) of this paper.)

I. STRINGHAM: (1) *On a Geometrical Interpretation of the Linear Bilateral Quaternion Equation;* (2) *On the Rotation of a Rigid System in Space of Four Dimensions;* (3) *On the Measure of Inclination of two Planes in Space of Four Dimensions.* Papers presented to Section A of the American Association for the Advancement of Science, the first two at the Philadelphia meeting of 1884, the third at the Cleveland meeting of 1888. Abstracts printed in Proceedings of the Association, 1884, pp. 54–56, and privately, 1888. These papers form the nucleus of the present investigation.

A. BUCHHEIM: *A Memoir on Biquaternions*, in the American Journal

* Presented to the Society (Chicago) December 27, 1900. Received for publication December 7, 1900.

of Mathematics, vol. 7 (1885). The geometrical part is devoted to the geometry of non-Euclidean space. The theory of parallels is discussed on pp. 301–306, 316–325.

W. KILLING: *Die nicht-Euklidischen Raumformen*, Leipzig, 1885. Section 8, pp. 148–160, discusses, under the title: "Die gegenseitige Lage zweier Ebene," the relations to one another of μ- and ν-dimensional planes in an n-dimensional space.

CAYLEY: *On the quaternion Equation* $qQ - Qq' = 0$, and *On the Matrical Equation* $qQ - Qq' = 0$; Messenger of Mathematics, vol. 14 (1885), pp. 108–112 and 300–304, or Mathematical Papers, vol. 12, pp. 300–304 and 311–313. The first paper discusses the character of the roots of the equation $qQ - Qq' = 0$; the second interprets q, q', Q as matrices.

KLEIN: *Vorlesungen über nicht-Euklidische Geometrie*, II., 1890 (Zweiter Abdruck, 1893). On pages 120–124 the author identifies quaternion multiplications with orthogonal substitutions in four variables. In particular the products $p \cdot q$ and $q \cdot p$ are called respectively "eine Schiebung erster Art" and "eine Schiebung zweiter Art"; these are the parallel and contra-parallel translations of Professor HATHAWAY'S paper number (1), cited below. CLIFFORD'S theory of parallels in elliptic space is explained in the Vorlesungen on pages 228–237. KLEIN'S theory of Schiebungen was first presented to the members of his Seminar in January and February, 1880.

G. VERONESE: *Fondamenti di Geometria*, Padua, 1891, Part II, Book I, p. 455 et sq. Chapter 1 discusses elementary theorems in the geometry of four-dimensional space, some of which deal with the perpendicularity, parallelism, and intersections of planes.

M. BRÜCKNER; *Die Elemente der vierdimensionalen Geometrie mit besonderer Berücksichtigung der Polytrope;* Jahresber. d. Ver. f. Naturk., Zwickau, 1893. I have not been able to consult a copy of this paper, but SCHLEGEL, in Fortschritte der Mathematik, vol. 25, p. 1028, says: "Die ··· Arbeit giebt eine auf gründlicher Litteraturkenntnis beruhende und durch grosse Klarheit in der Darstellung sich empfehlende Zusammenstellung der in der elementaren vierdimensionalen Geometrie erzielten Resultate, und füllt dadurch in erwünschter Weise eine Lücke in der deutschen Litteratur aus."

P. CASSINI: *Sulla geometria pura Euclidiana ad n dimensioni;* Atti del Reale Instituto Veneto (7), vol. 5 (1894), p. 820 et sq., discusses metrical geometry in four-dimensional space; but see Fortschritte der Mathematik, vol. 25, p. 1035.

A. S. HATHAWAY: (1) *Quaternions as Numbers of Four-Dimensional Space*, in Bulletin of the American Mathematical Society, vol. 4 (1897), pp. 54–57; (2) *Alternate Processes*, in Proceedings of the Indiana Academy of Sciences, Indianapolis, 1897, pp. 1–10; (3) *Linear*

Transformations in Four-Dimensional Space, in Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 93–94. The fundamental ideas and formulæ of these papers are the same as those that were used in my papers of 1884 and 1888. Specifically Professor HATHAWAY exemplifies the utility of the quaternion analysis for the following purposes: (1) To interpret CLIFFORD'S theory of parallels in elliptic space (KLEIN'S Schiebungen), there stated in terms of great circular arcs on the hypersphere of four-dimensional space (see this paper, §§ 30–32); (2) To determine certain angles, areas, and volumes in four-dimensional space; (3) To formulate the theory of certain four-dimensional space transformations, in particular rotations (cf. my paper numbered (2) above).

A. N. WHITEHEAD: *Universal Algebra*, vol. 1 (1898). At pp. 405–406, 409 the properties of parallels in elliptic space are explained by means of the Calculus of Extension (Ausdehnungslehre).

§§ 1–12. SOME FUNDAMENTAL CONSIDERATIONS.

1. *The Quaternionic Manifold.* The four-dimensional space here considered is a point-manifold whose point-elements are uniquely determined by the sets of real variable numbers w, x, y, z, regarded as rectangular coördinates. To real numbers shall correspond always and only real points.

The space may be defined as the domain of the continuous translational and rotational transformation groups expressed in terms of the coördinates. The translations are the linear transformations of the form:

$$w + g, \quad x + a, \quad y + b, \quad z + c;$$

and the rotations are the orthogonal transformations of the type:

$$g_i w + a_i x + b_i y + c_i z \qquad (i = 1, 2, 3, 4).$$

In the quaternion analysis these are respectively additions and multiplications, and the coördinates of the transformed point are, in every case, the resultant coefficients of the fundamental units $1, i, j, k$. The quaternion terms, or factors, being

$$q = g + ai + bj + ck, \quad p = w + xi + yj + zk,$$

translations are represented by the sum

$$p + q = (w + g) + (x + a)i + (y + b)j + (z + c)k,$$

rotations by the products: pq, qp, qpq'^{-1}, etc., provided (in the latter case) the condition $Tq = Tq' =$ a constant be assigned. Thus, in particular,

$$qp = W + Xi + Yj + Zk,$$

where

$$W = gw - ax + by - cz,$$
$$X = aw + gx - cy + bz,$$
$$Y = bw + cx + gy - az,$$
$$Z = cw - bx + ay + gz,$$

and in these equations all the conditions of orthogonality are satisfied if

$$g^2 + a^2 + b^2 + c^2 = 1.$$

Such multiplications obviously constitute an orthogonal group, that is a group of rotations in the quaternionic manifold, rotations having here the same meaning as when expressed in terms of the Cartesian analysis.*

Now it is well known that the continuous translational and rotational transformation groups (expressed in the Cartesian form) constitute the totality of the possible real movements, without distortion, in a parabolic space defined by the variables w, x, y, z; and the quaternion operations do actually reproduce such transformation groups, are therefore competent to interpret the geometry of a parabolic space of four dimensions. But do quaternion additions and multiplications suffice to produce *all* the movements (without distortion) of such a space?

In order to answer this question I assume that a rigid body is fixed by four points that have no special relation to one another, e. g., do not lie in a space of two dimensions, and I then show that, by quaternion additions and multiplications, any one set of four points o_1, a_1, b_1, c_1 can be moved into any second set o', a', b', c' congruent to the first, but otherwise arbitrarily placed. The letters here used denote quaternions except where statement to the contrary is made.

Place the origin at the point defined by o', so that o', regarded as a quaternion, has a zero tensor.

By a first operation of addition applied to each of the points o_1, a_1, b_1, c_1, the additive term being $o' - o_1$, we transform (o_1, a_1, b_1, c_1) into (o', a, b, c) where

$$a, b, c = -o_1 + a_1, -o_1 + b_1, -o_1 + c_1,$$

and the application of the proper test shows that (o', a, b, c) is actually congruent to (o_1, a_1, b_1, c_1)

By a second operation of multiplication (o', a, b, c) can now be transformed into (o', a', b', c'). In fact, the origin being still at o', if it be defined in general that

$$r(a, b, \cdots)r' = (rar', rbr', \cdots),$$

* Cf. KLEIN: loc. cit. and *Vorlesungen über das Ikosaeder* (1884), pp. 35, 36; also CAYLEY: *On Certain Results Relating to Quaternions* in Philosophical Magazine, vol. 26 (1845), pp. 141–145, or Mathematical Papers, vol. 1, pp. 123–126.

it is easily verified that

$$q(o', a, b, c)a^{-1}q^{-1}a' = (o', a', b', c'),$$

where

$$q = \beta'(\gamma' - \gamma) + (\gamma' - \gamma)\beta$$

and

$$\beta, \gamma, \beta', \gamma' = \mathrm{V}ba^{-1}, \mathrm{V}ca^{-1}, \mathrm{V}b'a'^{-1}, \mathrm{V}c'a'^{-1};$$

and again, the application of the proper test shows that (o', a', b', c') is actually congruent to (o', a, b, c).

By means of appropriate quaternion operations we may therefore transport without distortion the elements and configurations of our quaternionic space from any given position to any other.

The system of measurement is parabolic. But one should not forget that the quaternion analysis augmented into a calculus of biquaternions (CLIFFORD, BUCHHEIM), or of octonions (MCAULAY), may suffice to interpret both the elliptic and the hyperbolic forms of a four-dimensional space.

In concluding this preliminary discussion it is pertinent to remark that *the quaternion theory makes its interpretations in strict analogy with the vector interpretations of a parabolic space of three dimensions* and, so far as they are known, takes them for granted. The quaternionic four-dimensional space is therefore the analogue of that three-dimensional space which allows itself to be explained by the Hamiltonian vector analysis.

2. *Nomenclature.* The terms solid, surface, plane, sphere, curve, line, etc., will be used with their ordinary significations in Euclidean geometry.

By space, without qualification, is meant Euclidean space of three dimensions. When spoken of as a locus it may be conveniently called a Euclidean.

By the director of a point, or the quaternion of a point, is meant the directed straight line drawn from the origin to the point. Geometrically interpreted, a quaternion is a director.

The word perpendicular, when not qualified, is used in its ordinary sense; thus two planes are perpendicular to one another when a straight line can be found in one of them which is perpendicular to every straight line in the other.

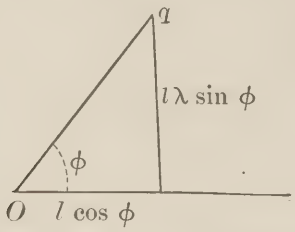
Two planes are said to be hyperperpendicular to one another when every straight line in the one plane is perpendicular to every straight line in the other.

In general, and unless specification to the contrary be made, the letters $a, b, c, d, e, p, q, r, s, t, u, v$ denote quaternions, $h, l, m, n, x, y, z, \theta, \phi, \psi, \chi$ scalars, $\alpha, \beta, \gamma, \delta, \rho, \sigma, \tau$ vectors. The four fundamental quaternion units are denoted by $1, i, j, k$ and their geometrical meaning is: four mutually perpendicular directors of unit length. The Hamiltonian notation is employed throughout.

3. *Amplitude.* When the quaternion is written in the form

$$q = l(\cos\phi + \lambda \sin\phi),$$

in which l is its tensor and λ is a unit vector, ϕ is called its amplitude. The scalar part, $l \cos \phi$, regarded as a director, lies along the scalar axis, and the vector part, $l\lambda \sin \phi$, is the director-perpendicular dropped from the extremity of q to the scalar axis; for this axis is, by definition, perpendicular to i, j, and k, and consequently also perpendicular to every vector. The amplitude is



therefore the angle (more strictly the arc-ratio of angle) between the quaternion, regarded as a director, and the scalar axis. When $\phi = \pi/2$, the quaternion director becomes a vector and is at right angles to the scalar axis; when $\phi = 0$, it is a part of the scalar axis itself.

4. *Geometric Addition.* As applied to a series of quaternions, interpreted as directors in a four-dimensional space, the law of geometric addition is: The sum of any two or more quaternion-directors (whether they be vectors, scalars, or any combinations of these) is the director that extends from the initial to the terminal extremity of the zig-zag formed by so disposing the several director-terms of the sum that all their intermediate extremities are conterminous.

By assuming the existence of a fourth independent direction in space, the geometrical interpretation of a quaternion (as given in § 2) makes this law of geometric addition a mere corollary of the law of vector addition in a three-dimensional domain. We require only to reiterate, for quadrinomials of the form $w + ix + jy + kz$, the statements that are valid for trinomials of the form $ix + jy + kz$.

Since quaternions obey the commutative law of addition, several geometrical steps lead to the same position in whatever order they may be taken, and two directors are identical when either one of them can be derived from the other by a simple translation unaccompanied by any rotation. If a rotation is required in order to bring the two directors into coincidence, they are distinguishable from one another by the fact that their versor parts are distinct. This commutative law in geometric addition implies the validity of the so-called parallel axiom and demands the existence of parallel elements (lines, planes, etc.), parallelism being determined by the usual Euclidean criteria.

5. *Relative Direction.* Let the quaternions q, q', which, for convenience and without loss of generality, may have unit tensors, be written in the binomial forms:

$$q = \cos \phi + \lambda \sin \phi,$$

$$q' = \cos \phi' + \lambda' \sin \phi',$$

and denote by θ the angle they form with one another. The ratio of q to q' is

$$q/q' = (\cos\phi + \lambda \sin\phi)(\cos\phi' - \lambda' \sin\phi'),$$

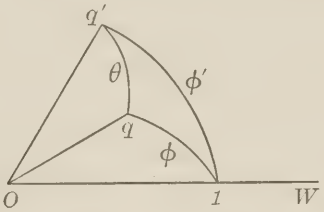
or, in its developed form,

$$q/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi' \mathrm{S}\lambda\lambda' + \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi'\mathrm{V}\lambda\lambda',$$

and

$$\mathrm{S}q/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi'\mathrm{S}\lambda\lambda' = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

We may represent the arcs ϕ, ϕ', θ by the sides of a spherical triangle on the surface of a unit sphere. In such a figure the intersection of the planes of the



arcs ϕ, ϕ' is the scalar axis and the angle of the vectors λ, λ' is the angle between these two planes. Hence, by the cosine formula of spherical trigonometry,

$$\cos\theta = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

Thus the measure of the inclination of q to q' is

$$\mathrm{S}q/q' = \cos\theta.*$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that $\theta = \pi/2$ or an odd multiple of $\pi/2$. Hence:

The necessary and sufficient condition in order that two directors q, q' may be perpendicular to one another is that their ratio shall be a vector. Symbolically,

$$\mathrm{S}\cdot q\mathrm{K}q' = 0.$$

To obtain the condition of parallelism we note that the vector part of q/q' is

$$\mathrm{V}q/q' = \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi'\mathrm{V}\lambda\lambda',$$

and that q and q' are parallel (or identical) if, and only if, their versor parts are either identical, or differ only in algebraic sign; that is, if

$$\phi' = \phi + n\pi, \quad \lambda' = \lambda, \quad \mathrm{V}\lambda\lambda' = 0,$$

where $n = 0$, or a positive integer. But these conditions are fully expressed in the equation

* Cf. HATHAWAY: loc. cit. (1) p. 55.

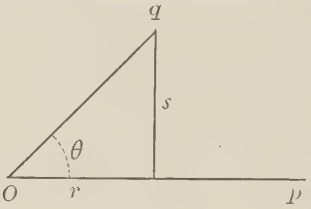
$$\mathrm{V}q/q' = \pm\lambda \sin(\phi - \phi') = 0\,.$$

Hence, the necessary and sufficient condition in order that two directors q, q' *may be parallel is that their ratio shall be a scalar.* Symbolically,

$$\mathrm{V}\cdot q\mathrm{K}q' = 0\,,$$

In the conditions $\mathrm{S}\cdot q\mathrm{K}p = 0$, $\mathrm{V}\cdot q\mathrm{K}p = 0$, the requirement that p and q shall have unit tensors is obviously not necessary.

6. *Binomial Form of* q/p. Let θ be the angle between any two quaternion directors p, q, let s be the director-perpendicular dropped from the terminal ex-



tremity of q to p, and r the director from the origin (intersection of p, q) to the foot of this perpendicular. Then, by geometric addition,

$$q = r + s$$

and

$$q/p = r/p + s/p\,;$$

and s/p, being the ratio of a pair of mutually perpendicular directors, is a vector. Hence, if ρ be defined as the ratio $\mathrm{U}s/\mathrm{U}p$, it is a unit vector and

$$q/p = \frac{\mathrm{T}q}{\mathrm{T}p}(\cos\theta + \rho\sin\theta)\,.$$

7. *Projection.* It is now evident that

$$\mathrm{S}\cdot q\mathrm{K}p = \mathrm{T}q\mathrm{T}p\cos(p, q)\,,$$

and that

$$\mathrm{TV}\cdot q\mathrm{K}p = \mathrm{T}q\mathrm{T}p\sin(p, q)\,;$$

and since obviously

$$\mathrm{S}(\mathrm{V}\cdot q\mathrm{K}p)(\mathrm{U}p\cdot\mathrm{K}p) = 0\,,$$

$(\mathrm{V}\cdot q\mathrm{K}p)\mathrm{U}p$ is perpendicular to p. Hence the identity

$$q \equiv (\mathrm{S}\cdot q\mathrm{K}p)(\mathrm{K}p)^{-1} + (\mathrm{V}\cdot q\mathrm{K}p)(\mathrm{K}p)^{-1}$$

has the following evident interpretation:

$$\frac{\mathrm{S}\cdot q\mathrm{K}p}{\mathrm{T}p}\mathrm{U}p = \text{projection of } q \text{ upon } p\,,$$

$$\frac{\mathrm{V}\cdot q\mathrm{K}p}{\mathrm{T}p}\mathrm{U}p = \text{projection of } q \text{ upon a director perpendicular to } p\,,$$

and the sum of these two projections is q itself.

8. *Rectangular Coördinates.* The propositions of § 5 might have been proved in terms of rectangular coördinates. Suppose the two quaternions given in the quadrinomial form:

$$q = w + ix + jy + kz,$$
$$q' = w' + ix' + jy' + kz'.$$

Their direction cosines are

$$w/l\ ,\ x/l\ ,\ y/l\ ,\ z/l\ ,$$
$$w'/l',\ x'/l',\ y'/l',\ z'/l',$$

where

$$l = \sqrt{w^2 + x^2 + y^2 + z^2},$$
$$l' = \sqrt{w'^2 + x'^2 + y'^2 + z'^2},$$

and the formula for the cosine of the angle between them is

$$\cos\theta = \frac{ww' + xx' + yy' + zz'}{ll'}.$$

The ratio of q to q' may be written in the form

$$q/q' = \frac{(w + ix + jy + kz)(w' - ix' - jy' - kz')}{w'^2 + x'^2 + y'^2 + z'^2}.$$

In the numerator of this fraction the coefficients of 1, i, j, and k are respectively,

$$W = ww' + xx' + yy' + zz',$$
$$X = w'x - wx' + yz' - y'z,$$
$$Y = w'y - wy' + zx' - z'x,$$
$$Z = w'z - wz' + xy' - x'y;$$

and in terms of these q/q' takes the form

$$q/q' = \frac{W + iX + jY + kZ}{l'^2}.$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that

$$ww' + xx' + yy' + zz' = 0, \quad \text{i. e.,} \quad W = 0,$$

or otherwise expressed,

$$q/q' = \text{a vector.}$$

In order that q and q' may be parallel to one another it is necessary and sufficient that

$$\frac{w}{w'} = \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'},$$

that is,

$$iX + jY + kZ = 0,$$

or otherwise expressed,

$$q/q' = \text{a scalar.}$$

9. *Equations of Loci.* In general it will be found that a quaternion equation in one variable has a definite, finite number of solutions and that, if it represent a locus, its coefficients must satisfy certain conditions, a remark in fact that is justified by the observation that the quaternion equation

$$f(q) = 0$$

is equivalent to a system of four scalar equations in four scalar variables. Thus a principal first step in the interpretation of the linear equation as a locus will be the assigning of the conditions necessary and sufficient in order that the equation may have an infinite number of solutions.

10. *Transference of Origin.* By virtue of the law of geometric addition, a locus, given in terms of a variable quaternion p, may be referred to a new origin, whose director is c, and be represented by an equation in terms of a new variable quaternion q, by substituting

$$p = c + q$$

in the given equation. Thus the equation

$$ap + pa^{-1}a\alpha + 2\alpha = 0 \qquad (a^2 = -1)$$

is satisfied by the value $p = a\alpha$ and is referred to a point in the locus itself, as a new origin, by writing

$$p = a\alpha + q;$$

and through this substitution it becomes

$$aq + qa^{-1}a\alpha = 0.$$

In comparing two or more loci, advantage may be taken of this principle in order to simplify one or more of the equations by transferring the origin to a point within one of the loci, or to a point common to two or more of them.

11. *Solution of* $a_1 p + p a_2 = c$. I indicate briefly HAMILTON'S solution of this equation.* Denote the conjugate of a_2 by $\bar{a}_2$† and multiply $a_1 p + p a_2$ progressively by a_1, regressively by $\bar{a}_2$ and add; there results

* *Lectures on Quaternions*, p. 565, or TAIT'S *Treatise on Quaternions*, 3d ed., p. 136.

† This notation is used throughout the paper.

$$\{a_1^2 + a_1(a_2 + \bar{a}_2) + a_2\bar{a}_2\}\, p = a_1 c + c\bar{a}_2 ,$$

or

$$p = (a_1^2 + 2a_1 \mathrm{S}a_2 + a_2\bar{a}_1)^{-1}(a_1 c + c\bar{a}_2) .$$

If nullitats (quaternions involving scalar $\sqrt{-1}$ and having zero tensors) be excluded the equation has in general but one root. For, denote any root by b; then

$$a_1 b + b a_2 = c ,$$

and the original equation may be written in the form

$$a_1(p - b) + (p - b)a_2 = 0 ;$$

and if p be supposed to have any other value than b, for such value $p - b$ does not vanish and

$$a_1 = -(p - b)a_2(p - b)^{-1}$$

is not indeterminate; therefore

$$\mathrm{S}a_1 = -\mathrm{S}a_2 , \quad \mathrm{T}a_1 = \mathrm{T}a_2 .$$

Thus the coefficients are not independent and the equation may be written

$$a_1 p + p a_2 = c ,$$

in which $a_1 = \mathrm{V}a_1$, $a_2 = \mathrm{V}a_2$, and $\mathrm{T}a_1 = \mathrm{T}a_2$.

It follows that unless the conditions: $\mathrm{S}a_1 = -\mathrm{S}a_2$, $\mathrm{T}a_1 = \mathrm{T}a_2$, be satisfied, the equation has but one root. But if it have more than one, it then has an infinite number of roots and represents a locus.

12. As an example of other equations in a variable quaternion p whose solution leads to this linear bilateral form, consider

$$\mathrm{S}ap = m , \quad \mathrm{S}bp = n ,$$

which obviously represent a pair of Euclideans (ordinary spaces of three dimensions). They are equivalent to

$$ap + \bar{p}\bar{a} = 2m , \quad bp + \bar{p}\bar{b} = 2n ,$$

from which is obtained, by a series of sufficiently simple operations,

$$\bar{b}ap - pb\bar{a} = 2(m\bar{b} - n\bar{a}) ,$$

and since $\mathrm{S}\bar{b}a = \mathrm{S}b\bar{a}$ and $\mathrm{T}\bar{b}a = \mathrm{T}b\bar{a} = \mathrm{T}ab$, this may be written

$$\mathrm{UV}\bar{b}a \cdot p - p \cdot \mathrm{UV}b\bar{a} + \frac{2(n\bar{a} - m\bar{b})}{\mathrm{TV}ab} = 0 .$$

Here again is a single equation in a variable quaternion p representing a locus, viz., the intersection of a pair of Euclideans.

§§ 13–16. Loci of Linear Equations.

13. *Planes through the Origin.* If a_1 and a_2 be any two unit vectors and p a variable quaternion such that

$$a_1 p + p a_2 = 0 ,$$

then

$$\mathrm{S}(a_1 + a_2)\mathrm{V}p = 0 .$$

Thus, under the conditions imposed, $\mathrm{V}p$ moves in a plane through the origin perpendicular to $a_1 + a_2$ and can therefore be written in the form

$$\mathrm{V}p = x(a_1 - a_2) + y\mathrm{V}a_1 a_2 ,$$

in which x and y are independent variable scalars; thus p has the form

$$p = x(a_1 - a_2) + y\mathrm{V}a_1 a_2 + \mathrm{S}p .$$

Subject this expression for p to the condition that it satisfy the equation $a_1 p + p a_2 = 0$ and note that the part involving $x(a_1 - a_2)$ vanishes identically. There remains

$$a_1(y\mathrm{V}a_1 a_2 + \mathrm{S}p) + (y\mathrm{V}a_1 a_2 + \mathrm{S}p)\, a_2 = 0 ,$$

and this solved for $\mathrm{S}p$ gives

$$\mathrm{S}p = y(a_1 - a_2)(a_1 + a_2)^{-1}\mathrm{V}a_1 a_2 = y(1 + \mathrm{S}a_1 a_2) ,$$

whence

$$p = x(a_1 - a_2) - y a_1(a_1 - a_2) .$$

Thus, as x and y vary p moves in the plane of the two directors $a_1 - a_2$, $a_1(a_1 - a_2)$ and the equation of this plane is

$$a_1 p + p a_2 = 0 .$$

Or this equation may be written in the form

$$a_1 p - p a^{-1} a_1 a = 0 ,$$

where a is any quaternion satisfying the condition $a_1 a + a a_2 = 0$.

14. *Planes through any Point.* If q satisfy the equation

$$a_1 q + q a_2 + 2a = 0 , \qquad (a_1^2 = a_2^2 = -1),$$

it is immediately evident that $a_1 a = a a_2$ and that therefore

$$a_1 \cdot a_1 a + a_1 a \cdot a_2 + 2a = 0 .$$

Hence the values of q that satisfy this equation can be written in the form

$$q = a_1 a + x(a_1 - a_2) - y a_1(a_1 - a_2) , \tag{1}$$

where x and y are arbitrary scalars. The equation represents a plane through the extremity of $a_1 a$.

* Stringham: loc. cit. (1) (1884), pp. 54–55.

These values of q may also be expressed in either of the forms:

$$q = a_1 a + l a_1^{z}(a_1 - a_2),$$
$$q = a_1 a + a_1 r - r a_2,$$

where l and z are arbitrary scalars and r is an arbitrary quaternion. The verifications are easily made by substitution in the equation.

It is important to observe that since $a_1 a = a a_2$, only two constants, a unit vector and a quaternion, are necessary for the complete determination of a plane. Thus, these constants being a_1 and a, the equation is

$$a_1 \cdot \rho + \rho \cdot a^{-1} a_1 a + 2a = 0.$$

15. *Planes containing given Elements.* Such an equation as the one last written represents any plane in four-dimensional space. For, suppose the plane to be determined by a point a_0 and two straight lines c, e through this point. We may find two unit vectors a_1, a_2 such that

$$a_1 c + c a_2 = 0, \quad a_1 e + e a_2 = 0.$$

In fact the solution of these equations for a_1, a_2 is

$$a_1 = \pm \mathrm{UV} e\bar{c}, \quad a_2 = \pm \mathrm{UV}\bar{e}c,$$

and by substitution in the equations it is found that these versors must have like signs. The vectors a_1, a_2 being thus determined, let $a = -a_1 a_0$ (equivalent to $a_0 = a_1 a$) and consider the equation

$$a_1 \rho + \rho a_2 + 2a = 0.$$

It is the equation of a plane and it is satisfied by a_0, by $a_0 + c$ and by $a_0 + e$, and thus, as required, it represents the plane that contains the point a_0 and the lines c, e.

If the plane be determined by three points, c, d, e, the equations of condition for a_1, a_2 may be written in the form

$$a_1(c - d) + (c - d)a_2 = 0, \quad a_1(c - e) + (c - e)a_2 = 0,$$

and then

$$a_1 = \mathrm{UV}(cd + de + e\bar{c}), \quad a_2 = \mathrm{UV}(\bar{c}d + de + e\bar{c}),$$
$$a = -\tfrac{1}{2}(a_1 c + c a_2) = -\tfrac{1}{2}(a_1 d + d a_2) = -\tfrac{1}{2}(a_1 e + e a_2).$$

16. *The director $a_1 a$ is perpendicular to the plane.* For since $a_1 a = a a_2$,

$$\mathrm{S} a_1 a\{x(a_2 - a_1) - y(a_1 - a_2)a_1\} = 0,$$

the condition for perpendicularity (§ 5). Thus, the equation of a plane through the extremity of, and perpendicular to, a given director a_0 is

$$a_1 q + q a_2 - 2 a_1 a_0 = 0,$$

in which a_1, a_2 are any two unit vectors, satisfying the condition $a_1 a_0 = a_0 a_2$.

It will be convenient to write $(a_1, a_2, 2a)$ as an abbreviation meaning: the plane whose equation is $a_1 q + q a_2 + 2a = 0$ $(a_2 = a^{-1} a_1 a)$.

§§ 17–22. Intersections.

17. *Director of Meeting-point.* It is here shown that two planes,

$$a_1 p + p a_2 + 2a = 0\,, \quad \beta_1 p + p \beta_2 + 2b = 0\,,$$

always meet in at least one point, and the director to this point is found in terms of the constants of the equation. Operate progressively and regressively on the first equation with β_1, β_2, on the second with a_1, a_2 and in each case take the difference of the products and then add these differences together; the result is

$$2p\mathrm{S}(a_1\beta_1 - a_2\beta_2) + 2(\beta_1 a - a\beta_2 + a_1 b - b a_2) = 0\,.$$

Hence, in general

$$p = \frac{\beta_1 a - a\beta_2 + a_1 b - b a_2}{\mathrm{S}(a_2\beta_2 - a_1\beta_1)}\,.$$

If the planes have two points in common they have a line of intersection and a relation between the coefficients is obviously necessary. Hence, in general, two planes meet in one and only one point.

18. *Intersection Lines at the Origin.—In order that the two planes* $(a_1, a_2, 0)$, $(\beta_1, \beta_2, 0)$, *which meet at the origin, may meet in a straight line, it is necessary and sufficient that* $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$.

It is necessary; for, if there be some value of p not zero that satisfies the two equations $a_1 p + p a_2 = 0$, $\beta_1 p + p\beta_2 = 0$, then

$$a_1 = -\,p a_2 p^{-1}\,, \quad \beta_1 = -\,p\beta_2 p^{-1}\,,$$

and therefore

$$\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2\,.$$

It is sufficient; for, if

$$c = \beta_1(x - y a_1)(a_1 + a_2) - (x - y a_1)(a_1 + a_2)\beta_2\,,$$

where x and y are arbitrary scalars, then the condition $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$ suffices to make $a_1 c + c a_2 = 0$, $\beta_1 c + c\beta_2 = 0$, and thus the two planes meet in c.

19. *Cosine of a Dihedral Angle.* Two planes are supposed to meet in a straight line at the origin. Let u, v be two unit directors drawn, u in the plane $(a_1, a_2, 0)$, v in the plane $(\beta_1, \beta_2, 0)$ perpendicular to their line of intersection. Then $a_1 u$ lies in $(a_1, a_2, 0)$ and $\beta_1 v$ in $(\beta_1, \beta_2, 0)$, and identically

$$\mathrm{S}a_1 u\bar{u} = \mathrm{S}\beta_1 v\bar{v} = 0\,,$$

conditions sufficient to make $a_1 u$ and $\beta_1 v$ perpendicular respectively to u and v.

Hence a_1u and β_1v are two expressions for the intersection line of the two planes and $a_1u = \pm\beta_1v$, whence

$$S u\bar{v} = \pm S a_1\beta_1 = \pm S a_2\beta_2 . \qquad (\S 18)$$

These are the cosines of the dihedral angles formed by the two planes.

The planes are perpendicular to one another if $Sa_1\beta_1 = Sa_2\beta_2 = 0$.

20. *Intersection Lines in General.—In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may meet in a straight line it is necessary and sufficient that*

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0 .$$

If the two planes meet in a straight line at a finite distance it will be possible to transfer the origin to a point in this line and thus reduce the equations to the form

$$a_1q + qa_2 = 0 , \quad \beta_1q + q\beta_2 = 0 .$$

Hence (§18) a necessary condition for a straight-line intersection is $Sa_1\beta_1 = Sa_2\beta_2$. But when this condition is satisfied the expression (§ 17)

$$\frac{a_1b - ba_2 + \beta_1a - a\beta_2}{S(a_2\beta_2 - a_1\beta_1)} ,$$

for a *point* of intersection of the two planes, is determinate and infinite unless the numerator vanish. Hence the condition

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0$$

is necessary.

This condition requires that $Sa_1\beta_1 = Sa_2\beta_2$, but the converse of this statement is not true; for if

$$f = a_1b - ba_2 , \quad g = \beta_1a - a\beta_2 , \quad m = S(a_2\beta_2 - a_1\beta_1) ,$$

then the identical relations

$$a_1f + fa_2 = 0 , \quad a_1g + ga_2 + 2am = 0 ,$$

$$\beta_1g + g\beta_2 = 0 , \quad \beta_1f + f\beta_2 + 2bm = 0 ,$$

put in immediate evidence the fact that, x being any scalar, $f = xg$ makes $m = 0$, and *vice versa*, but $m = 0$ does not involve $f + g = 0$.

The sufficiency of the condition $f + g = 0$ will now be proved if it be shown that the vanishing of $f + g$ makes it possible to assign a singly infinite series of quaternion directors which satisfy the equations of the two planes. But identically,

$$a_1V a\bar{b}\cdot f + V a\bar{b}\cdot fa_2 = af\bar{f} ,$$

$$\beta_1V b\bar{a}\cdot g + V b\bar{a}\cdot g\beta_2 = bg\bar{g} ,$$

and, if $f + g = 0$, also

$$\beta_1V a\bar{b}\cdot f + V a\bar{b}\cdot f\beta_2 = bf\bar{f} ,$$

$$a_1xf + xfa_2 = 0 , \quad \beta_1xf + xf\beta_2 = 0 ,$$

where x is an arbitrary scalar. Hence, *provided f and g be not separately zero*, $(x - 2\mathrm{V}ab)/f$ represents a singly infinite series of quaternions possessing the required property.

Thus the condition $f = -g \neq 0$ determines that the two planes meet in a straight line and enables us to assign as its equation

$$\rho = (x - 2\mathrm{V}ab)/f.$$

Here neither f nor $\mathrm{V}ab$ can vanish separately; for if $b = ya$, then $f = y(a_1a - aa_2) = 0$, and if $f = 0$ both a and b represent the same director, namely the intersection of the two planes $(a_1, -a_2, 0)$, $(\beta_1, -\beta_2, 0)$, which can only happen when b is a numerical multiple of a.

Suppose then that $f = g = 0$; there are two cases:

(1) $a_1b = ba_2 \neq \beta_1a = a\beta_2$; (2) $a_1b = ba_2 = \beta_1a = a\beta_2$.

(1) If $a_1b = ba_2 \neq \beta_1a = a\beta_2$, then the above expression for ρ may be replaced by

$$\rho = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1a - a_1b + xa)$$

which, as may be easily verified, reduces the equations of $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to identities for all (scalar) values of x. This is again the equation of a straight line. (I have not proved that $\mathrm{V}a_1\beta_1$ is not zero.)

(2) If $a_1b = ba_2 = \beta_1a = a\beta_2$, a formal solution is

$$\rho = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1 + x)a,$$

an expression which again reduces the equations of the two planes to identities. But here, since b is a multiple of a and since now $a_1\beta_1 = -ab^{-1}$, we have $\mathrm{V}a_1\beta_1 = 0$ and therefore

$$\beta_1 = \pm a_1, \quad \beta_2 = \pm a_2,$$

and the signs must be $++$ or $--$. Hence $\rho = \infty$. This is the case of parallel planes, as will be shown in the sequel (§ 29).

Thus, in whatever way $f + g$ becomes zero the two planes meet in a straight line, either at a finite or at an infinite distance.

Incidentally, either $-2\mathrm{V}ab/f$ or $(\mathrm{V}a_1\beta_1)^{-1}(\beta_1a - a_1b)$, the former in the general case, the latter when $a_1b = ba_2 \neq \beta_1a = a\beta_2$, is determined as the director-perpendicular from the origin to the intersection-line of the two planes. This is made evident by applying the test of (§ 5).

21. *Intersections at Infinity.* If the two planes meet only once at an infinite distance, we must have $(f + g)/m = 0$, and $f + g \neq 0$ (§ 20), and these conditions are obviously sufficient. Hence, taking account of § 20 (2):

For the two planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to meet at infinity, the necessary and sufficient conditions are:

if at a point, $\quad \mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2, \quad f + g \neq 0;$

if in a line, $\quad \beta_1 = \pm a_1, \quad \beta_2 = \pm a_2.$

22. *Normals.* It is easily verified that:

(1) The perpendicular distance (director-normal) between the two planes $(a_1, a_2, 2a)$ and $(a_1, a_2, 2b)$ is, in both magnitude and direction, $a_1(a - b)$.

(2) The director normal from the extremity of c to the plane $(a_1, a_2, 2b)$ is

$$\tfrac{1}{2}a_1(2a + a_1c + ca_2).$$

(3) It has been shown in § 21 that the director-normal from the origin to the intersection of $(a_1, a_2, 2a)$ and $(\beta_1, \beta_2, 2b)$ is

$$(ab - ba)/(ba_1 - a_2b)$$

the conditions $f + g = 0$, $f \neq g$ being here essential.

§§ 23–27. Divergence of Two Planes and its Measure.

23. *Isoclinal Angles.* Two planes meet, in general, in a point or in a straight line (§§ 17, 20). It is always possible, as will presently appear, to pass through any point common to two given planes other (transversal) planes which meet the former in straight lines and form with them equal opposite interior dihedral angles. The plane angle formed by the edges of the two dihedrals, whose plane as viewed from a point between the two given planes is equally inclined to them, will be called their *isoclinal angle.* This angle is an appropriate index (measure) of the amount (rapidity) of the divergence of the two planes under a special aspect, namely, from the vertex and in the direction of the sides of the isoclinal angle. Its variations afford the means for determining certain important relations of planes to one another.

The given planes being $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$, we transfer the origin to the point of, or to a point in, their intersection. Their equations become (§ 13)

$$a_1\rho + \rho a_2 = 0, \quad \beta_1\rho + \rho\beta_2 = 0;$$

and x, y being continuous scalar variables,

$$u = a_1'(a_1 - a_2), \quad v = \beta_1''(\beta_1 - \beta_2)$$

represent two directors lying in them, u in the first, v in the second (§ 14). I enquire whether and under what conditions the plane angle formed by u and v may have a maximal or a minimal value. Without loss of generality we may suppose that $\mathrm{T}u = \mathrm{T}v = 1$.

The first and second partial derivatives of $\mathrm{S}u\bar{v}$ with respect to x and y are

$$\frac{\partial \mathrm{S}u\bar{v}}{\partial x} = \frac{\pi}{2}\mathrm{S}a_1u\bar{v}, \quad \frac{\partial \mathrm{S}u\bar{v}}{\partial y} = -\frac{\pi}{2}\mathrm{S}\beta_1u\bar{v},$$

$$\frac{\partial^2 \mathrm{S}u\bar{v}}{\partial x^2} = -\frac{\pi^2}{4}\mathrm{S}u\bar{v}, \quad \frac{\partial^2 \mathrm{S}u\bar{v}}{\partial y^2} = -\frac{\pi^2}{4}\mathrm{S}u\bar{v},$$

and

$$\frac{\partial^2 \mathrm{S}u\bar{v}}{\partial x\,\partial y} = -\frac{\pi^2}{4}\mathrm{S}\beta_1a_1u\bar{v}.$$

The two second partial derivatives, with respect to x, and with respect to y, being negative as compared with the function itself, the conditions for a maximum or a minimum of $S u\bar{v}$ are

$$Sa_1 u\bar{v} = 0, \quad S\beta_1 u\bar{v} = 0,$$

and

$$S^2 u\bar{v} > S^2\beta_1 a_1 u\bar{v}, \quad \text{or} \quad S^2 u\bar{v} < S^2\beta_1 a_1 u\bar{v}.$$

Thus $S u\bar{v}$, *and therefore also the angle of* u, v, *will have maxima and minima if there exist an inequality of the form:*

$$|S u\bar{v}| \neq |Sa_1 u\bar{\beta_1 v}|.$$

Let it be supposed, for the moment, that the directors u, v are fixed by the conditions thus imposed (it will be shown presently that this is actually the case) and let u and v be turned in their respective planes about the origin through the same angle $\frac{1}{2}\pi\theta$ by multiplying the first by a_1^θ, and the second by β_1^θ, where θ is a scalar variable.

(1) The essential preliminary condition: *The plane of* $a_1^\theta u$ *and* $\beta_1^\theta v$ *shall intersect both* $(a_1, a_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *in straight lines*, is here fulfilled; for γ_1, γ_2 being $= UVa_1^\theta u\bar{v}\beta_1^{-\theta}$, $UV\bar{u}a_1^{-\theta}\beta_1^\theta v$, the equation of this plane is, by § 15 (1),

$$\gamma_1 \rho + \rho\gamma_2 = 0,$$

and the sufficient conditions (§ 18)

$$Sa_1\gamma_1 = Sa_2\gamma_2, \quad S\beta_1\gamma_1 = S\beta_2\gamma_2,$$

are here satisfied.

(2) *The angle of* $a_1^\theta u$ *and* $\beta_1^\theta v$ *is isoclinal to* $(a_1, a_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *for all values of* θ. For we may write

$$a_1^\theta = x + ya_1, \quad \beta_1^\theta = x + y\beta_1 \qquad (x^2 + y^2 = 1),$$

and then, the conditions $Sa_1 u\bar{v} = S\beta_1 u\bar{v} = 0$ being satisfied,

$$Sa_1\gamma_1 = xyS(\beta_1 a_1 u\bar{v} - u\bar{v}) = -S\beta_1\gamma_1$$
$$= Sa_2\gamma_2 = -S\beta_2\gamma_2,$$

that is, the corresponding exterior-interior dihedral angles on the same side of the transversal are equal. This isoclinal angle may thus be treated as a function of θ.

(3) Let the values of θ that make the angle of $a_1^\theta u$ and $\beta_1^\theta v$ a maximum or a minimum be sought. The first and second derivatives of $Sa_1^\theta u\bar{v}\beta_1^{-\theta}$ with respect to θ are

$$\frac{dSw}{d\theta} = \frac{\pi}{2} S(a_1 - \beta_1)w \qquad (w = a_1^\theta u\bar{v}\beta_1^{-\theta}),$$

$$\frac{d^2 Sw}{d\theta^2} = -\frac{\pi^2}{2} S(1 + \beta_1 a_1)w,$$

and by virtue of the previously assumed conditions $Sa_1u\bar{v} = S\beta_1u\bar{v} = 0$, the first derivative vanishes for all integral values of θ.

When $\theta = 0$ or an even number, the second derivative is a negative quantity if $Su\bar{v} > Sa_1u\overline{\beta_1v}$, and these (as previously found) are the conditions that make $Su\bar{v}$ a maximum.

When $\theta =$ an odd number, the second derivative is a positive quantity if $Su\bar{v} > Sa_1u\overline{\beta_1v}$, and these are the conditions that make $Su\bar{v}$ a minimum.

It is immediately evident that if there be maxima and minima, there are two of each, occurring alternately at intervals of ninety degrees; for the successive angles are:

$$\angle(a_1^\theta u\,,\ \beta_1^\theta v) \qquad (\theta = 0,\ 1,\ 2,\ 3),$$

and $Sa_1^\theta u\bar{u} = S\beta_1^\theta v\bar{v} = 0$ if θ be an odd integer.

It will be shown presently (§ 24) that non-integral values of θ do not give rise to maximal or minimal isoclinal angles. The above enumeration is therefore exhaustive.

(4) *The two planes of the maximal and minimal isoclinal angles, and these only, are orthogonal to both of the given planes.* For, in order that the conditions for perpendicularity (§ 19) may be satisfied, namely:

$$Sa_1\gamma_1 = Sa_2\gamma_2 = S\beta_1\gamma_1 = S\beta_2\gamma_2 = 0\,,$$

it is necessary and sufficient that x or $y = 0$ (in $a_1^\theta = x + ya_1$, $\beta_1^\theta = x + y\beta_1$, of (2)), and these are precisely the condition $\theta =$ an integer.

(5) If the terms of $Su\bar{v} > Sa_1u\beta_1\bar{v}$ be interchanged, $Su\bar{v}$ becomes a minimum and $Sa_1u\beta_1\bar{v}$ a maximum. Such an interchange, however, is merely equivalent to assigning $u' = a_1u$, $v' = \beta_1v$ as the two director boundaries of the minimal and a_1u', β_1v' as the director boundaries of the maximal angle. And unless $Su\bar{v} = Sa_1u\overline{\beta_1v}$, an inequality of the form $Su\bar{v} > Sa_1u\beta_1\bar{v}$ may be assumed to exist. Hence if the conditions

$$Sa_1u\bar{v} = 0\,,\ \cdot S\beta_1u\bar{v} = 0\,,\quad Su\bar{v} \neq Sa_1u\beta_1\bar{v}$$

be satisfied, either $Su\bar{v}$ is a maximum and $Sa_1u\overline{\beta_1v}$ a minimum, or *vice versa*.

(6) It remains to show that the conditions $Sa_1u\bar{v} = S\beta_1u\bar{v} = 0$ suffice to determine a convenient measure of the divergence of the two planes. In another form, $Sa_1u\bar{v} = 0$ is

$$a_1u\bar{v} - v\bar{u}a_1 = 0\,,$$

which is also

$$\beta_1a_1u\bar{v} - \beta_1v\bar{u}a_1 = 0\,.$$

But

$$\beta_1a_1u\bar{v} + v\bar{u}a_1\beta_1 = 2S\beta_1a_1u\bar{v} = \beta_1v\bar{u}a_1 + v\bar{u}a_1\beta_1\,.$$

Multiplied by $u\bar{v}$ this last equation is

$$2u\bar{v}\mathrm{S}\beta_1 a_1 u\bar{v} = u\bar{v}v\beta_2 a_2\bar{u} + v\bar{v}u\bar{u}a_1\beta_1,$$

whence, by equating scalar parts and dividing by 2,

$$\mathrm{S}u\bar{v}\cdot\mathrm{S}a_1 u\beta_1 v = -\tfrac{1}{2}\mathrm{S}(a_1\beta_1 + a_2\beta_2).*$$

There is thus obtained, as the measure of divergence of the two planes, the product of the cosines of their numerical maximal and minimal isoclinal angles.

24. *The Ultimate Criteria of Maxima and Minima.* The higher odd-order derivatives of $\mathrm{S}w$, $= \mathrm{S}a_1^\theta u\bar{v}\beta_1^{-\theta}$ are

$$\frac{d^{2n-1}\mathrm{S}w}{d\theta^{2n-1}} = (-1)^{n-1}\frac{\pi^{2n-1}}{2}\mathrm{S}(a_1 - \beta_1)w,$$

and the even-order derivatives are

$$\frac{d^{2n}\mathrm{S}w}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}\mathrm{S}(1 + \beta_1 a_1)w.$$

If we write

$$(a_1^\theta,\ \beta_1^\theta) = (x + ya_1,\ x + y\beta_1) \qquad (x^2 + y^2 = 1),$$

and maintain intact the conditions $\mathrm{S}a_1 u\bar{v} = \mathrm{S}\beta_1 u\bar{v} = 0$, these higher derivatives become

$$\frac{d^{2n-1}\mathrm{S}w}{d\theta^{2n-1}} = (-1)^{n-1}\pi^{2n-1}xy\mathrm{S}(a_1 u\overline{\beta_1 v} - u\bar{v}),$$

$$\frac{d^{2n}\mathrm{S}w}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}(y^2 - x^2)\mathrm{S}(a_1 u\overline{\beta_1 v} - u\bar{v}).$$

Hence the derivatives of odd order vanish if, and only if, $x = 0$, or $y = 0$, or $\mathrm{S}a_1 u\overline{\beta_1 v} = \mathrm{S}u\bar{v}$; and if $\mathrm{S}a_1 u\overline{\beta_1 v} = \mathrm{S}u\bar{v}$ all the derivatives of even order vanish. The immediate consequence is:

Non-integral values of θ (corresponding to $x \neq 0$, $y \neq 0$) give rise to neither maxima nor minima of the isoclinal angle $(a_1^\theta u, \beta_1^\theta v)$ and the enumeration of § 23 (3) *is complete.*

If the equations of the two planes be in the general form, referred to an arbitrary origin, the common vertex of their isoclinal angles is their point of meeting, or a point in their line of intersection, and c being the director to this point, we may assign as general solutions of their equations: $c + a_1^\theta u$ and $c + \beta_1^\theta v$ respectively, where u and v are determined by the foregoing conditions for maxima and minima. The final form of our criterion then is:

The conditions necessary and sufficient in order that the two planes

* STRINGHAM: loc. cit. (3) (1888), p. 64. Of course the formulæ leading up to this result had been determined prior to the date August, 1888. Cf. HATHAWAY: loc. cit. (2) (1897), § 45, p. 10.

$(a_1, \beta_1, 2a)$, $(\beta_1, \beta_2, 2b)$ *may have maximal and minimal isoclinal angles are: that there exist solutions* $c + a_1^{\theta}u$ *and* $c + \beta_1^{\theta}v$, *of their respective equations, such that*

$$\mathrm{S}a_1u\bar{v} = 0\,, \quad \mathrm{S}\beta_1u\bar{v} = 0\,, \quad \mathrm{S}u\bar{v} \neq \mathrm{S}a_1u\bar{\beta}_1\bar{v}\,.$$

25. *The Maximal and Minimal Angles Determined.* It has been seen (§ 23 (4)) that the planes of the maximal and minimal isoclinal angles, and these only, cut both the given planes orthogonally. But the conditions of perpendicularity will be here satisfied by writing as the equations of these two orthogonal transversals,

$$\gamma_1\rho + \rho\gamma_2 = 0, \quad \gamma_1\rho - \rho\gamma_2 = 0\,,$$

where

$$\gamma_1 = \mathrm{UV}a_1\beta_1\,, \quad \gamma_2 = \mathrm{UV}a_2\beta_2\,.$$

Hence the four intersections of these two planes with $(a_1, a_2, 0)$ and $(\beta_1, \beta_2, 0)$ must be the directors hitherto designated by u, v, a_1u, β_1v; and we may write (§ 18)

$$u = a_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)a_2\,,$$

$$v = \beta_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\beta_2\,,$$

$$a_1u = u' = a_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)a_2\,,$$

$$\beta_1v = v' = \beta_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)\beta_2\,;$$

and we may verify independently the necessary conditions

$$\mathrm{S}a_1u\bar{v} = 0\,, \quad \mathrm{S}\beta_1u\bar{v} = 0\,, \quad \mathrm{S}a_1u'\bar{v}' = 0\,, \quad \mathrm{S}\beta_1u'\bar{v}' = 0\,,$$

which are in fact identities if u, v, u', v' have the values above assigned.

The further conditions that u, v shall be perpendicular to u', v' (§ 23 (3)) are here also satisfied, for the planes $(\gamma_1, \gamma_2, 0)$, $(\gamma_1, -\gamma_2, 0)$ are hyperperpendicular to one another (§ 28); or it may be verified independently that

$$\mathrm{S}u\bar{u}' = 0\,, \quad \mathrm{S}v\bar{v}' = 0\,.$$

This determination fails if either $\beta_1 = \pm a_1$ or $\beta_2 = \pm a_2$, for then either $\gamma_1 = 0$, or $\gamma_2 = 0$; but these are the conditions for the failure of maxima and minima (§ 26).

26. *Failure of the Condition* $\mathrm{S}u\bar{v} \neq \mathrm{S}a_1u\bar{\beta}_1\bar{v}$. From the conditions:

$$\mathrm{S}a_1u\bar{v} = \mathrm{S}a_2\bar{v}u = 0\,, \quad \mathrm{S}\beta_1u\bar{v} = \mathrm{S}\beta_2\bar{v}u = 0\,,$$

it follows, provided $a_1 \neq \beta_1$ and $a_2 \neq \beta_2$, that $\mathrm{V}u\bar{v}$ and $\mathrm{V}\bar{v}u$ are numerical multiples of $\mathrm{V}\beta_1a_1$ and $\mathrm{V}a_2\beta_2$ respectively, and (assuming $\mathrm{T}u = \mathrm{T}v = 1$) we may write

$$u\bar{v} = \cos\phi \pm \delta_1 \sin\phi \qquad (\delta_1 = \mathrm{UV}\beta_1\alpha_1),$$

$$\bar{v}u = \cos\psi \pm \delta_2 \sin\psi \qquad (\delta_2 = \mathrm{UV}\alpha_2\beta_2).$$

We may also assume

$$\beta_1\alpha_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

$$\alpha_2\beta_2 = -\cos\epsilon_2 + \delta_2 \sin\epsilon_2.$$

Then

$$\mathrm{S}(1 + \beta_1\alpha_1)u\bar{v} = 2\sin\frac{\epsilon_1}{2}\sin\left(\frac{\epsilon_1}{2} \pm \phi\right),$$

$$\mathrm{S}(1 + \alpha_2\beta_2)\bar{v}u = 2\sin\frac{\epsilon_2}{2}\sin\left(\frac{\epsilon_2}{2} \pm \psi\right),$$

and, therefore, in order that the condition $\mathrm{S}u\bar{v} \neq \mathrm{S}\alpha_1 u\beta_1 v$ may fail it is necessary and sufficient that

$$\epsilon_1 = 2n_1\pi \text{ or } 2n_1\pi \pm \phi,$$

and

$$\epsilon_2 = 2n_2\pi \text{ or } 2n_2\pi \pm \psi,$$

or, in equivalent terms,

$$\beta_1\alpha_1 = -1 \text{ or } -(v\bar{u})^2 \text{ or } -(u\bar{v})^2,$$

and

$$\alpha_2\beta_2 = -1 \text{ or } -(\bar{u}v)^2 \text{ or } -(\bar{v}u)^2,$$

and there are nine combinations. Any combination of the type $\beta_1\alpha_1 = -(v\bar{u})^2$, $\alpha_2\beta_2 = -(\bar{u}v)^2$, however, leads to results identical with those obtained from $\beta_1\alpha_1 = -1$, $\alpha_2\beta_2 = -1$; for, from any of the pairs of equations of this type and from § 23 (6) follows

$$\mathrm{S}(u\bar{v})^2 = -\tfrac{1}{2}\mathrm{S}(\alpha_1\beta_1 + \alpha_2\beta_2)^2 = \mathrm{S}^2u\bar{v},$$

an interchange of factors under S being here permissible; thus $\mathrm{V}u\bar{v} = 0$, $u = \pm v$, and in fact

$$\beta_1\alpha_1 = -1, \quad \alpha_2\beta_2 = -1.$$

But also, this combination is a particular case (obtained by making $u = \pm v$) of either of the remaining two and thus the two alternative conditions

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2(\bar{u}v)^2 \text{ or } \alpha_2(\bar{v}u)^2,$$

and

$$\beta_2 = \alpha_2, \quad \beta_1 = (v\bar{u})^2\alpha_1 \text{ or } (u\bar{v})^2\alpha_1,$$

account for all the cases that can arise. The specific value of $u\bar{v}$ is here undetermined and may vary from a unit scalar to a unit vector value.

It has been tacitly assumed that α_1, β_1 in the one case, and α_2, β_2 in the other, are both essentially positive, but it is evident that the algebraic signs in the equations of the two planes can always be so disposed that this shall be the fact. The conclusion, in its general form, is:

(1) *In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may cease to have maximal and minimal isoclinal angles it is necessary that* $\beta_1 = \pm a_1$, *or* $\beta_2 = \pm a_2$ *and any one of these conditions is sufficient.*

(2) *The isoclinal angle* $(a_1 u, \beta_1 v)$ *is constant under all variations of* θ; this is apparent in the equations

$$\mathrm{S}a_1^\theta u\bar{v}\beta_1^{-\theta} = \mathrm{S}u\bar{v}\,, \quad \mathrm{S}ua_2^{-\theta}\beta_2^\theta\bar{v} = \mathrm{S}u\bar{v}\,,^*$$

the first of which is a consequence of $\beta_1 = a_1$, the second of $\beta_2 = a_2$.

26. *An Example.* The following example illustrates the failure of the conditions for maximal and minimal isoclinal angles. Suppose the equations of the two planes to be

$$ap + pa = 0\,, \quad ap + p\beta = 0\,.$$

The first is equivalent to

$$2a\mathrm{S}p + 2\mathrm{S}a\mathrm{V}p = 0\,,$$

which involves the two equations

$$\mathrm{S}p = 0\,, \quad \mathrm{S}a\mathrm{V}p = 0\,.$$

Hence the values of p that satisfy the first equation represent vectors perpendicular to a and may be written in the form

$$p = xa^\theta\gamma\,,$$

in which x and θ are scalar variables and γ is a vector constant, and the condition $\mathrm{S}a\gamma = 0$ is necessary.

The general values of p that satisfy the second equation may have the form

$$p = xa^\theta(a - \beta)\,.$$

If now $u = \gamma$ and $v = a - \beta$, then

$$\mathrm{S}au\bar{v} = \mathrm{S}a\gamma(\beta - a)\,,$$

which vanishes if $\mathrm{S}a\gamma\beta = 0$, that is, if γ be chosen coplanar with a and β. Hence, with this choice of γ made, the directors γ and $a - \beta$ satisfy the first two conditions for a maximal or a minimal isoclinal angle. But

$$\mathrm{S}a^\theta\gamma(\beta - a)a^{-\theta} = \mathrm{S}\gamma(\beta - a) = \mathrm{S}u\bar{v}$$

for all values of θ and the third condition fails. The isoclinal angle $(a^\theta u, a^\theta v)$ remains unchanged during any variations of θ. In other words, every line in either plane has the same inclination to the other plane.

* Note that $a_1^\theta u = ua_2^{-\theta}$, $\beta_1^\theta v = v\beta_2^{-\theta}$, and therefore $\mathrm{S}a_1^\theta u\bar{v}\beta_1^{-\theta} = \mathrm{S}ua_2^{-\theta}\beta_2^\theta\bar{v}$.

The angle of this inclination (constant isoclinal angle) is represented at C, in the accompanying figure. The equation

$$ap + pa = 0$$

represents a plane through O perpendicular to α, lying wholly within vector space (a space containing only vectors), and

$$ap + p\beta = 0$$

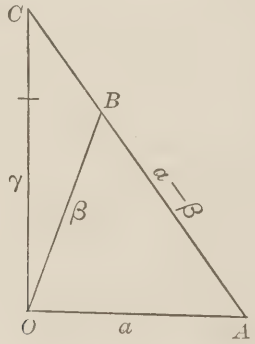
is the equation of a plane that intersects this vector space in a straight line through O parallel to $\alpha - \beta$.

§§ 28, 29. Some Special Cases.

28. *Perpendicularity.* If the maximal isoclinal angle, say $\angle(a_1 u, \beta_1 v)$, be a right angle, then not only $\mathrm{S}a_1 u\bar{v} = 0$, but also $\mathrm{S}a_1 u\bar{\beta}_1\bar{v} = 0$, and

$$\mathrm{S}a_1 u\bar{v}\beta_1^{-\theta} = 0$$

for all values of θ; that is, $a_1 u$, a director in one of the planes, is perpendicular to $\beta_1^{\theta} v$, any director (through their meeting point) in the other. This is perpendicularity according to the ordinary definition, though not of the ordinary sort along a line of intersection, for the planes may not meet in a line.



Conversely, if the two planes be perpendicular to one another it must be possible to assign in either of them a director that shall be perpendicular to every director in the other, say $a_1 u$ and $\beta_1 v$, such that

$$\mathrm{S}a_1 u\bar{v}\beta_1^{-y} = 0, \quad \mathrm{S}a_1^{x} u\bar{v}\beta_1 = 0,$$

where x and y are arbitrary scalars; and it follows (by assuming in succession $x, y = 0, 1$) that

$$\mathrm{S}a_1 u\bar{v} = \mathrm{S}\beta_1 u\bar{v} = \mathrm{S}\beta_1 a_1 u\bar{v} = 0.$$

These conditions make $\angle(a_1 u, \beta_1 v)$ a maximal isoclinal angle if $\mathrm{S}u\bar{v} \neq 0$.

But $S\beta_1 a_1 u\bar{v}$ is a factor in $S(a_1\beta_1 + a_2\beta_2)$ and therefore:

In order that the planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may satisfy the ordinary definition of perpendicularity it is necessary and sufficient that*

$$S(a_1\beta_1 + a_2\beta_2) = 0.$$

There are three further distinct criteria to be considered, giving rise to three kinds of perpendicularity.

(1) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} \neq 0$ or 1. Resume the equations of § 26 for $u\bar{v}$ and $\beta_1 a_1$, omitting the ambiguous sign not here needed; we have

$$u\bar{v} = \cos\phi + \delta_1 \sin\phi, \quad \beta_1 a_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

and therefore

$$Sa_1 u\overline{\beta_1 v} = -\cos(\epsilon_1 + \phi) = 0,$$

$$\epsilon_1 + \phi = \frac{\pi}{2}.$$

Hence $Sa_1\beta_1 \neq Sa_2\beta_2$ *and the planes meet only in a point.* According as $\angle(\beta_1, a_1)$ is large or small the minimal isoclinal angle is small or large. The condition may be stated in the form:

$$Sa_1\beta_1 = -Sa_2\beta_2 \neq 0, \text{ or } \pm 1.$$

(2) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} = 1$. Here $\phi = 0$, $\epsilon_1 = \pi/2$, and $\beta_1 a_1$ is a vector; hence

$$Sa_1\beta_1 = Sa_2\beta_2 = 0,$$

is the condition necessary and sufficient for simple perpendicularity along a line of intersection (the ordinary kind of perpendicularity); provided also, the planes being $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$, the further condition

$$a_1 b - b a_2 + \beta_1 a - a\beta_2 = 0$$

is satisfied (§§ 20, 21).

(3) $Sa_1 u\beta_1 v = Su\bar{v} = 0$. Here $u\bar{v} + v\bar{u} = 0$, $(u\bar{v})^2 = -1$, and therefore

$$\beta_1 a_1 = \pm 1, \quad a_2\beta_2 = \mp 1,$$

that is, $\beta_1 = \mp a_1$ and $\beta_2 = \pm a_2$, the combinations of signs being either $-+$ or $+-$.

If either of these pairs of conditions be assigned, that is, if u' and v' satisfy the equations

$$a_1 u' + u' a_2 = 0, \quad a_1 v' - v' a_2 = 0,$$

we may write (§ 14)

$$u' = a_1^x(a_1 - a_2), \quad v' = a_1^y(a_1 + a_2),$$

in which x and y are arbitrary scalars, and then

$$Su'\bar{v}' = -Sa_1^{x-y}(a_1 a_2 - a_2 a_1),$$

and this is zero for all values of x and y. The geometrical interpretation of this result is that *every line in either plane is perpendicular to every line in the other.* In other words:

The necessary and sufficient conditions for the hyperperpendicularity of the planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *are*

$$\beta_1 = \pm a_1, \quad \beta_2 = \mp a_2;$$

or, expressed in another form,

$$Sa_1\beta_1 = -Sa_2\beta_2 = \pm 1.$$

The conditions for maximal and minimal isoclinal angles fail.

29. *Parallelism.* If two planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ be parallel, the pairs of directors $a_1^\theta u$, $\beta_1^\theta v$* that form their isoclinal angles are parallel for all values of θ, and in particular, if $Tu = Tv = 1$,

$$S\bar{u}\bar{v} = Sa_1u\bar{\beta}_1v = 1,$$

whence

$$S(a_1\beta_1 + a_2\beta_2) = -2.$$

But also, they will have no point of intersection except at an infinite distance, and therefore

$$\frac{a_1b - ba_2 + \beta_1a - a\beta_2}{S(a_2\beta_2 - a_1\beta_1)} = \infty,$$

and the further condition $Sa_2\beta_2 = Sa_1\beta_1$ is necessary. Hence, for the two planes to be parallel it is necessary that

$$Sa_2\beta_2 = Sa_1\beta_1 = -1.$$

This condition is also sufficient, for, if it is satisfied, it immediately follows that $S(a_1\beta_1 + a_2\beta_2) = -2$ and $S\bar{u}\bar{v} = Sa_1u\beta_1v = 1$. Therefore:

(1) *In order that* $(a_1, a_2, 2a)$ *and* $(\beta_1, \beta_2, 2b)$ *may be parallel it is necessary and sufficient that*

$$a_1 = \beta_1, \quad a_2 = \beta_2.$$

(It is understood that the signs before a_1, a_2 in their respective equations are both positive.)

It is immediately evident that the condition

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0$$

is satisfied. This I interpret as meaning that two parallel planes meet in a straight line at infinity.

* These directors are supposed to be placed in their respective planes without reference to a particular origin.

(2) When the conditions for parallelism are satisfied,

$$a_1 a - a a_2 = 0, \quad a_1 b - b a_2 = 0,$$

and we may write

$$a = m a^{\theta}(a_1 + a_2), \quad b = n a_1^{\phi} a,$$

where m, n, θ, ϕ are arbitrary scalars, and the equation of $(\beta_1, \beta_2, 2b)$ becomes

$$a_1 \rho + \rho a_2 + 2 n a_1^{\phi} a = 0.$$

It represents a doubly infinite series of planes parallel to $(a_1, a_2, 2a)$, one series being obtained by varying n, the other by varying ϕ.

(3) When the point of intersection of a pair of planes is moved to an infinite distance, they have in general at the limit a point of intersection at infinity. But if the conditions for maximal and minimal isoclinal angles fail these angles become ultimately zero and the conditions for parallelism are satisfied. *Thus planes that have a constant isoclinal angle and meet at infinity are parallel and meet in a straight line.*

§§ 29–32. Isoclinal Systems.

30. *Two-dimensional Systems.* The following group of theorems (§§ 30–32) restate for planes through a point in four-dimensional space the fundamental parts of Clifford's theory of parallels in elliptic space of three dimensions (Hathaway's theory of parallel and contra-parallel great circles on the hypersphere).* The planes and lines here considered are, without exception, supposed to pass through the origin; and by any plane, or any line, is meant a plane or a line satisfying this condition.

Consider a series of planes meeting in a point no two of which have maximal and minimal isoclinal angles, and place the origin at their point of meeting. Corresponding to the two conditions $\beta_1 = \pm a_1$, $\beta_2 = \pm a_2$ of § 26, there are two systems of planes: an a-system whose equations may be written in the form (§ 13)

$$a\rho - \rho a^{-1} a a = 0,$$

and a β-system whose equations similarly constructed are

$$b\beta b^{-1}\rho - \rho\beta = 0,$$

where a and b are arbitrary versors whose different values produce the several

* It was Professor Hathaway's paper on *Quaternions as Numbers of Four-Dimensional Space* that forced upon my attention the essential identity of the two theories. It should be remarked that here, as also in Hathaway's paper, the theory is stated as applying to elliptic space in its antipodal (spherical) form.

The references to Clifford, Buchheim, Klein, Whitehead are here important.

individual planes of the systems. Any system is thus determined by a characteristic vector a, or β, and by the corresponding form of its equations.

(1) *Any two planes of a system have a unique (constant) isoclinal angle* (§ 23), a consequence of the failure of the conditions for maximal and minimal isoclinal angles (§ 26 (2)).

(2) *A transversal meeting any two planes of a system in straight lines makes equal dihedral angles with them.* For, the two planes being respectively $(a, -a^{-1}aa, 0)$, $(a, -a'^{-1}aa', 0)$ and their transversal being $(\gamma, -\gamma', 0)$, the conditions for straight-line intersections are

$$\mathrm{S}a\gamma = \mathrm{S}a^{-1}aa\gamma' = \mathrm{S}a'^{-1}aa'\gamma',$$

and in either case the cosine of the dihedral angle is $-\mathrm{S}a\gamma$ (§ 19). Here, the signs of the scalar functions being alike, the equal dihedrals are the alternate exterior-interior angles on the same side of the transversal.

Thus all the planes of a system determined by any unit vector are equally inclined to any transversal meeting them in straight lines. They constitute what may be called an *isoclinal system*, or a *system of isoclines*. Two planes may be said to be mutually isoclinal when their isoclinal angle is constant; and from this definition it immediately follows that *any two planes that are isoclinal to a third plane are isoclinal to each other.*

(3) *To a given plane through a given straight line there exist always two isoclines.* For, given a plane $(a, -\beta, 0)$ and a straight line c, both of the planes whose equations are

$$ap - pc^{-1}ac = 0, \quad c\beta c^{-1}p - p\beta = 0$$

are isoclinal to $(a, -\beta, 0)$, and they intersect in c.

(4) Given a and a, the equation $ap - pa^{-1}aa = 0$ is uniquely determined; but since we have identically

$$a^{-1}aa = (a^{\psi}a)^{-1}\,a\,(a^{\psi}a),$$

any pair of values a, $a^{\psi}a$ determines this same plane. Let $a = \beta^{\phi}$; the equation of the planes of the a-system has then the form

$$ap - p(a^{\psi}\beta^{\phi})^{-1}\,a\,(a^{\psi}\beta^{\phi}) = 0.$$

Changes in ψ give no new planes, but there is a distinct plane for each value of ϕ (within the limits 0, 4), and some (not all) of the differing values of β correspond to different planes.

If a change from β to β_1 makes no effective change in the equation, then

$$\beta_1^{-\phi}a\beta_1^{\phi} = \beta^{-\phi}a\beta^{\phi}, \quad a\beta_1^{\phi}\beta^{-\phi} = \beta_1^{\phi}\beta^{-\phi}a,$$

from which it follows that $\mathrm{V}\beta_1^{\phi}\beta^{-\phi}$ is parallel to a; then

$$\beta_1^{\phi} = a^z \beta^{\phi} .$$

Hence, corresponding to the varying scalar values of z, there is a singly infinite series of changes in β which give rise to no new planes.

But any unit vector is determined by two independent scalar parameters and thus one parameter in β is at our disposal for producing new planes of the system. Therefore:

A system of isoclines of given type (an α-system or a β-system), corresponding to a given vector, consists of a doubly infinite series of planes. It is a two-dimensional system.

31. *Cardinal and Ordinal Systems.** For brevity write $a_{\phi} = \beta^{-\phi} a \beta^{\phi}$ and define the two-dimensional system of a-isoclines by the equation

$$ap - pa_{\phi} = 0 .$$

Changes in ϕ produce a singly infinite series of planes whose isoclinal angles, formed by the successive members of the series with a fixed member of it, are all different; for the squared cosine of the isoclinal angle formed by $(a , -a_{\phi'} , 0)$, regarded as fixed, and $(a , -a_{\phi} , 0)$, any other plane of the series, is by § 23 (6)

$$S^2 u\bar{v} = \tfrac{1}{2}(1 - Sa_{\phi} a_{\phi'}) ,$$

and this varies with ϕ.

Now an infinite series of a-planes which form with a fixed member of the a-series the same (constant) isoclinal angle may be determined in the following manner:

Subject the variable p in the equation $ap - pa_{\phi} = 0$ to the rotational operation $a^{\theta}( )a_{\phi'}^{-\theta}$. This leaves undisturbed all the points of the fixed plane $(a , -a_{\phi'} , 0)$ and rotates, without distortion, all other configurations in our four-dimensional space through the angle $\pi\theta$.† The equation is transformed into

$$aa^{\theta} p a_{\phi'}^{-\theta} - a^{\theta} p a_{\phi'}^{-\theta} a_{\phi} = 0 ,$$

and p now satisfies the equation

$$ap - pa_{\phi'}^{-\theta} a_{\phi} a_{\phi'}^{\theta} = 0 ,$$

which belongs to the a-system; and the isoclinal angle formed by the plane represented by this equation, whose variations depend on θ, with the fixed plane $(a , -a_{\phi'} , 0)$ is the same for all values of θ; for its squared cosine is

$$S^2 u\bar{v} = \tfrac{1}{2}(1 - Sa_{\phi} a_{\phi'}) \qquad (\S 23\,(6)) .$$

Each new value of θ (within the limits 0 , 4) produces a new plane.

* So far as I know the characterization of one-dimensional systems of isoclines (parallels in elliptic space) as of two types, here called cardinal and ordinal, has been made in no previous investigation.

† STRINGHAM: loc. cit. (2) (1884), pp. 55–56. Compare also §1 of this paper and HATHAWAY: loc. cit. (3) (1898), pp. 93–94.

The doubly infinite series of a-isoclines is hereby determined. The planes of the system produced by the variations of ϕ are transmuted into one another by successive changes of the isoclinal angles (all different) which they form with a fixed member of the series; I call them *ordinals*. The planes of the system produced by the variations of θ pass into one another by rotations about the fixed plane $(a, -a_{\phi'}, 0)$, with which they form the same isoclinal angle; I call them *cardinals*. Thus the two-dimensional a-system is made up of one-dimensional ordinal and cardinal systems; and the same remark is also obviously true of a β-system.

Evidently any a-isocline belongs both to an ordinal system and to a cardinal system, but two planes in a system of either type belong to different systems of the other type. We may exhaust the doubly infinite series of a-isoclines by setting up an infinite number of systems of either type; for example, a fixed value of ϕ determines a cardinal system and the variations of θ produce the planes of this system; then a second value of ϕ determines a second cardinal system, and so on. Briefly expressed:

A complete system of isoclines consists indifferently of a simply infinite set of ordinal systems, or of a simply infinite set of cardinal systems.

I call attention to the forms of the expressions for ρ that satisfy respectively the two equations: namely, for the ordinal system $a^x\beta^\phi$, and for the cardinal system $a^y(a + a_\phi)a^\theta_{\phi'}$, where x, y are arbitrary scalar variables.*

32. *Conjugate Systems.* In general, a plane arbitrarily placed does not meet the isoclines of a system in straight lines.

(1) *If a plane have straight-line intersections with three members of a one-dimensional system (ordinal or cardinal) it meets them all in straight lines.*

We may assign as the mono-parametric equation of the system

$$a\rho - \rho\lambda_\phi \doteq 0 \qquad (\lambda_\phi = \mu^{-\phi}\lambda\mu^\phi),$$

where λ and μ are constant unit vectors which may be so determined as to fit the case of either an ordinal or a cardinal system. The cutting plane being $(\gamma, -\gamma', 0)$, the condition is that $S\gamma a = S\gamma'\lambda_\phi$ for three different values of ϕ. This makes γ' the axis of a cone on whose surface the three vectors λ_{ϕ_i} $(i = 1, 2, 3)$ lie. But μ is also the axis of this cone and therefore $\gamma' = \mu$ for all values of ϕ, and the condition becomes

$$S\gamma a = S\gamma'\lambda_\phi = S\lambda\mu;$$

and to make $S\gamma a = S\lambda\mu$ it suffices to determine γ by the condition that its angle with a shall be equal to the angle of λ, μ. This condition being assigned write $\gamma = a^\psi\beta a^{-\psi}$, so that $S\gamma a = Sa\beta = S\lambda\mu$. The transversal plane is now an isocline of the ψ-μ-system

* Cf. HATHAWAY: loc. cit. (1) (1897), p. 55.

$$\beta_\psi p - p\mu = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}),$$

every member of which meets in straight lines all the planes of the a-ϕ-system. Thus (1) is proved not merely for one plane but for an entire one-dimensional system.

(2) *No two planes of the same system intersect in a straight line;* for $Sa^2 \neq S\lambda_\phi \lambda_{\phi'}$, unless $\phi = \phi'$.

The two systems are, in the proper sense, *conjugate* to one another.

Note that one plane, $(a, -\mu, 0)$ is isoclinal to all the planes of both systems.

(3) *The corresponding exterior-interior dihedral angles formed at their intersections by the planes of two conjugate systems are all equal to one another;* for (see also § 30 (2)), independently of the values of ϕ and ψ,

$$Sa\beta_\psi = Sa\beta = S\lambda\mu = S\mu\lambda_\phi .$$

(4) *Conjugate Systems of Ordinals.* By assigning $\lambda = a$ and $\mu = \beta$ we obtain at once, as the equations of the conjugate systems of ordinals

$$ap - pa_\phi = 0 \qquad (a_\phi = \beta^{-\phi} a \beta^\phi),$$

$$\beta_\psi p - p\beta = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}).$$

(5) *Conjugate Systems of Cardinals.* If any three planes of the cardinal system,

$$ap - pa_{\phi'}^{-\theta} a_\phi a_{\phi'}^{\theta} = 0 ,$$

be met by $(\gamma, -\gamma', 0)$ in straight lines we must have

$$S\gamma a = S\gamma' a_{\phi'}^{-\theta} a_\phi a_{\phi'}^{\theta},$$

for three distinct values of θ; whence follows $\gamma' = a_{\phi'}$ and then

$$S\gamma a = Sa_\phi a_{\phi'} = Sa a_{\phi - \phi'}.$$

For the determination of γ it suffices to write $\gamma = a^\epsilon a_{\phi-\phi'} a^{-\epsilon}$. The equation of the conjugate system of cardinals then takes the form

$$a^\epsilon a_{\phi-\phi'} a^{-\epsilon} p - p a_{\phi'} = 0 .$$

If the equation of the initial system be

$$\beta_{\psi'}^{\epsilon} \beta_\psi \beta_{\psi'}^{-\epsilon} p - p\beta = 0 ,$$

the corresponding equation of the conjugate system is

$$\beta_{\psi'} p - p\beta^\theta \beta_{\psi-\psi'} \beta^{-\theta} = 0 .$$

32. *The Transition to Elliptic Space.* The foregoing theory of isoclinal systems is clearly a three-dimensional geometry with planes as elements; it is, in fact, the geometry of the sheaf of planes in four dimensional space. All of these planes meet the hypersphere in great circles, the straight lines of an elliptic space (antipodal) whose aggregate is a "space of lines." Hence in order to

translate the propositions of isoclinal systems into their equivalents in elliptic space we have merely to take note of the following dualistic correspondences:

In parabolic four-dimensional space:		In elliptic three-dimensional space:
straight lines	~	points,
planes	~	straight lines,
isoclines	~	parallels,
dihedral angles	~	plane angles,
isoclinal angle of two planes	~	perpendicular distance between two straight lines.

UNIVERSITY OF CALIFORNIA,
November, 1900.

LIBRARY
OF THE
UNIVERSITY of ILLINOIS.

ON THE CONVERGENCE OF CONTINUED FRACTIONS WITH COMPLEX ELEMENTS*

BY

E. B. VAN VLECK

Up to the present time few theorems of a general character for the convergence of continued fractions with complex elements have been obtained, and these few are of very recent date. In the first section of this paper such theorems upon the subject as are known to the writer are brought together for the purpose of indicating the present state of our knowledge, and the scope of the paper is also explained. Some new criteria for convergence are then deduced in the succeeding sections. The results obtained are summed up in theorems 1–10, which may be read independently of the rest of the paper. The demonstration of these theorems is based upon certain equations, Nos. 3–8, 11, and 12, which seem to be new and of a fundamental character.

I. *Summary of previous criteria for convergence.*

§ 1. Two classes of continued fractions

$$\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}+\frac{\mu_3}{\lambda_3}+\cdots$$

with real elements are commonly treated in mathematical writings. In the first class all the elements are positive; in the second the numerators of the partial quotients are negative and the denominators positive.

Complete criteria have been obtained for the convergence and divergence of continued fractions of the first kind. If, namely, the fraction is thrown into the form

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots \qquad (a_n \gtrless 0),$$

it will converge when $\sum a_n$ is divergent; on the other hand, if $\sum a_n$ is convergent, the even convergents have one limit and the odd convergents have another. No criteria of equal generality have been obtained for the second class of con-

* Presented to the Society February 23, 1901. Received for publication April 15, 1901.

tinued fractions. The principal theorem, however, establishes the convergence of the fraction when $\lambda_n \geqq 1 + |\mu_n|$.*

§ 2. The convergence-theorems for continued fractions with imaginary elements may be regarded as generalizations of these theorems for real continued fractions. In 1898 PRINGSHEIM† showed that the condition $|\lambda_n| \geqq 1 + |\mu_n|$ was sufficient to ensure the convergence not only of continued fractions with real elements but also of those containing complex elements. If, furthermore, the continued fraction has the customary normal form in which $\mu_n = 1$, this condition may be replaced by the less restricted one:

$$\frac{1}{|\lambda_{2n-1}|} + \frac{1}{|\lambda_{2n}|} \leqq 1.$$

In this modified form PRINGSHEIM's criterion is applicable to the continued fractions of HURWITZ,‡ in which the partial denominators are complex integers.

The only extension of the convergence-theorem for the first class of real continued fractions is contained in a remarkable and extended memoir by STIELTJES,§ published in 1894. STIELTJES considers the fraction

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \cdots \qquad (a_n > 0, \quad n = 1, 2, 3, \cdots),$$

and shows that when $\sum a_n$ is divergent, it converges over the entire plane of z with the exception of the negative half of the real axis. On the other hand, when $\sum a_n$ is convergent, the even and the odd convergents separately converge, the limits of their numerators and denominators being holomorphic functions of *genre* 0 whose roots all lie upon the negative half of the axis.

From a function-theoretical standpoint this result is of great importance. It also calls attention anew to the fact that algebraic continued fractions commonly have a more extended region of convergence than the corresponding infinite series, and hence have, at least prospectively, a wide field of usefulness. The convergence-proofs of STIELTJES are most admirable, both by themselves and in relation to the theory of functions. When, however, they are considered solely with respect to the general theory of continued fractions, they are not entirely satisfactory. Not only are his proofs based upon the properties of the conver-

* For reference to the possibilities when this condition is not fulfilled, see PRINGSHEIM'S report in the *Encyklopädie der mathematischen Wissenschaften.*

† Sitzungsberichte der mathematisch-physikalischen Classe der Münchener Akademie, vol. 28, p. 295.

‡ Acta Mathematica, vol. 11.

§ Annales de la Faculté des Sciences de Toulouse, vol. 8, with continuation in vol. 9.

gents as functions of z,* but the form of his continued fraction is also very restricted, inasmuch as the ratio of the real to the imaginary component is the same in all the denominators of the alternate partial quotients.

These considerations have led to the present paper. It comprises such results as have suggested themselves in seeking to extend the convergence-theorems for the first class of real continued fractions to complex continued fractions.

To complete the historical review, mention should be made of the theorems of PRINGSHEIM and STIELTJES relating to continued fractions in which the denominators of the partial quotients are equal to unity. PRINGSHEIM† proves that such a fraction will converge if

$$|\mu_2| < \tfrac{1}{2}, \quad |\mu_{2n+2}| + |\mu_{2n+1}| \leqq \tfrac{1}{2} \qquad (n = 1, 2, \cdots).$$

STIELTJES‡ treats the fraction

$$\frac{b_1 z}{1} + \frac{b_2 z}{1} + \frac{b_3 z}{1} + \cdots \qquad (b_n > 0),$$

and shows that it represents a meromorphic function when b_n approaches zero as its limit, and only then.

Various other theorems pertaining to continued fractions are reducible to the foregoing by transformation of the fraction.

II. *Fundamental equations.*

§ 3. Let

$$\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} + \cdots$$

be any continued fraction whose elements are complex numbers. Denote by p_n and q_n the numerator and denominator of the n-th convergent, by M_n the modulus of q_n. If then we put

$$\lambda_n = \alpha_n + i\beta_n, \quad \mu_n = \gamma_n + i\delta_n,$$

$$p_n = r'_n + is'_n, \quad q_n = r_n + is_n,$$

the familiar laws for the formation of the successive convergents,

$$(1) \qquad p_n = \lambda_n p_{n-1} + \mu_n p_{n-2}, \quad q_n = \lambda_n q_{n-1} + \mu_n q_{n-2},$$

---

*See, for example, the remarkable preliminary theorem which he establishes in order to prove the convergence of the continued fraction when $\sum a_n$ is divergent and the real part of z is negative. Loc. cit., vol. 8, p. J56.

†Sitzungsberichte der Münchener Akademie, vol. 28.

‡Annales de Toulouse, vol. 9, p. A, 42–47.

may be replaced by

$$r_n = a_n r_{n-1} + \gamma_n r_{n-2} - \beta_n s_{n-1} - \delta_n s_{n-2},$$
$$s_n = a_n s_{n-1} + \gamma_n s_{n-2} + \beta_n r_{n-1} + \delta_n r_{n-2}, \tag{2}$$

with two similar equations for r_n' and s_n'.

Squaring and adding equations (2) we obtain next the following fundamental relation between the moduli of three consecutive denominators:

$$M_n^2 = (a_n^2 + \beta_n^2) M_{n-1}^2 + (\gamma_n^2 + \delta_n^2) M_{n-2}^2$$
$$+ 2(a_n\gamma_n + \beta_n\delta_n) C_{n-1} + 2(\beta_n\gamma_n - a_n\delta_n) D_{n-1}, \tag{3}$$

in which

$$C_{n-1} = r_{n-1}r_{n-2} + s_{n-1}s_{n-2}, \tag{4}$$

$$D_{n-1} = r_{n-1}s_{n-2} - s_{n-1}r_{n-2}. \tag{5}$$

The form of the last two equations may be modified advantageously by introducing in place of r_{n-1} and s_{n-1} the expressions which are obtained by the substitution of $n-1$ for n in (2). The equations then become

$$C_{n-1} = a_{n-1}M_{n-2}^2 + \gamma_{n-1}C_{n-2} - \delta_{n-1}D_{n-2}, \tag{6}$$

$$D_{n-1} = -\beta_{n-1}M_{n-2}^2 - \gamma_{n-1}D_{n-2} - \delta_{n-1}C_{n-2}. \tag{7}$$

In this form they may be used with (3) to compute M_{n-1}, C_{n-1}, D_{n-1} from the corresponding quantities of next lower subscript. If also we take

$$r_0 = 1, \quad r_0' = s_0 = s_0' = 0,$$

we shall have

$$M_0^2 = 1, \quad C_1 = r_1 = a_1 M_0^2, \quad D_1 = -s_1 = -\beta_1 M_0^2.$$

Hence C_{n-1}, D_{n-1}, M_{n-1}^2 are each homogeneous linear functions of the squares of the moduli M_i with lower subscripts (ultimately of M_0^2 and M_1^2), in which the coefficients are entire functions of a_i, β_i, γ_i, $\delta_i (i = 1, 2, \cdots, n-1)$.

From (4) and (5) we also get the important relation:

$$C_{n-1}^2 + D_{n-1}^2 = M_{n-1}^2 M_{n-2}^2. \tag{8}$$

§ 4. To investigate the convergence of a continued fraction recourse is sometimes had to the equation:*

$$\frac{p_{n+m}}{q_{n+m}} = \frac{p_n}{q_n} + (-1)^n \mu_1\mu_2\cdots\mu_{n+1}\left(\frac{1}{q_n q_{n+1}} - \frac{\mu_{n+2}}{q_{n+1}q_{n+2}}\right.$$
$$\left. + \frac{\mu_{n+2}\mu_{n+3}}{q_{n+2}q_{n+3}} + \cdots + (-1)^{m-1}\frac{\mu_{n+2}\mu_{n+3}\cdots\mu_{n+m}}{q_{n+m-1}q_{n+m}}\right), \tag{9}$$

* HEINE'S *Kugelfunctionen*, vol. 1, p. 263.

in which m is allowed to increase indefinitely. The convergence of a continued fraction is thus made to depend upon that of an infinite series. As, however, this form of the series is not well adapted to the consideration of cases of alternating convergence, it will be better to group by twos the terms in the parenthesis of (9). With the aid of the relation

$$\frac{1}{q_n q_{n+1}} - \frac{\mu_{n+2}}{q_{n+1} q_{n+2}} = \frac{\lambda_{n+2}}{q_n q_{n+2}},$$

the series may then be thrown into the form:

$$\frac{p_n}{q_n} + (-1)^n \mu_1 \mu_2 \cdots \mu_{n+1} \left(\frac{\lambda_{n+2}}{q_n q_{n+2}} + \frac{\lambda_{n+4} \mu_{n+2} \mu_{n+3}}{q_{n+2} q_{n+4}} + \frac{\lambda_{n+6} \mu_{n+2} \mu_{n+3} \cdots \mu_{n+5}}{q_{n+4} q_{n+6}} + \cdots \right). \tag{10}$$

Two different cases arise according as n is even or odd. If the series converges in both cases and the continued product of the μ_i is convergent and greater than zero, *the necessary and sufficient condition for the coincidence of the two limiting values of the series is that* $M_{n-1} M_{n-2}$ *shall increase indefinitely with* n. The continued fraction then converges in the ordinary sense of the term. On the other hand, if $M_{n-1} M_{n-2}$ has a finite limit, the convergence is of an alternating character.

The series (10) has the advantage over (9) that it is frequently absolutely convergent when the latter is only conditionally convergent.

Either series loses its significance if q_i vanishes in one or more of its terms. Should this, however, happen only a finite number of times, the continued fraction may still converge, for the objectionable terms can be removed from the series by increasing n sufficiently.

In one other particular the convergence of a continued fraction requires special comment. The removal of the first r partial quotients μ_i / λ_i may change a convergent continued fraction into a divergent one, or *vica versa*. Adopting a term proposed by PRINGSHEIM,* we shall say that the convergence is *unconditional* when it is not affected by the removal of any finite number of partial quotients, beginning with the first.

§ 5. Besides discussing the convergence of the fraction, we shall in certain cases determine the signs of its real and imaginary parts. For this purpose let p_n / q_n be thrown into the form:

$$\frac{r_n r_n' + s_n s_n' + i(r_n s_n' - r_n' s_n)}{M_n^2}.$$

*For a further discussion of unconditional convergence consult the memoir of PRINGSHEIM previously cited.

If we then place

$$K_n = r_n r'_n + s_n s'_n,$$
$$L_n = r_n s'_n - r'_n s_n,$$

and combine these equations with (2) and the two analogous equations for r'_n and s'_n, we obtain

$$K_n = (a_n^2 + \beta_n^2) K_{n-1} + (\gamma_n^2 + \delta_n^2) K_{n-2} + 2(a_n\gamma_n + \beta_n\delta_n) k_{\alpha\gamma}^{(n-1)} + 2(\beta_n\gamma_n - a_n\delta_n) k_{\beta\gamma}^{(n-1)}, \tag{11}$$

$$L_n = (a_n^2 + \beta_n^2) L_{n-1} + (\gamma_n^2 + \delta_n^2) L_{n-2} + 2(a_n\gamma_n + \beta_n\delta_n) l_{\alpha\gamma}^{(n-1)} + 2(\beta_n\gamma_n - a_n\delta_n) l_{\beta\gamma}^{(n-1)}, \tag{12}$$

in which

$$k_{\alpha\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}r'_{n-2} + r'_{n-1}r_{n-2} + s_{n-1}s'_{n-2} + s'_{n-1}s_{n-2}),$$
$$k_{\beta\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}s'_{n-2} + r'_{n-1}s_{n-2} - s_{n-1}r'_{n-2} - s'_{n-1}r_{n-2}),$$
$$l_{\alpha\gamma}^{(n-1)} = \tfrac{1}{2}(r_{n-1}s'_{n-2} - r'_{n-1}s_{n-2} - s_{n-1}r'_{n-2} + s'_{n-1}r_{n-2}),$$
$$l_{\beta\gamma}^{(n-1)} = \tfrac{1}{2}(-r_{n-1}r'_{n-2} + r'_{n-1}r_{n-2} - s_{n-1}s'_{n-2} + s'_{n-1}s_{n-2}).$$

The last set of equations may be simplified in the same manner as (4) and (5). They then become

$$k_{\alpha\gamma}^{(n-1)} = a_{n-1}K_{n-2} + \gamma_{n-1}k_{\alpha\gamma}^{(n-2)} - \delta_{n-1}k_{\beta\gamma}^{(n-2)},$$
$$k_{\beta\gamma}^{(n-1)} = -\beta_{n-1}K_{n-2} - \gamma_{n-1}k_{\beta\gamma}^{(n-2)} - \delta_{n-1}k_{\alpha\gamma}^{(n-2)},$$
$$l_{\alpha\gamma}^{(n-1)} = a_{n-1}L_{n-2} + \gamma_{n-1}l_{\alpha\gamma}^{(n-2)} - \delta_{n-1}l_{\beta\gamma}^{(n-2)},$$
$$l_{\beta\gamma}^{(n-1)} = -\beta_{n-1}L_{n-2} - \gamma_{n-1}l_{\beta\gamma}^{(n-2)} - \delta_{n-1}l_{\alpha\gamma}^{(n-2)},$$

and may be used in conjunction with (11) and (12) to find K_n and L_n for successive values of n.

It will, of course, be observed that the above equations are similar in their construction to equations 3–7.

III. *On the convergence and properties of a certain class of continued fractions.*

§ 6. Our attention in this section will be confined entirely to continued fractions which have been reduced to the normal form for which $\gamma_n = 1$, $\delta_n = 0$. Equations (3), (6), and (7) then take the simpler form:

$$M_n^2 = (a_n^2 + \beta_n^2) M_{n-1}^2 + M_{n-2}^2 + 2a_n C_{n-1} + 2\beta_n D_{n-1}, \tag{13}$$

$$C_{n-1} = a_{n-1}M_{n-2}^2 + C_{n-2}, \tag{14}$$

$$D_{n-1} = -\beta_{n-1}M_{n-2}^2 - D_{n-2}, \tag{15}$$

and the series (10) simultaneously becomes

$$\frac{P_n}{Q_n} + (-1)^n \left(\frac{a_{n+2} + i\beta_{n+2}}{Q_n Q_{n+2}} + \frac{a_{n+4} + i\beta_{n+4}}{Q_{n+2} Q_{n+4}} + \cdots \right). \tag{16}$$

We shall first determine what signs for a_n and β_n are most favorable to the absolute convergence of (16), supposing their absolute values to be given. Since

$$C_{n-1} = \sum_{i=1}^{n-1} a_i M_{i-1}^2, \quad D_{n-1} = -\sum_{i=1}^{n-1} (-1)^{n-i+1} \beta_i M_{i-1}^2, \tag{17}$$

C_{n-1} will increase in absolute value with increasing n if the a_n have a common sign, and D_{n-1} will similarly increase if the β_n are alternately of opposite sign (each zero value of β_n being included and reckoned with an appropriate sign). Moreover $a_n C_{n-1}$ and $\beta_n D_{n-1}$ will then never be negative. These conditions are clearly as favorable as possible for the increase of $M_n = |Q_n|$ and hence also for the absolute convergence of (16).

§ 7. In considering the convergence of the continued fraction two cases are to be distinguished according as $\sum |\lambda_n|$—that is, $\sum |a_n + i\beta_n|$—if convergent or divergent.

Consider first the former case and suppose either that the a_n have a common sign or that the β_n alternate in sign. It is immaterial which supposition is made because a_n and $-\beta_n$ enter in like manner into the fundamental equations. In either case we have from (1):

$$M_m \leqq |\lambda_m| M_{m-1} + M_{m-2} \leqq (1 + |\lambda_m|) M',$$

in which M' is used to represent the larger of the two moduli M_{m-1}, M_{m-2}. This shows that

$$M_n < M' \prod_{i=m}^{n} (1 + |\lambda_i|) \qquad (n > m).$$

But by a well-known theorem $\prod (1 + |\lambda_i|)$, $(i = 1, 2, \cdots)$, converges with $\sum |\lambda_i|$. There must accordingly be an upper limit to the value of M_n. On the other hand, by virtue of the hypothesis concerning a_n or β_n, either $|C_{n-1}|$ or $|D_{n-1}|$ will increase with n, or at least not decrease. It follows then from (8) that $M_{n-1} M_{n-2}$ has a lower limit. This can only be if M_n, which has an upper limit, has also a lower limit.

Turning next to the series (16) we see now that it must be absolutely convergent, since the numerators of its terms form an absolutely convergent series while the moduli of the denominators have a lower limit. As also $M_{n-1} M_{n-2}$ does not indefinitely increase, two distinct values for the series will be obtained according as n is odd or even. We have therefore the following result:

THEOREM I. *In any continued fraction*

$$\frac{1}{a_1 + i\beta_1} + \frac{1}{a_2 + i\beta_2} + \frac{1}{a_3 + i\beta_3} + \cdots \tag{18}$$

the even and the odd convergents will both converge and their limits will be distinct if (1) $\sum |a_n + i\beta_n|$ *is convergent and* (2) *either the* α*-constituents have a common sign or the* β*-constituents alternate in sign.**

§ 8. In § 7 it was shown, and without making use of condition (1), that $M_{n-1}M_{n-2}$ had a lower limit. An immediate inference from this is that M_n cannot vanish for any finite value of n. A like conclusion obviously holds for the numerator of the nth convergent, inasmuch as the equations for the formation of the successive numerators are similar to those for the denominators of the convergents. We have thus a very simple proof of the following theorem:

THEOREM 2. *Neither the numerator nor the denominator of any convergent of the continued fraction* (18) *will vanish if the* α*-constituents have a common sign or if the* β*-constituents alternate in sign.*

§ 9. We pass now to the consideration of the case in which $\sum |a_n + i\beta_n|$ diverges. Let it be first supposed that both a_n and β_n are restricted in sign in the manner indicated in the preceding theorem. Every term in (13) will then be positive or zero, and M_n must be equal to or greater than M_{n-2} for every value of n. Accordingly M_n has a lower limit. Furthermore, by our initial hypothesis either $\sum |a_n|$ or $\sum |\beta_n|$ must diverge, or both. It follows therefore from (17) that at least one of the two quantities, C_{n-1} and D_{n-1} increases indefinitely with n. But by (8) $M_{n-1}M_{n-2}$ must increase in the same manner. If, therefore, the series (16) converges, it must have the same limit whether n be odd or even.

To prove that (16) does in fact converge, we shall establish separately the convergence of the two series:

$$\sum_{n=1}^{\infty} \frac{|a_n|}{M_{n-2}M_n}, \quad \sum_{n=1}^{\infty} \frac{|\beta_n|}{M_{n-2}M_n}.$$

Consider first the former. If in computing M_n by formula (13) for successive values of n we should equate each β_n to zero, smaller values for M_n would be obtained, which we will denote by M'_n. Obviously these are successively connected by the relation:

$$M'_n = |a_n|\, M'_{n-1} + M'_{n-2},$$

or by its equivalent

$$\frac{1}{M'_{n-2}M'_{n-1}} = \frac{|a_n|}{M'_{n-2}M'_n} + \frac{1}{M'_{n-1}M'_n}.$$

* Zero-constituents are admissible in this and the following theorems, being taken as having the signs which belong to the places in which they are found.

If in the latter we set in turn $n = m, m+1 \cdots$ and then add we get,

$$\frac{1}{M'_{m-2}M'_{m-1}} \geqq \sum_{n=m}^{\infty} \frac{|a_n|}{M'_{n-2}M'_n}.$$

This proves that

$$\sum \frac{|a_n|}{M'_{n-2}M'_n}$$

converges, and hence also $\sum |a_n| M_{n-2}^{-1} M_n^{-1}$. Similar considerations apply to prove the convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$.

We conclude, therefore, that the continued fraction has a definite limit, and the reasoning evidently holds if the first n partial quotients are omitted. This result may be restated in

THEOREM 3. *If the α-constituents in* (18) *all have the same sign and the β-constituents alternate in sign, the continued fraction will converge unconditionally when $\sum |a_n + i\beta_n|$ is divergent.*

§ 10. The signs of the real and imaginary parts of the convergents can be quickly determined. Equations 13–15 hold when M_i^2, C_i, D_i are replaced by $K_i, k_{\alpha\gamma}^{(i)}, k_{\beta\gamma}^{(i)}$ or by $L_i, l_{\alpha\gamma}^{(i)}, l_{\beta\gamma}^{(i)}$. The initial values of these quantities are as follows:

$$K_0 = L_0 = 0, \quad K_1 = a_1, \quad L_1 = -\beta_1,$$

$$k_{\alpha\gamma}^{(1)} = l_{\beta\gamma}^{(1)} = \tfrac{1}{2}, \quad k_{\beta\gamma}^{(1)} = l_{\alpha\gamma}^{(1)} = 0.$$

An easy mathematical induction then shows that all the terms in (13) after either of the above substitutions have a common sign. From this we get at once

THEOREM 4. *If the α-constituents have a common sign and the β-constituents alternate in sign, the real part of any convergent will have the same sign as a_1 and the imaginary part the same sign as $-\beta_1$.*

§ 11. Returning now to the continued fraction of theorem 3, we proceed to consider the effect of the admission of a finite number of α-constituents or of β-constituents which fail to satisfy the conditions there imposed. It is immaterial whether the exceptional elements occur among the a_n or the β_n; let them occur among the former.

When n is increased, there must be a value—call it m—subsequent to which a_n and C_{n-1} have each an invariable sign. If they have the same sign, all the steps of the proof of theorem 3 will apply after the m-th convergent. This will necessarily be the case when $\sum |a_n| M_{n-1}^2 (n = 1, 2, \cdots)$ is divergent.

There remains for consideration only the special case in which $\sum |a_n| M_{n-1}^2$ converges and a_n and C_{n-1} have opposite signs when n exceeds m. Equation (13) then gives

$$M_n^2 \geqq a_n^2 M_{n-1}^2 - 2|a_n C_{n-1}| + M_{n-2}^2,$$

whence by combination with (8) we obtain

$$M_n^2 \geqq M_{n-2}^2 \left(1 - \frac{|a_n C_{n-1}|\, M_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2}\right)^2. \tag{19}$$

Now since $\sum |a_n| M_{n-1}^2$ converges, $|C_{n-1}|$ has an upper limit. On the other hand, $|D_{n-1}|$, which never diminishes, will have a finite lower limit after some fixed value of n, unless it is equal to zero for every value of n. This exceptional case will not occur if there is an infinite number of values of n for which $|\beta_n| > 0$, as we will suppose, neither can it occur if there is a single such value greater than m, inasmuch as $M_{m+i} > 0$. An upper limit must therefore exist for $|C_{n-1}|/(C_{n-1}^2 + D_{n-1}^2)$. Since also $|a_n| M_{n-1}^2$ diminishes indefinitely with increasing n, the second term of the binomial in the last inequality may be made as small as desired. Let m be taken large enough to ensure that it shall always be less than unity when $n > m$. We then have

$$M_{m+2n} > M_m \prod_i \left(1 - \frac{|a_i C_{i-1}|\, M_{i-1}^2}{C_{i-1}^2 + D_{i-1}^2}\right) \quad (i = m+2,\ m+4,\ \cdots,\ m+2n).$$

If in the last expression n is indefinitely increased, the product $\prod$ must be convergent since $\sum |a_i M_{i-1}^2|$ is convergent and $|C_{i-1}|/(C_{i-1}^2 + D_{i-1}^2)$ has an upper limit. Moreover, by a well-known theorem the limiting value of $\prod$ must be greater than zero. Hence M_{m+2n} has a lower limit, and the same is true of M_n since m may be taken to be either odd or even.

The convergence of $\sum |a_n|$ may next be quickly argued. This follows from the fact that the terms in the convergent series $\sum a_n M_{n-1}^2$ have ultimately a common sign while M_n has a lower limit. Now by our original hypothesis $\sum |a_n + i\beta_n|$ was a divergent series. This necessitates that $\sum |\beta_n|$ should likewise diverge. But when this diverges, $|D_{n-1}|$ increases indefinitely and with it simultaneously $M_{n-1} M_{n-2}$.

To demonstrate the convergence of the continued fraction, it remains only to prove that (16) converges. This can be quickly done. For, on the one hand, $\sum |a_n| M_{n-2}^{-1} M_n^{-1}$ obviously converges. On the other hand, if each a_n is replaced by zero, $\sum |\beta_n| M_{n-2}^{-1} M_n$ will converge, as we saw in § 9. But from (19) it is clear that the presence of the a-terms in (13), when it does not actually increase the value of M_n, will cut it down in a ratio not exceeding

$$1 : \left(1 - \frac{|a_n C_{n-1}|\, M_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2}\right).$$

It has also been shown that the product of all such ratios for which n exceeds m is convergent. The restoration of the a-terms will therefore not reduce the ~~If in the latter we set in turn $n = m,\ m+1,\ \cdots$, and then add, we get~~

value of $M_n (n > m)$ by more than a certain fixed part of itself. Since this in no wise affects the convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$, the series must still converge. This completes the proof.

The conclusion reached can be summed up as follows:

THEOREM 5. *The continued fraction of theorem* 3 *will still converge if a finite number of the* a_n *or of the* β_n *are admitted which fail to satisfy the conditions there imposed, provided that after the last irregularity there is at least one value of* n *for which* β_n, *respectively* a_n *is not equal to zero.*

§ 12. The criterion of theorem 5 differs from most or all criteria hitherto given for the convergence of continued fractions in that it admits a finite number of irregularities whose positions in the continued fraction are entirely arbitrary. This raises the question whether an infinite number of exceptional values of a_n might not have been admitted into the continued fraction, or, in other words, whether the condition imposed upon β_n was not in itself sufficient to ensure the convergence of the continued fraction, as in the case when $\sum |a_n + i\beta_n|$ converged. That this, however, is in no wise true, can be shown by constructing divergent fractions in which all the β-constitutents fulfill the conditions imposed in the theorem.

Consider, for example, a continued fraction in which β_n is equal to zero for an infinite number of values of n but not for all values. When $\beta_n = 0$, we have

$$M_n^2 = \left(M_{n-2} + \frac{a_n C_{n-1}}{M_{n-2}} \right)^2 + \frac{a_n^2 D_{n-1}^2}{M_{n-2}^2}.$$

Let now a_n be so chosen that the term in the parenthesis shall vanish. This makes $|D_{n-1}^{-1}| = |a_n| M_{n-2}^{-1} M_n^{-1}$. If the values of the β_n which do not vanish fulfill the requirements concerning their signs and are also so chosen that $|D_{n-1}|$ shall have an upper limit, the terms of (16) cannot all decrease indefinitely with increasing n. The series (16) and the continued fraction are in consequence divergent.

§ 13. The restriction upon the signs of a_n in theorem 3 may, however, be dispensed with if we assume that $|\beta_n| / |a_n|$ has a finite lower limit. The proof of the convergence of the continued fraction is then not so simple.

The first step will be to show that $|D_n|$ increases indefinitely with n. Suppose, if possible, that the contrary is the case. It will be recalled that the signs of the β_n were so restricted that $|D_n| = \sum |\beta_n| M_{n-1}^2$. If $\sum |\beta_n| M_{n-1}^2$ converges, $\sum |a_n| M_{n-1}^2$ must likewise converge in consequence of the above assumption. Now it was shown in § 11 that when the latter converged, M_n had a finite lower limit. It follows that $\sum |\beta_n|$ should be convergent. But, on the other hand,

the assumption of a lower limit for $|\beta_n|/|a_n|$ and the hypothesis that $\sum|a_n + i\beta_n|$ diverges, together necessitate the divergence of $\sum|\beta_n|$. This involves a contradiction. We conclude, therefore, that $|D_n|$ must increase indefinitely, as stated. Equation (8) shows that $M_{n-1}M_{n-2}$ also increases without limit.

To establish the convergence of (16) it clearly suffices to demonstrate the convergence of $\sum|\beta_n|M_{n-2}^{-1}M_n^{-1}$. Let k^{-1} be a lower limit to the value of $|\beta_n|/|a_n|$. Equations (17) show at once that $|C_{n-1}| \leqq k|D_{n-1}|$, whence it follows by (8) that

$$|D_{n-1}| > \frac{M_{n-2}M_{n-1}}{k+1}.$$

When a_n and C_{n-1} are of like sign for any value of n, we have from (13),

$$M_n^2 > \beta_n^2 M_{n-1}^2 + 2\beta_n D_{n-1} + M_{n-2}^2.$$

The combination of this inequality with the preceding gives

$$M_n > \frac{|\beta_n|\,M_{n-1}}{k+1} + M_{n-2}, \tag{20}$$

or

$$\frac{1}{M_{n-2}M_{n-1}} > \frac{|\beta_n|}{(k+1)M_{n-2}M_n} + \frac{1}{M_{n-1}M_n}. \tag{21}$$

The last inequality will also hold when a_n and C_{n-1} have opposite signs if the expression $a_n^2M_{n-1}^2 + 2a_nC_{n-1}$ in (13) is positive. The only possible exceptions therefore occur when a_n differs in sign from C_{n-1} and

$$|a_n| < \frac{2|C_{n-1}|}{M_{n-1}^2}.$$

In this case, if we put

$$a_n = -(1+x_n)\frac{C_{n-1}}{M_{n-1}^2}, \tag{22}$$

the absolute value of x_n will be less than unity. The combination of the last equation with

$$C_n = -a_nM_{n-1}^2 + C_{n-1}$$

then gives

$$C_n = -x_nC_{n-1}.$$

Hence the necessary and sufficient condition that $a_n^2M_{n-1}^2 + 2a_nC_{n-1}$ shall be negative is that $|C_n|$ shall be less than $|C_{n-1}|$.

Even when this condition is fulfilled, the inequality (21) may yet hold. This will, for example, be the case whenever $|C_{n-1}|$ falls below a certain fixed part of $|D_{n-1}|$. For suppose $C_{n-1} < k'|D_{n-1}|$, where k' denotes the positive number less than unity which satisfies the equation

$$\frac{1-kk'}{1+k'} = \frac{1}{1+k}.$$

Then simultaneously

$$|a_n| < k|\beta_n|, \qquad |D_{n-1}| > \frac{M_{n-2}M_{n-1}}{1+k'}.$$

Since also $kk' < 1$, it follows that

$$2(a_n C_{n-1} + \beta_n D_{n-1}) > 2|\beta_n D_{n-1}|(1-kk') > \frac{2|\beta_n| M_{n-1}M_{n-2}}{1+k}.$$

The combination of the last inequality with (13) shows immediately that (20) and (21) also hold.

When for any value of n the inequality (21) is not valid, we shall substitute the expression (22) for a_n in equation (13). This equation then takes the form:

$$M_n^2 = \beta_n^2 M_{n-1}^2 + 2\beta_n D_{n-1} + M_{n-2}^2 - \frac{(1-x_n^2) C_{n-1}^2}{M_{n-1}^2}.$$

Since also

$$\frac{C_{n-1}^2}{M_{n-1}^2} = \frac{C_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2} M_{n-2}^2,$$

this gives

$$M_n^2 > \beta_n^2 M_{n-1}^2 + \frac{2|\beta_n|}{k+1} M_{n-1}M_{n-2} + y_n^2 M_{n-2}^2,$$

in which

$$y_n = \sqrt{1 - \frac{(1-x_n^2) C_{n-1}^2}{C_{n-1}^2 + D_{n-1}^2}}. \tag{23}$$

It follows then *a fortiori* that

$$\frac{1}{M_{n-2}M_{n-1}} > \frac{|\beta_n|}{(k+1) M_{n-2}M_n} + \frac{y_n}{M_{n-1}M_n}. \tag{24}$$

§ 14. For every value of n there is, then, an inequality of the form (21) or (24). Let these inequalities be arranged in order for successive values of n. When thus arranged they may be divided into sets according to the value of $|D_n|$ in the following manner: Let c be any fixed value greater than unity. Denote also by n_2 the first value of n for which $|D_n| \geqq c|D_1|$, by n_3 the first succeeding value for which $|D_n| \geqq c^2 D_1$, etc. Then the first set is to comprise all the inequalities which precede the one in which $n = n_2$; the second set is to begin with the latter inequality and to continue until $n = n_3$, and so on.

Since $|D_n| < M_{n-1}M_n$, the left hand member of the first inequality of the $(r+1)$-th set can not be greater than $1/|D_1|c^r$, and the sum of all such left hand members will therefore be less than a convergent geometrical progression. Consider now any set by itself, and let each inequality contained therein be

multiplied by the product of the y_n which occur in the preceding inequalities of the set. If the inequalities thus modified are added together, each left hand member except the first will be canceled by the last term on the right hand side of the preceding inequality. The sum of the uncanceled left hand members will accordingly be less than the above geometrical progression.

It will be shown immediately that the product of all the y_n which occur in any set has a finite lower limit L independent of the number r which specifies the set. Granting this to be true, if we add together the various sets of inequalities, the sum of all the first terms on the right hand side of the inequalities will be greater than

$$\frac{L}{k+1}\sum_{n=1}^{\infty}\frac{|\beta_n|}{M_{n-2}M_n}.$$

As this sum is less than the above geometric progression, $\sum|\beta_n|\,M_{n-2}^{-1}M_n^{-1}$ has a finite limit. The same must be true of the series (16). Since also $M_{n-2}M_{n-1}$ increases indefinitely, the limit will be the same whether n is odd or even. The continued fraction will therefore converge.

It remains now only to establish the existence of the lower limit L. Let $m_1, m_2, m_3, \cdots$ denote in order the values of n which specify for any set the inequalities which are of the form (24). From inspection of (23) it is clear that the product of the y_n will have a finite lower limit if

$$\sum\frac{(1-x_n)^2C_{n-1}^2}{C_{n-1}^2+D_{n-1}^2} \qquad (n=m_1, m_2, m_3, \cdots) \tag{25}$$

has an upper limit. Since $|D_n|$ never diminishes,

$$C_{n-1}^2+D_{n-1}^2>D_{m_1-1}^2$$

for each term of this sum, while in the first term

$$|C_{m_1-1}|<k'\,|D_{m_1-1}|.$$

We have also previously seen that $C_n=-x_nC_{n-1}$ for each of the above values of n. Hence if the ratio

$$|C_{m_{r+1}-1}|:|C_{m_r}|$$

is denoted by κ_r, the series (25) will term by term be less than

$$(k')^2\{(1-x_{m_1}^2)+(1-x_{m_2}^2)x_{m_1}^2\kappa_1^2+(1-x_{m_3}^2)x_{m_1}^2x_{m_2}^2\kappa_1^2\kappa_2^2+\cdots\}.$$

The expression here in parenthesis obviously has a value not greater than 1 when each κ_n is placed equal to unity. Hence its original value can not exceed the product of all the κ_n^2 which are greater than unity. Our problem is thus reduced to showing that this product has a common upper limit for the various sets of inequalities. This can be quickly proved. For when $|C_n|$ increases

from $|C_{m_r}|$ to $|C_{m_{r+1}-1}| \equiv \kappa_r |C_{m_r}|$, an increment is given to $|D_n|$ which is at least equal to the increment of $|C_n|$ divided by k. Since also for each of the values of n under consideration $|C_n|$ can not fall below $k'|D_n|$, the sum of the increments given to $|D_n|$ must exceed

$$\frac{k'}{k}|D_{m_1}|\sum(\kappa_r - 1).$$

On the other hand, as soon as this sum reaches $c|D_{m_1}|$, or before, a new set of inequalities must be begun. Hence for each set

$$\sum(\kappa_r - 1) < \frac{kc}{k'}.$$

But the product $\Pi\kappa_r$ has an upper limit simultaneously with $\sum(\kappa_r - 1)$. This completes the proof.

It should be noticed that the values of these upper limits and hence of L depend only upon k and c.

The result which has thus been reached may be recapitulated in the following theorem.

THEOREM 6. *When $\sum|a_n + i\beta_n|$ is divergent, the continued fraction* (18) *will converge if the β-constituents alternate in sign and the ratio $|\beta_n| : |a_n|$ has a finite lower limit,* also if a_n never varies in sign and $|a_n| : |\beta_n|$ has such a limit.*

§ 15. When the second set of conditions of the last theorem are fulfilled and $|a_n| \geqq |\beta_n|$ for every value of n, the signs of the real parts of any convergent and hence of the continued fraction can be determined without difficulty. For consider the equations obtained by substituting K_i, $k^{(i)}_{\alpha\gamma}$, $k^{(i)}_{\beta\gamma}$ for M_i^2, C_i, D_i in (13). It will be found that the only term which varies in sign is $2\beta_n k^{(n-1)}_{\beta\gamma}$, and the absolute value of this term is less than that of $2a_n k^{(n-1)}_{\alpha\gamma}$. The following result will then be evident.

THEOREM 7. *If a_n has a constant sign and $|a_n| \geqq |\beta_n|$ for all values of n, the real part of any convergent and hence of the limit of continued fraction will have the same sign as a_1; if β_n alternates in sign and $|\beta_n| \geqq |a_n|$, the sign of the imaginary part is opposite to that of β_1.*

IV. *On a certain class of algebraic continued fractions.*

§ 16. Application of the results of section III. may now be made to algebraic continued fractions. Consider first any fraction of the form

* "*Untere Grenze*" (WEIERSTRASS); not necessarily a value which the ratio approaches when n increases indefinitely.

$$\frac{1}{d_1 z} + \frac{1}{d_2 z} + \frac{1}{d_3 z} + \cdots$$

in which the d_n are real numbers alternately positive and negative.

Theorem 6 is applicable only when $\sum |d_n|$ is divergent, and shows that the continued fraction converges over the entire plane of z with the exception of the whole or a part of the real axis. Theorem 2 proves that the roots of the numerators and denominators lie only upon the real axis. Above and below the axis the convergents are therefore analytic. That their limit is also an analytic function can be seen from the series (16). For since the individual terms of (16) are analytic, it suffices, by a well-known theorem of WEIERSTRASS,* to prove that the convergence is uniform. Take any circle (R) whose boundary lies wholly above or below the axis, and let $1/\bar{k}$ denote the smallest value of $|y|/|x|$ for this circle. Then since $|a_n| \leqq \bar{k}|\beta_n|$ throughout (R), we need only to demonstrate the uniform convergence of $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$. If in §§ 13 and 14, k is replaced by $\bar{k}$, its maximum value within (R), the reasoning of these paragraphs will be seen to hold for every value of z within the circle or upon its boundary. The number L, which depends only on $\bar{k}$ and c, is moreover independent of z. If $|\bar{D}_1|$ denotes the smallest value of $|D_1|$ for the circle (R), the series

$$\frac{L}{\bar{k}+1} \sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$$

will be less, *term by term*, than a geometrical progression whose first term is $|\bar{D}_1^{-1}|$ and whose ratio c^{-1} is less than unity. It follows that $\sum |\beta_n| M_{n-2}^{-1} M_n^{-1}$ is uniformly convergent. The continued fraction therefore represents an analytic function.

The position of the zeros of this analytic function can be quickly determined from theorem 2. For when a sequence of rational functions—in the present case the convergents—converge uniformly to an analytic function as their limit, the zeros of this function are the condensation-points of the zeros of the rational fractions.† The analytic function defined by the continued fraction therefore can vanish only for real values of z.

§ 17. When $\sum |d_n|$ is convergent, the even and the odd convergents define distinct functions. Theorem 1 shows that they converge over the entire plane with the possible exception of the real axis. Not only is this true, but in either sequence of convergents the numerators and denominators separately converge, and their limits are holomorphic functions. This may be demonstrated as follows:

* WEIERSTRASS, *Abhandlungen aus der Functionenlehre*, p. 73 ff.; or HARKNESS AND MORLEY, *Analytic Functions*, § 81.

† HURWITZ, Mathematische Annalen, vol. 33, p. 249.

The denominators of the convergents, also the numerators, contain alternately only odd and even powers of z, and in each one the coefficient of z^r, when it does not vanish, consists of certain products of the d_n taken r at a time. By the aid of equation (1) it can be easily shown that the products which figure in the coefficient of z^r in the denominator (numerator) of the nth convergent are included among the products which make up the corresponding coefficient of the second succeeding denominator (numerator). Hence as n is indefinitely increased, the coefficient of any power of z tends to take a definite limiting form. But by virtue of our initial hypothesis $\prod(1 + |d_n|z)$ converges for every finite value of z. It may therefore be expressed as a series $\sum c_r z^r$ which likewise converges over the finite plane, and the coefficient c_r is the limit of the sum of all the partial products of the $|d_n|$ taken r at a time. As c_r is obviously greater than the limit of the coefficient of z^r in either sequence of alternate numerators or denominators, the limits of the latter must be holomorphic functions.

It will be observed that *this conclusion always holds when* $\sum|d_n|$ *is convergent, irrespective of the conditions imposed upon* d_n *in our continued fraction.*

In the special case under discussion the position of the roots of the limiting functions can be inferred from theorem 2 in the same manner as when $\sum|d_n|$ was divergent. Theorem 7 may also be applied for a part of the plane.

Before summing up our results, it will be advantageous to transform the continued fraction so as to make the coefficients of the denominators of the partial quotients positive. We have then the following theorem.

THEOREM 8. *If any algebraic continued fraction has the form*

$$\frac{1}{b_1 z} - \frac{1}{b_2 z} - \frac{1}{b_3 z} - \cdots$$

and every element b_n *is positive, the continued fraction will have the following properties:*

(*a*) *The zeros of the numerators and denominators of its convergents all lie upon the real axis of* z.

(*b*) *If* $\sum|b_n|$ *is divergent, it represents a function which is analytic over the entire plane with the possible exception of the axis and which vanishes only for real values of* z.

(*c*) *If* $\sum|b_n|$ *is convergent, the limits of the numerators and denominators of the even, also of the odd convergents are holomorphic functions whose zeros lie upon the real axis.*

(*d*) *The imaginary part of each convergent or of the limiting function will have a sign opposite to the imaginary part of* z *in the two quarter planes for which* $|y| \geqq |x|$.

The continued fraction of the last theorem can readily be transformed, by neglecting a numerical factor, so as to take either the form

$$\frac{1}{a_1 z'} + \frac{1}{a_2 z'} + \frac{1}{a_3 z'} + \cdots \qquad (a_n > 0),$$

or the form preferred by STIELTJES and cited in § 1. To throw it into the first form one has only to put $z = iz'$. If it is to be reduced to the form of STIELTJES, first put $z = w^{-1}$ and transform so as to remove w from the denominators of the partial quotients. Each numerator after the first becomes equal to $-w^2$. The desired form is then immediately obtained by setting $-w^{-2} = z'$ and clearing the numerators of z'.

§ 18. The considerations by which theorem 8 was established apply with little change to continued fractions having the form

$$\frac{1}{b_1 z + c_1} + \frac{1}{-b_2 z - c_2} + \frac{1}{b_3 z + c_3} + \frac{1}{-b_4 z - c_4} + \cdots$$

in which the coefficients b_n have a common sign. If both $\sum b_n$ and $\sum c_n$ are convergent, $\prod (1 + |b_n z| + |c_n|)$ is also convergent and can be expressed as a power series which converges over the finite plane. As in § 17, the numerators and denominators of the alternate convergents tend to assume a definite limiting form, in which the coefficient of each power of z is the sum of certain products of the b_n and c_n whose moduli are included among the products which make up the coefficient of the corresponding power of z in the series for the above product $\prod$. The limits of the numerators and denominators of the alternate convergents are therefore again holomorphic functions.

If $\sum b_n$ or $\sum |c_n|$ is divergent, some additional restriction must be added. Suppose $|c_n|/|b_n|$ to have an upper limit L. Then at least $\sum b_n$ is divergent. Since

$$\frac{|\beta_n|}{|a_n|} = \frac{|y b_n|}{|x b_n + c_n|} \geqq \frac{|y b_n|}{|x b_n| + L|b_n|},$$

it is evident that $|\beta_n|/|a_n|$ has a finite lower limit for any value of z for which $y > 0$. The conclusions of (b), theorem 8, then follow. Hence

THEOREM 9. *If a continued fraction has the form*

$$\frac{1}{b_1 z + c_1} - \frac{1}{b_2 z + c_2} - \frac{1}{b_3 z + c_3} - \frac{1}{b_4 z + c_4} - \cdots$$

in which the coefficients b_n have a common sign, it will have the following properties:

(a) The roots of the numerators and denominators of its convergents all lie upon the real axis of z.

(b) If $\sum b_n$ and $\sum |c_n|$ are convergent, the limits of the numerators and denominators of the odd, also of the even convergents are holomorphic functions whose zeros lie upon the real axis.

(*c*) *If* $\sum b_n$ *is divergent and* $|c_n|/|b_n|$ *has an upper limit, the continued fraction converges for all values of* z *which are not real, and it represents a function which is analytic over the entire plane with the exception of the whole or a part of the real axis and which vanishes only for real values.*

The reality of the roots of the continued fraction has been previously proved by SYLVESTER,* who also shows that the roots of the denominators of two consecutive convergents alternate with each other. He does not consider the question of the convergence of the continued fraction when the number of elements is infinite.

§ 19. The theorems of III. may also be applied to certain classes of irregular continued fractions. It will suffice here to give as illustrations two applications of this nature.

THEOREM 10. *If the elements of an algebraic continued fraction*

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots$$

are all either positive numbers multiplied by z *or complex numbers* $\alpha_n + i\beta_n$ *in which* $|\alpha_n|/|\beta_n|$ *exceeds some fixed positive number for all values of* n, *the continued fraction will have the following properties:*

(*a*) *None of the roots of the numerators and denominators of its convergents will lie within the half plane in which the real part of* z *is positive, or upon its boundary.*

(*b*) *If* $\sum |\lambda_n|$ *is divergent, the continued fraction represents within this half plane an analytic function which does not vanish at any point in its interior.*

(*c*) *If* $\sum |\lambda_n|$ *is convergent, the limits of the numerators and denominators of the even, also of the odd convergents are holomorphic functions whose roots lie without this half plane.*

The second class of irregular algebraic continued fractions is that in which the even (odd) denominators of the partial quotients are all positive numbers and the odd (even) partial denominators are positive numbers or such numbers multiplied by z. The region of convergence covers the same half plane and the zeros of the numerators and denominators of the convergents and of the limiting functions lie upon the negative half of the axis. The continued fraction of STIELTJES is the special case in which z appears in *every* odd denominator.

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN.

* Philosophical Transactions, vol. 143, part I, p. 497.

GEOMETRY WITHIN A LINEAR SPHERICAL COMPLEX*

BY

PERCEY F. SMITH

Introduction.

The following pages are devoted to the study of a point-sphere correspondence of involutory character, which appears as direct generalization from a certain point of view of the well known point-sphere correspondence arising in a dilatation and the point-point correspondence of spherical inversion. Problems resulting from the establishment by this correspondence of a spherical contact transformation are discussed.

Elsewhere in the Transactions† I have shown the importance to the higher spherical geometry of the involutory contact-transformation here studied, for its relation to the group of all spherical contact transformations is precisely that of spherical inversion to the group of the geometry of reciprocal radii.

The titles of the sections indicate to a great extent their contents.

§1. *Point-sphere correspondence: definition and discussion.*

The *æquatio directrix* in the rectangular point coördinates x, y, z, X, Y, Z,

$$\begin{aligned}\Phi \equiv\ & (X^2 + Y^2 + Z^2 - r_0^2)(x^2 + y^2 + z^2 - r_0^2) \\ & + r_0^2(1 - \kappa_0^2)[(X - x)^2 + (Y - y)^2 + (Z - z)^2] = 0\,,\end{aligned} \tag{1}$$

establishes in space R_3 the most general involutory point-sphere correspondence Γ such that the ∞^3 coaxial systems determined each by a point and its associated sphere intersect in a fundamental sphere σ, viz., $x^2 + y^2 + z^2 = r_0^2$. Under (1) the point $P(x, y, z)$ corresponds to the sphere S of center x', y', z' and radius r' where

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{r_0^2(1 - \kappa_0^2)}{x^2 + y^2 + z^2 - r_0^2\kappa_0^2}, \tag{2}$$

* Presented to the American Mathematical Society October 27, 1900. Received for publication February 23, 1901.

† *On surfaces enveloped by spheres belonging to a linear spherical complex*, vol. 1 (1900), p. 371.

(3)* $$r' = \frac{r_0\kappa_0(x^2 + y^2 + z^2 - r_0^2)}{x^2 + y^2 + z^2 - r_0^2\kappa_0^2}.$$

The spheres S in question are subjected to a single condition. The elimination of x, y, z from (2) and (3) gives

$$x'^2 + y'^2 + z'^2 - r_0^2 - r'^2 - 2r_0r'\kappa = 0,$$

if

$$\kappa = \frac{1 + \kappa_0^2}{2\kappa_0}.$$

The ∞^3 spheres S, therefore, belong to a linear spherical complex A whose fundamental sphere is σ and whose "constant" equals κ.

Special features of Γ are the following:—Points on σ correspond to themselves. Points on the sphere $\sigma_0(0, 0, 0, -r_0\kappa_0)$ give the ∞^2 planes of A, and these all touch the sphere $\sigma_0'(0, 0, 0, -r_0\kappa)$. The sphere σ_0 belongs to A. *Any* point of R_3 at infinity and *not* on the imaginary circle C corresponds to σ_0, while a point P of C gives the ∞^1 spheres of radius $-r_0\kappa_0$ whose centers lie upon the line joining P with the origin. Therefore the *fundamental configurations* of Γ are the points of C in R_3 and the sphere σ_0 of A.

A property of the correspondence of prime importance is established by the equation

(4)
$$(x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2 - (r_1' - r_2')^2 = \frac{r_0^2(1 - \kappa_0^2)[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]}{(x_1^2 + y_1^2 + z_1^2 - r_0^2\kappa_0^2)(x_2^2 + y_2^2 + z_2^2 - r_0^2\kappa_0^2)},$$

in which $S_1(x_1', y_1', z_1', r_1')$ and $S_2(x_2', y_2', z_2', r_2')$ are taken as corresponding to $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, respectively. This gives

THEOREM 1. *The ∞^1 points P of a minimum line intersecting σ in P_0 become, under Γ, ∞^1 spheres of A touching one another at P_0.*

By considering the minimum cone with vertex P, we easily get a second result:

A sphere S of a linear spherical complex A is touched by ∞^2 spheres of A along its circle of intersection with the fundamental sphere.†

This circle we call the *trajectory circle* of S.

Let E be any plane passing through P, and let the two minimum lines drawn in E through P intersect σ in p_1, p_2. Then p_1, p_2 are vertices of a skew quadrilateral $p_1p_2P_1P_2$ of minimum lines lying in S, while p_1p_2, P_1P_2 are polar lines with respect to S, since the planes $p_1P_1p_2$, $p_1P_2p_2$ touch S at P_1 and P_2 respectively. Denoting these planes by E_1, E_2, we have the result:

*Cf. Transactions, p. 383. For $\kappa_0 = 0$, the correspondence is the usual inversion in σ. Evidently the case $\kappa_0^2 = 1$ is of no interest.

†LIE, Mathematische Annalen, vol. 5, p. 207.

THEOREM 2. *A surface element* (P, E) *corresponds under* Γ *to two elements* (P_1, E_1), (P_2, E_2) *of the sphere* S. *The planes* E, E_1, E_2 *intersect in the radical plane of* S *and* σ, *while the circle drawn through* P, P_1, P_2 *is orthogonal to* E, S, σ.

For example, if P is at infinity in E and not on the imaginary circle, then (P_1, E_1) and (P_2, E_2) are on σ_0 while E_1, E_2, E are parallel.

If, now, P moves along the minimum line Pp_1, then P_1 and P_2 are displaced on p_1P_1 and p_1P_2 respectively, this being true for any plane E through Pp_1. Therefore,

A minimum line regarded as defined by its ∞^2 *surface elements becomes under* Γ *two minimum lines whose plane belongs to* A, *the three lines intersecting on* σ.

The result just stated is of fundamental importance, for we may at once deduce the following properties:

A sphere s becomes under Γ two spheres s_1 and s_2, and s_1, s_2, s intersect on σ. If s degenerates into a plane, then s_1 and s_2 touch the sphere σ_0.

The spheres determined by $P_1p_1p_2P$ and $P_2p_1p_2P$ are invariant. Hence the planes tangent to either along its intersection with σ must be invariant, i. e., must touch the sphere σ_0. The conclusion is:

THEOREM 3. *The spheres of a linear spherical complex* A_0 *with fundamental sphere* σ *and constant* κ_0 *are invariant under* Γ. *The* ∞^2 *spheres of* A_0 *tangent to a given sphere* s *also touch each of two spheres* s_1 *and* s_2, *into which* s *goes over by* Γ.

From the properties enunciated in this last theorem, I have called the transformation of space which Γ establishes *inversion in the complex* A_0,* and have symbolized it by (A_0).

Theorem 2 may be given another interesting form. For if we denote the two spherical images of the plane E on the sphere S by p_0 and p_0', i. e., the line joining p_0 and p_0' is that diameter of S which is perpendicular to E, then the spherical inversion in the center P which leaves S invariant transforms p_0 and p_0' into P_1 and P_2 respectively. Thus we have the result:

By the spherical inversion with center P *which transforms* S *into itself, the spherical image on* S *of any surface element* (P, E) *inverts into the point* P_1 *of the corresponding surface element* (P_1, E_1).

Next consider all surface elements with common P whose planes intersect in a line L. The special representation of these elements on S gives the points of the great circle C whose axis is parallel to L. Then the transformed elements lie along the small circle c on S into which C inverts in the center P. The circle c intersects the trajectory circle of S orthogonally, since its plane passes through the pole of the radical plane of P and S with respect to S.

*Cf. Transactions, vol. 1, p. 376.

The results established in this section hold with certain obvious exceptions for any number of dimensions, although the reasoning here given has been restricted to the case of ordinary space.

§ 2. *The corresponding contact transformation.*

We proceed now to find the equations of (A_0) as a contact transformation.

For convenience, we write in (1) x_1, x_2, x_3 for x, y, z, and similarly for X, Y, Z. As coördinates of the surface element (P, E) we may take

$$x_1, x_2, x_3; \quad p_1, p_2, p_3, \quad \text{or} \quad (x, p)$$

if $p_1 : p_2 : p_3$ determine the direction of the normal to E. Let (P, E) become (X, P) by (A_0); then we have to determine X and P from the equations †

$$\Phi = 0, \quad \rho p_i - \frac{\partial \Phi}{\partial x_i} = 0, \quad \rho P_i + \frac{\partial \Phi}{\partial X_i} = 0 \qquad (i = 1, 2, 3), \tag{5}$$

after elimination of ρ.

The result is

$$\left.\begin{aligned} X_i &= x_i - \frac{(x_i s - p_i \delta)}{s(u + \delta^2) - 2\delta l}(u - r_0^2), \\ P_i &= \frac{p_i(u - \delta^2) - 2x_i(l - \delta s)}{r_0^2 - \delta^2} \qquad (i = 1, 2, 3), \end{aligned}\right\} \tag{6}*$$

where

$$u = \sum_i x_i^2, \quad s^2 = \sum_i p_i^2, \quad l = \sum_i p_i x_i, \quad \delta = -r_0 k_0.$$

To these may be added, if $S^2 = \sum_i P_i^2$,

$$S = \frac{s(u + \delta^2) - 2\delta l}{r_0^2 - \delta^2}. \tag{7}$$

The equation $s(u + \delta^2) - 2\delta l = 0$ defines the congruence of spheres of radius δ and with centers on the minimum cone whose vertex is the center of σ.

The element (P, E) is invariant when, and only when, P is on $\sigma\,(u = r_0^2)$ and E is tangent to $\sigma_0\,(l - \delta s = 0)$. This result shows that the spheres of the complex A_0 are unchanged by (A_0) as already stated.

The following special results are to be noted, and may be readily verified.

An element (P, E) such that P is on σ and E belongs to A transforms into ∞^1 (P', E')'s, where E' is a minimum plane, and P' *any* point on the minimum line in E' through P.

An element (P, E) belonging to σ_0 gives the ∞^1 elements of E at infinity.

---

* Cf. LIE-ENGEL, *Transformationsgruppen*, vol. II, p. 151.

† Transactions, loc. cit., p. 385. As remarked above, $\delta = 0$ gives inversion in σ.

An element (P, E) of the imaginary circle at infinity goes over into the ∞^2 elements of the sphere into which P inverts under (A_0) (§ 1).

§ 3. *Duality of the geometry of reciprocal radii and geometry within a linear spherical complex.*

The group of the geometry within a linear spherical complex A is that subgroup G_{10} of the general G_{15} of all sphere-sphere contact transformations under which A is invariant. Any transformation of G_{10} is compounded of inversions in complexes in involution with A. The contact transformation (A_0) which carries A over into the complex of all points transforms G_{10} into the group of the geometry of reciprocal radii, every transformation of which is compounded of spherical inversions. Thus within the general higher spherical geometry, *the two geometries mentioned appear equally justified.* This result is fundamental for what follows. Let us now assign to the infinite region the characteristics demanded by the geometry of reciprocal radii. Then the points on the imaginary circle are to be excluded and the infinite region becomes the "point at infinity." The fundamental configurations of the correspondence Γ (§ 1) disappear, while for (A_0) the $\infty^2(P, E)$'s whose points are on σ and whose planes touch σ_0' constitute the fundamental configuration. These go over into the ∞^3 minimum elements (finite).

We may now mention a number of the more interesting correspondences in the two geometries; in each case the first statement refers to the geometry of reciprocal radii.

Surface f, point locus.	Surface F, enveloped by ∞^2 spheres of A.
Curve of intersection c of f and σ.	Double curve c of F on σ.
Curve of intersection c' of f and σ_0.	Developable of bitangent planes to F, enveloping σ_0'.
Principal spheres of f.	Principal spheres of F.
Minimum developable intersecting σ in c_0.	Curve c_0.
Singular line of curvature on f, viz., the curve of contact of f with the circumscribing minimum developable f'.	Line of curvature on F lying on σ, along which the principal spheres have four-point contact; this line of curvature and the double curve c above mentioned constitute the entire intersection of F and σ.*

* This singular line of curvature on any surface f is determined by differentiation and elimination. These operations suffice also for the determination of the line of curvature in question on F.

Confocal surfaces, i. e., surfaces inscribed in a minimum developable f'.	Surfaces enveloped by ∞^2 spheres of A and inscribed in a developable, whose planes belong to A, along a line of curvature on σ.
Lines of curvature on f.	Lines of curvature on F.
Minimum lines on f.	Minimum lines on F.

The congruence of ∞^2 circles orthogonal at once to f and the fundamental sphere σ intersect F also orthogonally, such that each circle cuts F in the two points of contact of an enveloping sphere (§ 3). The ∞^1 circles of this congruence intersecting f along any curve trace out upon F the corresponding curve.

It will be remarked that the developables whose planes belong to A, i. e., touch the sphere σ_0', play an analogous rôle in the geometry here discussed to that of minimum developables in the geometry of reciprocal radii. The first mentioned surfaces are remarkable for the fact that their lines of curvature (non-linear) lie upon spheres concentric with the fundamental sphere.

Continuing the discussion of the duality, we have as corresponding:

Curve C, point locus.	Surface F enveloped by ∞^1 spheres of A, i. e., an annular surface.
Point of intersection of C and σ.	Nodes on F.
Points of intersection of C and σ_0.	Planes touching F along circles.
Minimum developable passed through C and intersecting σ in c.	Line of curvature c on F, along which the principal spheres have four-point contact, the complete insection of F and σ.
Minimum curve, cuspidal edge of minimum developable intersecting σ in c.	Curve c.

The locus of centers (x', y', z') of the spheres of A enveloping F is found from f by the *point* transformation defined by equations (2), § 1.*

The conformal property of transformations of the group of the geometry of reciprocal radii is expressed in the geometry of surface elements as follows:— (P, E) and (P, E') go over into (P', E_1) and (P', E_1') such that the angle of EE' equals that of E_1E_1'. For definiteness orientation must be assigned to the planes in question. Now by inversion in the complex (A_0) which transforms P into S, (P, E) and (P, E') go over into the elements of S, $(P_1 E_1)$ and (P_2, E_2), and P_1 and P_2 lie on a circle of S orthogonal to the trajectory circle

* For a discussion of the locus of centers, see Transactions, vol. 1, p. 383, seq.

(§ 1), while the minimum elements of the pencil $(P, E), (P, E')$, invert into the elements common to these two circles. We have then from the results of the last part of § 1 the

THEOREM. *If the linear spherical complex A invert by (A_0) into the complex of all points so that $(P, E), (P, E')$ become $(P_1, E_1), (P_1', E_1')$ on the sphere S, and if the circle on S through P_1 and P_1' orthogonal to the trajectory circle intersect the latter in P_0 and P_0', then the logarithm of the cross ratio $(P_1P_0, P_1'P_0')$ divided by $2\sqrt{-1}$ equals one-half the angle of the planes E and E', and this cross ratio is unaltered by every transformation of G_{10}.*

Let, now, a curve C invert in (A_0) into the annular surface F enveloped by ∞^1 spheres of A. Any curvature strip* along C becomes under (A_0) a curvature strip on F, say along the curve c on F. By the theorem of JOACHIMSTHAL, any strip along C intersecting that mentioned under a constant angle is also a curvature strip, therefore inverts under (A_0) into a curvature strip on F, and in this way *all* non-circular lines of curvatures c on F are found. Hence the result:

Theorem of Joachimsthal. Two curvature strips along a curve intersect under a constant angle.

The non-circular lines of curvature of an annular surface whose spheres belong to a linear spherical complex determine projective ranges on the circles of curvature.†

In the same way we get as dual theorems:

Theorem of Dupin. Any system of ∞^3 surfaces mutually orthogonal intersect along lines of curvature.

If in any system of ∞^3 surfaces enveloped by the spheres of a linear spherical complex A the six points of contact of every sphere of the complex are the intersections of three mutually orthogonal circles on that sphere whose common orthogonal circle is the trajectory circle, then the annular surface enveloped by ∞^1 spheres of A and tangent to a surface of the system along a line of curvature will touch also a second surface of the system along a line of curvature.

* i. e., a system of ∞^1 elements belonging to C whose normals form a developable surface.

† This theorem holds for all annular surfaces. But when the enveloping spheres belong to a linear spherical complex the non-circular lines of curvature are determined by *quadratures*, for a *particular* solution of the corresponding RICCATI equation is known, viz., the line of curvature c above mentioned.

Other theorems of interest might be added to those already given, but enough has been given to justify the study of the duality.

It is evident that the general surface within a linear spherical complex will intersect the plane at infinity only along the imaginary circle. Furthermore, such a surface will contain no *finite* minimum element, whence the result:

A general algebraic surface in the geometry within a general linear spherical complex is of equal order and class, cutting the plane at infinity only in the imaginary circle, and the enveloping minimum developable is singular.

§ 4. *The group of inversions in all linear spherical complexes with common fundamental sphere.*

If (A) and (A') are inversions in complexes A and A' having a common fundamental sphere σ, then *

$$(AA') \equiv (\Omega A''),$$

where (Ω) denotes change in orientation and A'' also has σ for fundamental sphere. If $\kappa, \kappa', \kappa''$ are the constants of A, A', A'' respectively, then

$$\kappa'' = \pm \frac{1 - \kappa\kappa'}{\kappa - \kappa'}. \tag{8}$$

Now (Ω) results from (6) and (7) when $\delta = \infty$, for then $(x; p)$ becomes $(x; p)$ *but* s *changes sign*. Hence (Ω) plays here the rôle of the identical transformation, and we have

THEOREM 4. *The equations* (6) *define a group* G_1 *of contact transformations in the parameter* $\kappa_0 (\delta = -r_0\kappa_0)$. *The composition law is given by* (8).

We find the infinitesimal transformation $U(f)$ of G_1 in the usual way by setting $\delta = \infty + \delta t$, and obtain

$$\frac{\delta x_i}{\delta t} = \frac{p_i}{s}(u - r_0^2), \qquad \frac{\delta p_i}{\delta t} = -2x_i s \qquad (i = 1, 2, 3);$$

therefore G_1 is generated by

$$U(f) \equiv \frac{u - r_0^2}{s}\left(p_1\frac{\partial f}{\partial x_1} + p_2\frac{\partial f}{\partial x_2} + p_3\frac{\partial f}{\partial x_3}\right) \cdot - 2s\left(x_1\frac{\partial f}{\partial p_1} + x_2\frac{\partial f}{\partial p_2} + x_3\frac{\partial f}{\partial p_3}\right). \tag{9}$$

We determine now the *invariants* of G_1. The equation $U(f) = 0$ has the six solutions

* This follows from Theorem 4, p. 378, Transactions, vol. 1 (1900).

$$(10)\qquad \begin{cases} a_3 \equiv x_1p_2 - x_2p_1, \quad a_1 \equiv x_2p_3 - x_3p_2, \quad a_2 \equiv x_3p_1 - x_1p_3, \\ \beta_i \equiv x_i l - \tfrac{1}{2}p_i(u + r_0^2) \qquad (i = 1, 2, 3), \end{cases}$$

connected by the identical relation

$$(11)\qquad a_1\beta_1 + a_2\beta_2 + a_3\beta_3 \equiv 0,$$

but otherwise independent. Hence

THEOREM 5. *Every function invariant under G_1 has the form*

$$F(a_1, a_2, a_3, \beta_1, \beta_2, \beta_3),$$

where F is homogeneous of dimension zero.

The a, β have an interesting geometrical significance. For, if L is the line of intersection of the plane E of (P, E) $[(x; p)]$ with the radical plane of P and the fundamental sphere σ, then the equations of L in current coördinates X_1, X_2, X_3 are

$$a_3X_1 = a_1X_3 + \beta_2, \quad a_3X_2 = a_2X_3 + \beta_1,$$

i. e., the a, β are the *line coördinates* of L. The plane through P perpendicular to L is

$$a_1X_1 + a_2X_2 + a_3X_3 = 0;$$

and if C be its intersection with L, then the coördinates of C are

$$\frac{a_2\beta_3 - a_3\beta_2}{a_1^2 + a_2^2 + a_3^2}, \quad \text{etc.},$$

and

$$CP^2 = \frac{\sum_i \beta_i^2}{\sum_i a_i^2} - r_0^2.$$

These results give

THEOREM 6. *By the transformations of G_1 upon any surface element (P, E) the point P is made to describe the circle orthogonal at once to E and the fundamental sphere, while E turns round the axis of this circle.*

For the coördinates of the axis, the center, and the radius of the circle in question are invariants.

The most general *system* of ∞^4 elements invariant under G_1 will be defined by an equation of the form

$$(12)\qquad F(a_1, a_2, a_3, \beta_1, \beta_2, \beta_3) = 0,$$

F being homogeneous.

This equation, however, also defines a line complex, the locus of ∞^3 axes $(a; \beta)$ of the circles of Theorem 5. We are therefore led to a solution of the problem:—*To determine all partial differential equations of the first order*

the integral surfaces of which cut orthogonally a given system of ∞^3 *circles orthogonal to a given sphere* σ.

For the given system of circles is uniquely defined by the line complex formed of their axes, (α, β) i. e., by an equation (11). If then we replace the α and β from equations (10) the problem is solved. *

The "characteristic function" of the infinitesimal homogeneous contact transformation $U(f)$ is

$$H \equiv s(u - r_0^2),$$

since †

$$U(f) \equiv \sum_{i=1}^{3} \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial f}{\partial p_i} \right).$$

From theorem 5 we see that the ∞^1 surfaces resulting from transformation of an *arbitrary* surface by the group G_1 belong, by the theorem of RIBAUCOUR, to an orthogonal system. Furthermore, under $U(f)$ the lines of curvature go over into lines of curvature. Finally, SOPHUS LIE has found that the most general infinitesimal contact transformation possessing these properties ‡ has for characteristic function

$$H \equiv s(a_0 u + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4),$$

into which $U(f)$ transforms by a translation.

§ 5. *Generalization to space of n dimensions; in particular, to* $n = 2$.

The preceding discussion may readily be generalized to n dimensions. Theorems 1–5 hold, with trivial changes, in general.

Confining ourselves for brevity to the discussion of the problem mentioned in § 4 for the case $n = 2$, we may state the result:

The most general differential equation of the first order whose integral curves cut orthogonally a given system of ∞^1 circles orthogonal to a given circle σ is determined thus:—

The center of σ being the origin and its radius r_0, let the given system of circles be defined by the locus of centers

$$F(\alpha, \beta, \gamma) = 0,$$

F being homogeneous in the homogeneous rectangular coördinates α, β, γ. Then setting

---

* Writing, of course,

$$p = \frac{\partial x_3}{\partial x_1} = -\frac{p_1}{p_3}, \quad q = \frac{\partial x_3}{\partial x_2} = -\frac{p_2}{p_3}.$$

The integration problem of such a differential equation is much simplified by observing that its integral surfaces admit of $U(f)$.

† Cf. LIE-ENGEL, *Transformationsgruppen*, vol. 2, p. 263.

‡ Vide Leipziger Berichte, 1889, p. 153. Cf. also DARBOUX, *Leçons sur les systèmes orthogonaux*, vol. 1, pp. 56, 72, in which (A_0) is called the *transformation de Ribaucour*.

$$\gamma = x_1 p_2 - x_2 p_1, \quad \alpha = \tfrac{1}{2} p_2 (u + r_0^2) - x_2 l, \quad \beta = x_1 l - \tfrac{1}{2} p_1 (u + r_0^2),$$

$$y' = \frac{dx_2}{dx_1} = -\frac{p_1}{p_2},$$

where

$$u = x_1^2 + x_2^2, \quad l = p_1 x_1 + p_2 x_2,$$

we obtain the required equation,

$$F\left(\tfrac{1}{2}(x^2 + y^2 + r_0^2) + y(xy' - y),\ \frac{y'}{2}(x^2 + y^2 + r_0^2) - x(xy' - y),\ x + yy'\right) = 0,$$

wherein x and y have been written for x_1 and x_2, and F is an *arbitrary homogeneous function.*

The characteristic function for the infinitesimal contact transformation $U(f)$ now takes the form*

$$W \equiv -(x^2 + y^2 - r_0^2)\sqrt{1 + y'^2}.$$

§ 6. *Some special cases of the preceding results.*

Inversion in a general linear spherical complex is transformed by a spherical inversion whose center lies on the fundamental sphere into inversion in a plane spherical complex, i. e., into the transformation of LAGUERRE.† The fundamental sphere σ is replaced by a fundamental plane π. If π be taken for $z = 0$, the point-sphere correspondence Γ of § 1 becomes

$$(13) \qquad \Phi' \equiv (1 - \kappa_0^2)\{(X - x)^2 + (Y - y)^2 + (Z - z)^2\} + 4zZ = 0,$$

and is *the most general involutory point-sphere correspondence such that a point P and its associated sphere S have a common radical plane.*

The ∞^3 spheres S belong to a plane complex E with fundamental plane π and constant $\kappa = 1 + \kappa_0^2/2\kappa_0$, as before.

Characteristic properties of the transformation are (1), planes transform into planes, the plane at infinity being invariant, and (2), the distance of two points P and P' equals the length of a common tangent of the corresponding spheres.

Theorems 1, 2, 3 hold here.

The equations of the contact transformation, i. e., inversion in the complex E_0 with fundamental plane $z = 0$ and constant κ_0 are readily found as before (§ 2). As *fundamental configurations* of (E_0) appear (1), the ∞^2 surface elements whose points lie in π and whose planes make with π the angle $\cos^{-1}\kappa$, and (2), the ∞^2 elements of the imaginary circle at infinity. An element of (1) transforms into the ∞^1 minimum elements along a minimum line, while to an

---

* Cf. SOPHUS LIE, *Geometrie der Berührungstransformationen*, vol. 1, p. 150.

† SMITH, Annals of Mathematics, July, 1900.

element of (2) correspond the ∞^2 elements of a plane belonging to E. Especially noteworthy is the fact that two elements (P_1, E), (P_2, E) having the same plane E go over into elements of the same sort, (P_1', E'), (P_2', E') such that $P_1P_2 = P_1'P_2'$. This property is expressed thus:

Tangential distance is invariant under inversion in a plane spherical complex.

By a spherical inversion elements having a common point (P, E_1), (P, E_2) transform into elements of like nature (P', E_1'), (P', E_2'), and such that the angle of E_1 and E_2 equals that between E_1' and E_2'. I have elsewhere bespoken for the transformation of LAGUERRE an attention it has not as yet received,* and the analogy just remarked would seem to warrant this.

Within the general G_{15} (§ 3) of all sphere-sphere contact transformations in space is contained as subgroup a G_{10}' of all such transformations as are compounded of inversions in plane spherical complexes. This G_{10}' is the group of the geometry of reciprocal directions. The contact transformation (E_0) inverts the group G_6 of all transformations compounded of plane reflections into that subgroup G_6' of G_{10}' under which the plane spherical complex E remains invariant. In this way we establish a *duality of the Euclidean geometry* (excluding similitude) *to spherical geometry within a plane spherical complex.*

It is not proposed to develop this question. There is of course much in § 3 which is valid here, but we have in addition the very interesting correspondence of distance to tangential distance.

The group of ∞^1 contact transformations made up of inversions in all plane complexes having the same fundamental plane $x_3 = 0$ is generated by the infinitesimal contact transformation $U(f)$ whose characteristic function is

$$H \equiv 2x_3 s,$$

i. e.,

$$U(f) = \frac{2x_3}{s}\left(p_1\frac{\partial f}{\partial x_1} + p_2\frac{\partial f}{\partial x_2} + p_3\frac{\partial f}{\partial x_3}\right) - 2s\frac{\partial f}{\partial p_3}.$$

The general invariant function has the form

$$F(p_1, p_2, p_1x_1 + p_2x_2 + p_3x_3, x_1p_2 - x_2p_1, x_3s),$$

F being homogeneous of dimension zero, while every invariant system of ∞^4 elements is defined by setting any homogeneous function of these arguments equal to zero.

Under the present G_1 carried out on the surface element (P, E) the point P describes a circle c whose center lies in the fundamental plane while E turns around the axis l of this circle.

The center of c is

*In the Annals of Mathematics, July, 1900.

$$\left(\frac{p_1 l + p_2(x_1 p_2 - x_2 p_1)}{p_1^2 + p_2^2},\quad \frac{p_2 l - p_1(x_1 p_2 + x_2 p_1)}{p_1^2 + p_2^2}\right),$$

and its radius equals $x_3 s\, p_1 p_2 (p_1^2 + p_2^2)^{-\frac{1}{2}}$. The axis l intercepts on OX and OY the distances

$$\frac{l}{p_1},\quad \frac{l}{p_2}.$$

Finally, the preceding discussion put us in a position to solve the problem:

To determine all partial differential equations of the first order whose integral surfaces intersect orthogonally a given system of ∞^3 circles whose centers lie in a given plane and whose planes are perpendicular to that plane.

If now the fundamental plane of Γ is the plane at infinity, the correspondence becomes

$$(X - x)^2 + (Y - y)^2 + (Z - z)^2 = a^2.$$

The corresponding contact transformation is a dilatation of a point P into the sphere S of constant radius a and center at P, while the problem mentioned above becomes this: *To determine all partial differential equations of the first order whose integral surfaces have for normals the lines of a given line complex.**

§ 7. *Geometrical definition of the correspondence* Γ.

The subject under discussion appears in a new light from a definition of the fundamental correspondence (§ 1) now to be established.

Let P be any point in space R_n, and O a point in R_{n+1} external to R_n. Then by a spherical inversion in the center O the minimum cone V of R_{n+1} whose vertex is P transforms into a minimum cone V' of vertex P'. The cone V' intersects R_n in a sphere S, while the sphere S_0 into which R_n inverts cuts R_n in a sphere σ. Also V and S intersect on σ. Reciprocally, the minimum cone in R_{n+1} with vertex any point P'' on S, passes through P', and hence inverts into a minimum cone containing P. We see, therefore, that the correspondence of P and S possesses the characteristics of Γ. This gives

THEOREM 7. *The point-sphere correspondence Γ in R_n may be thus defined: A spherical inversion in a center in R_{n+1} external to R_n inverts the minimum cone of R_{n+1} whose vertex is a point P of R_n into a minimum cone whose intersection with R_n is the corresponding sphere S.*

To proceed to the other properties of Γ, we choose the sphere as element of R_n. The ∞^{n+1} points P' of R_{n+1} project by *minimal projection* into the ∞^{n+1} spheres S of R_n. In fact, the minimum cone in R_{n+1} with vertex P' intersects R_n in the sphere S. We shall lose nothing in generality by confining the discussion to $n = 2$; then R_2 becomes a plane π; and the ∞^3 points

* Cf. SOPHUS LIE, Mathematische Annalen, vol. 5 (1872), p. 204.

P' of space project into the ∞^3 circles C of π, such that the center of C is the orthogonal projection of P' on π while the radius of C equals the ordinate of P' relative to C multiplied by $\sqrt{-1}$.*

The ∞^2 points of any surface F in R_3 project into the ∞^2 spheres of a circular complex whose curve of singularities is the intersection with π of the minimum developable circumscribed to F. In particular, let F be a sphere S_0. If then S_0 cuts π in the circle σ of radius r_0 and center O, a simple calculation shows that any point on S_0 projects into a circle in π of center c' and radius r' such that

$$\overline{cc'}^2 - r_0^2 - r'^2 - 2r_0 r'\kappa = 0\,,$$

where

$$R_0^2 = r_0^2(1 - \kappa^2),$$

R_0 being the radius of S_0. We see then

THEOREM 8. *The points of a sphere S_0 in R_{n+1} project by minimal-projection on R_n into the spheres of a linear spherical complex whose fundamental sphere is the intersection of S_0 and R_n.*

Spherical inversion in the sphere S_0 in R_3 goes over into inversion in the corresponding circular complex in π. Hence the conformal group of point transformations in R_3 transforms by minimal projection into the group of all circular contact transformations in π. A plane reflection in R_3 becomes inversion in a line complex, which, if the plane of the reflexion is parallel to π, reduces to a dilatation. The theorems proved in §§ 2, 3 in the paper already quoted † are readily verified by the known properties of inversions in space.

If now we consider either of the two commutative spherical inversions (I_0) in space by which π transforms into the sphere S_0, then to (I_0) corresponds in π by the minimal-projection, the inversion in the circular complex A_0 to which Γ gives rise as a contact transformation. The main results of the previous sections are thus easily established.

Let C be any curve on S_0, and denote the minimum developable which is the envelope of the minimum cone V whose vertex describes C by F'. Then F' intersects π in a curve c which is enveloped by ∞^1 circles in the circular complex A. The locus of centers of the circles enveloping c is the orthogonal projection of C on π, and c is derived by inversion in a circular complex A_0 from a primitive curve c', regarded as point locus, which is the stereographic projection of C on π. In particular, if C is a general sphero-quartic, the minimum developable in question envelops a system of confocal cyclides, and is of order 16 passing 8 times through the imaginary

* Cf. for further details, LIE, *Geometrie der Berührungstransformationen*, vol. 1, p. 437, or KLEIN, *Höhere Geometrie*, p. 472.

† SMITH, Transactions, vol. 1 (1900).

circle at infinity and containing five double sphero-quartics of which C is one. The curve c in π is therefore the curve of singularities of the general quadratic circular complex, and the properties enumerated in the paper already quoted * follow at once.

We may now state

THEOREM 9. *Given an M_r on a sphere S_0 in R_{n+1} and let the manifoldness which is the envelope of a minimum cone whose vertex describes M_r intersect R_n in M'_{n-1}. Then M'_{n-1} is enveloped by ∞^r spheres of a linear spherical complex in R_n, this complex being the locus of the spheres into which the points of S_0 pass over by minimal projection. The locus of centers of the spheres enveloping M'_{n-1} is the orthogonal projection of M_r on R_n, and M'_{n-1} may be derived by inversion in a linear spherical complex from a primitive m_r regarded as point locus, viz. the stereographic projection of M_r on R_n.*

SHEFFIELD SCIENTIFIC SCHOOL, NEW HAVEN, CONN., *February*, 1901.

* SMITH, Transactions, vol. 1, p. 390.

A NEW DETERMINATION OF THE PRIMITIVE CONTINUOUS GROUPS IN TWO VARIABLES*

BY

H. F. BLICHFELDT

The primitive continuous groups of point-transformations in two variables can, by a proper choice of the variables, be transformed into *projective groups* of the plane, a result LIE obtains after determining the canonical forms of the primitive groups.† This fact can, however, be established from the general properties of such groups, and its use leads to a new determination of these primitive groups, to show which is the object of this paper.

A primitive group will be defined as a group which does not leave invariant a differential equation of the first order.‡ Such a group is at least three-parametric, as a two-parametric group possesses a differential invariant of the first order, J say, and therefore an invariant differential equation of the first order, $f(J) = \text{constant}$.§

§ 1.

It will first be necessary to show that *any group in two variables and of more than two parameters leaves invariant at least one differential equation, which is integral and algebraic in the derivatives* dy/dx, d^2y/dx^2, *etc., involved.*

Let $Xf \equiv \xi\, \partial f/\partial x + \eta\, \partial f/\partial y$ indicate any one of the infinitesimal transformations of the group. If we "extend" these n times,‖ and write $y_1, y_2, \cdots, y_n$ for $dy/dx, d^2y/dx^2, \cdots, d^ny/dx^n$, and $p, q, p_1, p_2, \cdots, p_n$ for $\partial f/\partial x, \partial f/\partial y, \partial f/\partial y_1, \partial f/\partial y_2, \cdots, \partial f/\partial y_n$, respectively, we easily find these extended transformations to be of the form

$$\xi p + \eta q + [\eta_x + (\eta_y - \xi_x)y_1 - \xi_y y_1^2]p_1 + (P_2 y_2 + Q_2)p_2 + \cdots + (P_n y_n + Q_n)p_n,$$

* Presented to the Society April 27, 1901. Received for publication March 3, 1901.

† S. LIE: *Vorlesungen über continuierliche Gruppen*, herausgegeben von Dr. SCHEFFERS, p. 359. This book will hereafter be designated *Continuierliche Gruppen*.

‡ *Invariant differential equation* defined in *Continuierliche Gruppen*, pp. 213–214, and in *Differentialgleichungen mit bekannten infinitesimalen Transformationen*, by S. LIE, p. 277.

§ *Continuierliche Gruppen*, p. 228.

‖ *Continuierliche Gruppen*,, pp. 213–214.

where

$$P_m = -(m+1)\xi_y y_1 - m\xi_x + \eta_y,$$

$$Q_2 = -\xi_{yy}y_1^3 + (-2\xi_{xy} + \eta_{yy})y_1^2 + (-\xi_{xx} + 2\eta_{xy})y_1 + \eta_{xx},$$

$$Q_3 = -3\xi_y y_2^2 + y_2 R_3(x, y, y_1) + S_3(x, y, y_1),$$

$$Q_m = -\tfrac{1}{2}m(m-1)\,\xi_y y_2 y_{m-1} + y_{m-1}R_m(x, y, y_1) + S_m(x, y, y_1, \cdots, y_{m-2}) \quad (m>3),$$

the quantities R and S being integral algebraic functions of $y_1, y_2, \cdots, y_{m-2}$.

We shall indicate the n times extended infinitesimal transformation Xf by

$$X^n f \equiv \xi p + \eta q + \sum_{i=1}^{n} \eta^{(i)} p_i.$$

LIE has proved that an m-parametric group possesses one, and only one, differential invariant, J, of order less than m.*

First, let J be of order $m-1$. Then must the m equations

$$X^{m-2}f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \sum_{i=1}^{m-2} \eta^{(i)} \frac{\partial f}{\partial y_i} = 0 \tag{1}$$

be independent of each other, $\xi\,\partial f/\partial x + \eta\,\partial f/\partial y$ being a type of the m independent infinitesimal transformations of the group. For, were they not independent, we should have a complete system of fewer than m equations in the m variables $x, y, y_1, \cdots, y_{m-2}$, which would have as solution a differential invariant of degree $m-2$ at most, contrary to hypothesis.

The determinant D of the m^2 coefficients $\xi, \eta, \eta^{(i)}$ of the equations (1) will then not vanish identically. If it contains any one of the variables $y_1, y_2, \cdots, y_{m-2}$, then will $D=0$ be an invariant differential equation,† and by examining the form of the quantities $\eta^{(i)}$ given above, we find that D must be an integral algebraic function of $y_1, y_2, \cdots, y_{m-2}$.

The determinant D may be a function of x, y only, say $\phi(x, y)$, which is easily seen to be the determinant of the m^2 quantities $\xi, \eta, \eta_x, \eta_{xx}, \cdots, \partial^{m-2}\eta/\partial x^{m-2}$.‡ Now, by choosing y for the independent variable instead of x, the transformed determinant D of (1) will be seen to contain $\pm\phi\,(dx/dy)^{(m+1)(m-2)/2}$ as the highest power of dx/dy. In this case then, *if* $m>2$, $D=0$ will be an invariant differential equation, integral and algebraic in the derivatives involved.

---

* *Continuierliche Gruppen*, page 230. In the text J is given by J_{m-1}.

†S. LIE, *Theorie der Transformationsgruppen*, bearbeitet unter Mitwirkung von Dr. F. ENGEL; I. Abschnitt, Kapitel 14.

‡ These quantities are the terms of $X^{m-2}f$ free from the derivatives $y_1, y^2, \cdots, y_{m-2}$.

Next, let J be of less than the $(m-1)$-th order. Every differential equation $J=$ constant is invariant, and therefore the equation $dJ/dx=0$ is invariant and is of the nth order, say, where $n \leqq m-1$.

The various equations:

$$(2) \qquad X^n\left(\frac{dJ}{dx}\right) \equiv \xi\frac{\partial}{\partial x}\left(\frac{dJ}{dx}\right) + \eta\frac{\partial}{\partial y}\left(\frac{dJ}{dx}\right) + \sum_{i=1}^{n}\eta^{(i)}\frac{\partial}{\partial y_i}\left(\frac{dJ}{dx}\right) = 0$$

cannot be identities, since dJ/dx is not a differential invariant, there being only one such of order less than m, which is here J. Since this is of the $(n-1)$-th order, only n of the equations

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \sum_{i=1}^{n-1}\eta^{(i)}\frac{\partial f}{\partial y_i} = 0$$

can be independent, and it is therefore possible to eliminate the quantities

$$\frac{\partial}{\partial x}\left(\frac{dJ}{dx}\right), \quad \frac{\partial}{\partial y}\left(\frac{dJ}{dy}\right), \quad \frac{\partial}{\partial y_i}\left(\frac{dJ}{dx}\right) \qquad (i=1, 2, \cdots, n-1),$$

in various ways from (2) and thus to obtain equations of the form

$$M\frac{\partial}{\partial y_n}\left(\frac{dJ}{dx}\right) = 0,$$

where the quantities M are determinants of the coefficients $\xi, \eta, \eta', \cdots, \eta^{(n)}$ and are not all zero identically, but must evidently all vanish on account of $dJ/dx=0$. Since the determinants M are integral and algebraic in $y_1, y_2, \cdots, y_n$, this must be the case with $dJ/dx=0$, after a possible reduction.

In all cases then, we have an invariant differential equation, integral and algebraic in the derivatives involved, as stated above.

§ 2.

Let now

$$\phi \equiv Ay_n^r + By_n^{r-1} + \cdots = 0$$

be such an equation of least order n, left invariant by a *primitive group*, which, according to definition, does not leave invariant a differential equation of the first order. It is supposed that ϕ does not contain any factor which is rational in $y_1, \cdots, y_n$. *We can show that* $n=2$.

The conditions for invariance are that $X^n\phi=0$ on account of $\phi=0$ for every infinitesimal transformation Xf of the group. Hence, as

$$X^n f \equiv (P_n y_n + Q_n)\frac{\partial f}{\partial y_n} + X^{n-1}f \qquad (n>1),$$

we have

$$AX^n\phi \equiv \phi(aAP_n + X^{n-1}A).$$

By taking the $(a-1)$-th partial derivative with respect to y_n of both members of these identities we get, after reduction,

$$AX^n(aAy_n + B) \equiv (aAy_n + B)(AP_n + X^{n-1}A). \tag{3}$$

The equation

$$aAy_n + B = 0$$

is therefore invariant. We may suppose that A and B have no factors in common containing derivatives.

Then follows from (3):

$$BX^{n-1}A \equiv A(aAQ_n + X^{n-1}B - BP_n). \tag{4}$$

Thus $A = 0$ is an invariant differential equation of less than the n-th order, contrary to hypothesis, unless A is a function of x, y only. Let us therefore put $B = aAC$, C being integral in $y_1, y_2, \cdots, y_{n-1}$. The invariant equation is therefore reducible to

$$y_n + C = 0,$$

and the equations (4) reduce to

$$-P_nC + Q_n + X^{n-1}C \equiv 0. \tag{5}$$

Suppose $n > 3$. Putting $C = ay'^{-1}_{n-1} + \beta y'^{-1}_{n-1} + \cdots$, we find that, if $t > 1$,

$$(tP_{n-1} - P_n)a + X^{n-2}a = 0,$$

so that $a = 0$ would be an invariant differential equation, unless $a \equiv 0$. Indeed, a could not be a function of x, y merely, as not every

$$tP_{n-1} - P_n \equiv \xi_y y_1 + \xi_x - (t-1)(n\xi_y y_1 + n\xi_x - \eta_y)$$

could be free from y_1 for a primitive group, where not every ξ is a function of x alone.*

Thus,

$$C \equiv ay_{n-1} + \beta.$$

Substituting in (5) we find the conditions

$$(P_{n-1} - P_n)a - \frac{n(n+1)}{2}\xi_y y_2 + R_n(x, y, y_1) + X^{n-2}a \equiv 0. \tag{6}$$

By assuming a, which is integral in $y_1, y_2, \cdots, y_{n-2}$, to be of the form

$$\gamma y_k^s + \delta y_k^{s-1} + \cdots,$$

* If every ξ were a function of x only, the group would leave invariant a differential equation of the first order, namely, $dx/dy = 0$.

y_k being the derivative of highest order contained in a, and substituting in (6), we find that $k = 2$ and $s = 1$, besides the conditions

$$\gamma(-P_n + P_{n-1} + P_2) - \frac{n(n+1)}{2}\xi_y + X^1\gamma = 0. \tag{7}$$

Let

$$\gamma = \epsilon y_1^r + \zeta y_1^{r-1} + \cdots.$$

Substituting in (7) and selecting the coefficients of y_1^{r+1} we get the equations

$$-(2+r)\xi_y\epsilon = 0,$$

which are impossibilities for every ξ_y that is not zero. It is to be noticed that ϵ could not be zero and therefore also γ, as then (7) would be impossible.

Thus $n > 3$ is not the least order of an invariant differential equation of the required form for a primitive group.

The case $n = 3$, when treated in the same manner, will give similar results. The coefficients of γ and ξ_y in the equations corresponding to (7) will here be slightly different from those in (7).

Thus $n = 2$. Equations (5) are here

$$-P_2C + Q_2 + X^1C \equiv 0,$$

from which we find

$$C \equiv ay_1^3 + \beta y_1^2 + \gamma y_1 + \delta,$$

a, β, γ, δ being functions of x, y only. The invariant differential equation is therefore

$$y_2 + ay_1^3 + \beta y_1^2 + \gamma y_1 + \delta = 0, \tag{8}$$

the reduction of which is our next object.

§ 3.

A three-parametric group, whose infinitesimal transformations are indicated by $Xf \equiv \xi p + \eta q$, leaves invariant a differential equation of the first order. For, if the determinant of the nine quantities ξ, η, η' is not zero identically, we have such an invariant equation as explained in the beginning of § 1, which will be of the first order in this case; and if the determinant vanishes identically, we have at least one differential invariant of the first order, J say, and therefore an invariant equation $f(J) = 0$.

A primitive group must therefore be at least four-parametric. Now, Dr. F. ENGEL has shown* that a group with more than three parameters contains at least one pair of independent infinitesimal transformations which are permutable.

* *Zur Theorie der Zusammensetzung der endlichen continuirlichen Transformationsgruppen*, Leipziger Berichte, 1886, pp. 83–94.

Such a pair, forming a two-parametric subgroup, can, by a proper choice of the variables, be given by

$$q, xq; \quad \text{or} \quad p, q.^*$$

In the first case it appears from a theorem by LIE† that the equation (8) can, by a transformation of the form

$$\bar{x} = x, \quad \bar{y} = y - \phi(x),$$

be changed into the following:

$$\bar{y}_2 = 0.$$

This transformation will change q, xq into $\bar{q}$, $\bar{x}\bar{q}$, where $\bar{q}$ stands for $\partial f/\partial \bar{y}$.

In the second case, by imposing upon the equation (8) the conditions that it should be left invariant by the infinitesimal transformations p, q, we find that the coefficients a, β, γ, δ are constants. Then, if $a \neq 0$, by choosing for new variables

(9) $$x = y - sx, \quad \bar{y} = x,$$

where $$s^3 a + s^2\beta + s\gamma + \delta = 0,$$

we get a differential equation of the form

(10) $$\bar{y}_2 + a'\bar{y}_1^2 + \beta'\bar{y}_1 + \gamma' = 0$$

in place of (8), a', β', γ' being constants.

The complete integral of this equation is readily found, and can in all cases be thrown into the form

$$\phi(\bar{x}, \bar{y}) + k_1\psi(\bar{x}, \bar{y}) + k_2 = 0,$$

k_1 and k_2 being the constants of integration. Then by changing to new variables x, y according to the relations

(11) $$y = \phi(\bar{x}, \bar{y}), \quad x = \psi(\bar{x}, \bar{y}),$$

the equation (10) will reduce to

$$y_2 = 0.$$

Applying the combined transformations (9) and (11) to the subgroup p, q, we find this changed into one or other of the following subgroups, corresponding to the different forms of the functions ϕ and ψ of (11):

$$p + axq, q; \quad xp, yq; \quad p, yq; \quad xp, q;$$

a being a constant.

---

* *Differentialgleichungen mit bekannten infinitesimalen Transformationen*, p. 425.

† *Differentialgleichungen mit* etc., p. 429.

Hence, *by a proper transformation of the variables* x, y, *the primitive groups of the plane can be transformed into groups leaving invariant the differential equation*

$$\frac{d^2y}{dx^2} = 0,$$

and possessing one or other of the following subgroups:

$$q, xq; \quad q, p + axq; \quad xp, yq; \quad p, yq; \quad xp, q.$$

On account of the invariance of the equation $d^2y/dx^2 = 0$, *these transformed groups are projective.**

§ 4.

The problem of finding the primitive groups of the plane is thus reduced to one of finding all the different subgroups of the general projective group

$$p, \quad q, \quad yp, \quad xq, \quad xp, \quad yq, \quad x^2p + xyq, \quad xyp + y^2q$$

which do not leave invariant a differential equation of the first order, and which contain the two-parametric subgroups given above.

Before we proceed it might be well to state a few well known propositions about the infinitesimal transformations of a group.

A set of r independent infinitesimal transformations generate an r-parameter group when, and only when, the commutator (Klammerausdruck) of any pair of these, as Xf and Yf:

$$(Xf, Yf) \equiv X(Yf) - Y(Xf),$$

is expressible as a sum of the given infinitesimal transformations multiplied by constants.

If Xf and Yf are any two infinitesimal transformations of a group, then is $aXf + bYf$ also an infinitesimal transformation of the group, a and b being constants.

A primitive group, which is projective, cannot leave invariant a given point in the plane, for otherwise the differential equation of the first order defining the ∞^1 straight lines through that point would be invariant.

Now, let the primitive group required contain the subgroup q, xq. The transformation

$$x = \frac{x'}{y'}, \quad y = \frac{1}{y'}$$

does not alter the form of the differential equation $d^2y/dx^2 = 0$ and it changes the

**Continuierliche Gruppen*, p. 35.

subgroup q, xq into the subgroup $x^2p + xyq, xyp + y^2q$, after we drop the accents.

One of the additional infinitesimal transformations of the group must be of the form

$$ap + bq + (cx + ey)p + (gx + hy)q \equiv Xf,$$

where a and b are not both zero, since it is necessary to have at least one infinitesimal transformation which does not leave invariant the point $(0, 0)$.

A linear transformation of the variables can now be chosen such that $ap + bq$ is changed into p and the subgroup $x^2p + xyq, xyp + y^2q$ is left unaltered, besides the differential equation $d^2y/dx^2 = 0$.

If now

$$Xf \equiv p + (cx + ey)p + (gx + hy)q,$$

we find by forming the following commutators

$$(Xf, x^2p + xyq) \equiv 2xp + yq + \alpha(x^2p + xyq) + \beta(xyp + y^2q) \equiv Yf,$$

$$(Xf, xyp + y^2q) \equiv yp + \alpha'(x^2p + xyq) + \beta'(xyp + y^2q),$$

$$\big((Xf, Yf) + Xf, Yf\big) \equiv 6p + 3\alpha(2xp + yq) + (4\beta + 2e)yp + \alpha''(x^2p + xyq) + \beta''(xyp + y^2q),$$

that the group required must contain the infinitesimal transformations

$$p, yp, 2xp + yq, x^2p + xyq, xyp + y^2q,$$

which will be found to generate a group. This group does not leave invariant a differential equation of the first order and is therefore primitive.

If the required group contains more infinitesimal transformations, we can assume these to be of the form $(a + bx + cy)q$. By building commutators as above we find that two cases present themselves, viz:

one additional infinitesimal transformation only, namely, yq;

three additional infinitesimal transformations, q, xq, yq.

We have now three types of primitive groups, which, by the transformation

$$x = \frac{x'}{y'}, \quad y = \frac{1}{y'},$$

become the following (dropping the accents):

$$\begin{aligned} &p, \quad q, \quad yp, \quad xq, \quad xp - yq; \\ &p, \quad q, \quad yp, \quad xq, \quad xp, \quad yq; \\ &p, \quad q, \quad yp, \quad xq, \quad xp, \quad yq, \quad x^2p + xyq, \quad xyp + y^2q. \end{aligned} \tag{12}$$

The two subgroups p, yq; xp, q are transformed one into the other by exchanging the variables. We need therefore consider only one of these, as p, yq.

The transformation $x = x'/y'$, $y = 1/y'$, used above, changes this subgroup into yp, $xp + yq$, after dropping the accents. As in the case considered above, we must have an infinitesimal transformation of the form

$$ap + bq + (cx + ey)q + g(x^2p + xyq) + h(xyp + y^2q) \equiv Xf,$$

where a and b are not both zero. Then from

$$\big(xp + yq, (xp + yq, Xf) - Xf\big) \equiv 2(ap + bq),$$
$$(yp, ap + bq) \equiv -bp,$$

we see that the required group must contain the infinitesimal transformation p, whether b be zero or not. We have now the subgroup p, yp which, by exchanging the variables, becomes the subgroup q, xq. This has been dealt with above.

Next, let the required group contain the subgroup xp, yq. We must have an infinitesimal transformation of the form

$$ap + bq + cyp + exq + g(x^2p + xyq) + h(xyp + y^2q) \equiv Xf,$$

a and b not being both zero.

From the commutators

$$\big(xp + yq, (xp + yq, Xf) - Xf\big) \equiv 2(ap + bq),$$
$$(xp, ap + bq) \equiv -ap, (yq, ap + bq) \equiv -bq,$$

we find that the group must contain one of the subgroups p, yq, or q, xp, which have just been considered.

Finally, let the subgroup $p + axq$, q belong to the group required. One of the additional infinitesimal transformations may be of the form

$$b(x^2p + xyq) + c(xyp + y^2q) + (gx + hy)p + (lx + ky)q \equiv Xf,$$

where b and c are not both zero.

Forming the commutators

$$\big(p + axq, (p + axq, Xf)\big) - (2b + ah)(p + axq)$$
$$\equiv -3a(b + ah)xq + 3acxp + aq \equiv Yf,$$
$$(p + axq, Yf) - 3ac(p + axq) \equiv -6a^2cxq + \beta q,$$
$$(q, Xf) - h(p + axq) \equiv (b - ah)xq + c(xp + 2yq) + lq,$$

we find that, if $a \neq 0$, we have the subgroup q, xq; and if $a = 0$, we see from

$$(-cp + bq, Xf) \equiv (bx + cy)(-cp + bq) + \gamma p + \delta q,$$

that we have the subgroup $-cp + bq$, $(bx + cy)(-cp + bq)$, which can be transformed into the subgroup q, xq by a linear transformation of the variables. But the primitive groups containing the subgroup q, xq have been determined above.

There remains to be determined the primitive groups whose infinitesimal transformations—at least four in number (§ 3)—are of the form

$$p + axq,\ q,\ (bx + cy)p + (ex + gy)q.$$

If we bear in mind that all the coefficients of p cannot be functions of x only, as we should then have an invariant differential equation of the first order $dx/dy = 0$, and that all the coefficients of q cannot be functions of y only, as then the equation $dy/dx = 0$ would be invariant, we find by building commutators that the first two groups of (12) will result. Accordingly:

All the primitive groups of the plane are represented by the types (12).*

* Cf. *Continuierliche Gruppen*, p. 360.

DETERMINATION OF ALL THE GROUPS OF ORDER p^m WHICH CONTAIN THE ABELIAN GROUP OF TYPE $(m-2, 1)$, p BEING ANY PRIME*

BY

G. A. MILLER

Introduction.

It is well known that there are just three abelian groups of order p^m which contain the group of type $(m-2, 1)$; viz. the groups of types $(m-1, 1)$, $(m-2, 2)$, $(m-2, 1, 1)$.† In what follows we shall therefore assume that the groups under consideration are non-abelian. The common abelian subgroup of type $(m-2, 1)$ will be represented by H. The operators of the required groups must transform those of H according to a subgroup of order p in the group of isomorphisms (I) of H. Our first problem is to determine all the operators of order p which are contained in I. Since the groups of order p^4 are well known we shall always assume $m > 4$. The groups which contain a cyclic invariant subgroup of order p^{m-2} are also known ‡, but we shall re-determine them in connection with the other groups since this will not materially affect the work.

The group H contains one cyclic characteristic subgroup of each order of the form p^a $(a = 1, 2, \cdots, m-3)$. It also contains one non-cyclic characteristic subgroup of each order of the form p^{a_1} $(a_1 = 2, 3, \cdots, m-2)$. Its other subgroups are cyclic and non-characteristic when p is odd. There are $p-1$ of these of each order of the form p^{a_2} $(a_2 = 2, 3, \cdots, m-3)$ while there are p such subgroups of each of the orders p and p^{m-2}. When $p = 2$ these subgroups of orders 2 and 2^{m-2} are the only non-characteristic subgroups of H. All the non-characteristic subgroups of the same order are conjugate under the holomorph of H. In fact, all the operators of the highest order in the non-characteristic subgroups of the same order are conjugate under this holomorph.

We proceed to prove several theorems applying to every abelian group (A), which will be employed in what follows. The operator of the group of isomor-

* Presented to the Society (Chicago) April 6, 1901. Received for publication March 13, 1901.

† FROBENIUS and STICKELBERGER, Crelle's Journal, vol. 84, 1879, p. 217.

‡ BURNSIDE, *Theory of groups of finite order*, 1897, p. 75.

phisms (A') of A, which transforms each operator of highest order in A into its β-th power must transform every operator of A into its β-th power since A is generated by its operators of highest order. Let s_1, s_2 be two operators of highest order (γ) in A and let $t^{-1}s_1t = s_1^{\alpha}$, $t^{-1}s_2t = s_2^{\beta}$ ($\beta \not\equiv \alpha$, mod γ). The independent generators of the smallest group containing s_1 and s_2 may be represented by s_1 and s_3. From the equations just mentioned we have $t^{-1}s_3t = s_3^{\beta'}$ ($\beta' \not\equiv \alpha$, mod the order of s_3). Since $t^{-1}s_1s_3t = s_1^{\alpha}s_3^{\beta'}$ is not a power of s_1s_3, t does not transform each subgroup of A into itself. Hence it follows that *the only operators of A' which transform each subgroup of A into itself are those which transform each operator of A into the same power.*

It will now be proved that the only invariant operators of A' are those which transform each operator of A into the same power of itself.* Let t' be any operator of A' which does not transform each operator of A into the same power. If t' permutes some of the subgroups of highest order in A it is not commutative† with the operator of A' which transforms into itself a generator of one of these subgroups (K') without transforming into itself a generator of the subgroup into which t' transforms K'. If t' does not permute any of the subgroups of highest order in A it must transform the generator of at least two of them into different powers and hence it is not commutative with the operator which transforms one of these two generators into the other. *The number of invariant operators of A' is therefore equal to the number of natural numbers which do not exceed the highest order of the operators of A and are prime to this order*, and the quotient group of A' with respect to these invariant operators is the group according to which the subgroups of A are transformed by A'.

§ 1. *Group of isomorphisms of H.*

Since each of the $p^{m-2}(p-1)$ operators of order p^{m-2} in H can be made to correspond with every other operator of this order‡ and any holomorphism of H is completely determined by such a correspondence and a correspondence between two of the $p(p-1)$ operators of order p which are not found in the characteristic subgroup of order p, the order of I is $p^{m-1}(p-1)^2$. By raising all the operators of H to each one of the $p^{m-3}(p-1)$ powers which are prime to p and not greater than p^{m-2} we obtain the holomorphisms of H which correspond to the cyclic subgroup of I all of whose operators are invariant under I.§ To determine the remaining operators of I we employ a method which

* Cf. Transactions of the American Mathematical Society, vol. 1, 1900, p. 396.

† Cf. Bochert, Mathematische Annalen, vol. 49, 1897, p. 134.

‡ Transactions of the American Mathematical Society, vol. 1, 1900, p. 397.

§ Transactions of the American Mathematical Society, vol. 1, 1900, p. 396. When $p = 2$ this group of order 2^{m-3} is of type $(m-4, 1)$. In what follows p is assumed to be odd unless the contrary is stated.

was explained in a recent number of the Bulletin of the American Mathematical Society.*

The group I contains an invariant subgroup (I_2) of order p^2 which is composed of all the operators of I which are commutative with each operator of the characteristic subgroup (in H) of order p^{m-2}. These isomorphisms can be obtained by making H isomorphic with its subgroups of order p, the characteristic subgroup of order p^{m-2} corresponding to identity. Hence I_2 is of type $(1, 1)$. By making H isomorphic with its characteristic subgroup of order p with respect to one of its cyclic subgroups of order p^{m-2} we observe that I contains operators of order p which are not contained in I_2. This proves that I contains a subgroup (I_3) of order p^3 which includes no operator of order p^2. We shall now prove that I_3 is the non-abelian group of order p^3 which contains no operator of order p^2.

Let P_1, P_2 represent two independent generators of H, the order of P_1 being p^{m-2}, and let $P_1^{p^{m-3}} = P_3$. The first four of the following equations result directly from the given holomorphisms of H while the last two show that two operators (t_1, t_2) of I_3 are not commutative and hence that I_3 is non-abelian.

$$t_1^{-1}P_2t_1 = P_2, \quad t_1^{-1}P_1t_1 = P_2P_1, \quad t_2^{-1}P_1t_2 = P_1, \quad t_2^{-1}P_2t_1 = P_3P_2,$$

$$t_2^{-1}t_1^{-1}P_1t_1t_2 = P_3P_2P_1, \quad t_1^{-1}t_2^{-1}P_1t_2t_1 = P_2P_1.$$

It has been observed that I contains a cyclic group of order $p^{m-3}(p-1)$ all of whose operators are invariant. Each operator of this group transforms every subgroup of H into itself. The corresponding quotient group of I is of order $p^2(p-1)$. We proceed to prove that this group can be represented as an intransitive substitution group of degree $2p$, formed by establishing a (p, p) correspondence between two metacyclic groups of degree p. Each of the two sets of p non-characteristic subgroups of the same order in H is transformed by I according to the metacyclic group of degree p. As cyclic groups of order p^{m-2} are transformed into each other by operators of I which transform each one of the cyclic groups of order p into itself, the subgroup of H must be transformed according to the said intransitive substitution group. Since the order of this group is the order of the quotient group of I with respect to its operators which transform each subgroup of H into itself, this intransitive group is the quotient group in question. As this quotient group contains only one subgroup of order p^2, I contains only one subgroup (K) of order p^{m-1}.

The subgroup K contains no invariant operators besides those of the given cyclic subgroup of order p^{m-3} since it must include I_3. The group of cogredient isomorphisms of K is therefore the same as that of I_3. Hence it contains the abelian subgroup of type $(m-3, 1)$ which includes I_2. The remaining operators of K are the products of this abelian subgroup and the first $p-1$ powers

* Vol. 6, 1900, p. 337.

of any operator (t_4) of I_3 which is not included in I_2. Let t_5 be any operator of the group of type $(m-3, 1)$ first mentioned. From the properties of the operators of I_3 it follows that

$$(t_5t_4)^p = t_5t_4t_5t_4\cdots p \text{ times} = t_5t_4t_5t_4^{-1}t_4^2t_5t_4^{-2}\cdots t_4^{-1}t_5t_4 = t_5^p u u^2 u^3 \cdots u^{p-1},$$

u being an invariant operator of I_3 which may be identity. Since t_5t_4 is of the same order as t_5 whenever t_5 is not identity I *contains no operator of order* p *besides those of* I_3.

As it is necessary to consider all the operators of I_3 in the construction of the groups under consideration we proceed to determine its sets of operators which are conjugate under I. It has been observed that $p-1$ of its operators are invariant under I and that its subgroup I_2 is also invariant under I. It is clear that I_3 contains p other subgroups of order p^2. It will be proved that one of these is always invariant under I while the other $p-1$ are conjugate when $p > 3$. From the quotient group of I with respect to its invariant operators and from the properties of I_3 it follows that each of the p^3-p non-invariant operators of I belongs to a set of $p(p-1)$ conjugates under I. Hence all the non-invariant operators of the invariant subgroups of order p^2 are conjugate.

We proceed to consider the holomorphisms obtained by making H isomorphic with its non-cyclic subgroup of order p^2, the cyclic characteristic subgroup of order p^{m-3} corresponding to identity and the non-cyclic subgroup of order p^{m-2} corresponding to the group generated by P_3. From the following equations it may be observed that each one of these $p(p-1)^2$ holomorphisms gives rise to an operator of order p in I. In these equations t_6 is the operator of I which corresponds to the holomorphism under consideration.

$$t_6^{-1}P_1t_6 = P_2P_1, \quad t_6^{-2}P_1t_6^2 = P_3P_2^2P_1, \quad t_6^{-3}P_1t_6^3 = P_3^3P_2^3P_1,$$

$$t_6^{-n}P_1t_6^n = P_3^{1+2+\cdots+n-1}P_2^nP_1, \; t_6^{-p}P_1t_6^p = P_1, \; t_6^{-1}P_2t_6 = P_3P_2, \; t_6^{-p}P_2t_6^p = P_3^pP_2.$$

We have now found all the holomorphisms of H which correspond to the operators of I_3. In the p^2 holomorphisms which correspond to I_2 each of the operators of the non-cyclic subgroup of order p^{m-2} corresponds to itself. In $p(p-1)$ others each operator of a cyclic group of order p^{m-2} corresponds to itself while in $p(p-1)^2$ just found the only operators which correspond to themselves are those of the cyclic characteristic subgroup of order p^{m-3}. The $p(p-1)$ operators, each of which is commutative with some operator of order p^{m-2}, together with the invariant operators of I_3 must correspond to an invariant subgroup in the quotient group of I with respect to its invariant operators. Hence *they form an invariant subgroup of order* p^2.

We shall now prove that the $p(p-1)$ subgroups, generated by the $p(p-1)^2$ operators of I_3 which transform only p^{m-3} operators of H into themselves, are

conjugate under I whenever $p > 3$. Let t_7 be any operator of I which satisfies the two conditions

$$t_7^{-1} P_1 t_7 = P_1, \quad t_7^{-1} P_2 t_7 = P_3^{\alpha} P_2^{\beta}, \qquad \beta \not\equiv 0, \mod p.$$

The transform of t_6 with respect to t_7 satisfies the following condition:

$$(t_7^{-1} t_6 t_7)^{-1} P_1 t_7^{-1} t_6 t^7 = P_3^{\alpha} P_2^{\beta} P_1.$$

If we assign to t_7 all its values subject to the given conditions we obtain the $p(p-1)$ holomorphisms which correspond to the conjugates of t_6. It remains to prove that the one which corresponds to t_6^2 is not in this set. If it were we should have $P_3^{\alpha} P_2^{\beta} = P_3 P_2^2$ and hence $\alpha = 1$, $\beta = 2$. This would lead to the following equations which are impossible when $p > 3$:

$$(t_7^{-1} t_6 t_7)^{-1} P_2^2 t_7^{-1} t_6 t_7 = P_3 P_2^2 = P_3^4 P_2^2 = t_6^{-2} P_2^2 t_6^2.$$

Since I does not transform t_6 into t_6^2, its quotient group with respect to its invariant operators can be obtained by establishing such a (p, p) correspondence between two metacyclic groups of degree p as leads to a group having only two invariant subgroups of order p.* This is clearly impossible when $p = 3$. In this special case each of the subgroups of order p^2 in I_3 is therefore invariant under I while, in all other cases, I_3 contains only two invariant subgroups of order p^2 and a set of $p-1$ conjugate subgroups of this order.

The rest of this section will be devoted to the study of I when $p = 2$.

We may assume $m > 5$ since all the groups of order 32 are known. Now I is of order 2^{m-1} and it is isomorphic to the abelian group of type $(m-5, 1)$ with respect to its operators which are commutative with each operator of the cyclic subgroup of order 2^{m-3} in H which is generated by P_1. Hence I_2 and I_3 may be found in the same manner as when p is odd, the latter being the group of order 8 containing five operators of order 2 and two of order 4 since it is non-abelian and contains I_2.

It has been observed that the 2^{m-3} invariant operators of I form a group of type $(m-4, 1)$. Since this cannot contain more than 2 operators of I_3, I and I_3 must have the same group of cogredient isomorphisms. Hence I contains an abelian group of type $(m-4, 1, 1)$ including I_2, and the rest of its operators transform just half of the operators of this group into themselves multiplied by the invariant operator (s_4) of I_3. Since s_4 is commutative with all the operators

* By establishing a (p, p) correspondence between two metacyclic groups of degree p, written in distinct sets of letters, we obtain as many intransitive groups of order $p^2(p-1)$ as there are operators in the group of isomorphisms of the cyclic group of order $p-1$ diminished by one-half of the number of its operators whose orders exceed two. Cf. American Journal of Mathematics, vol. 21, 1899, pp. 292 and 293. In one of these groups each subgroup of order p is invariant while each of the others contains only two invariant subgroups of this order. The latter exists only whenever $p > 3$.

of H whose order is less than 2^{m-2} it is the square of the operators of order four in the given group of type $(m-4, 1, 1)$. We shall soon observe that all the operators of order 4 in I have the same square. Let s_5 be any operator of the mentioned group of type $(m-4, 1, 1)$ and suppose that s_5 is not invariant in I. Also let s_6 represent an operator of order 2 in I_3 that is not also contained in I_2. From the equation $(s_5 s_6)^2 = s_5 s_6 s_5 s_6 = s_4 s_5^2$ it follows that we obtain additional operators of order two when s_5 is of order four and only then. When s_5 is of order two we obtain the additional operators of order four. Since s_4 is the square of all of these, *all the operators of order four in I have the same square.*

From the results of the preceding paragraph it follows that all the operators of order two in I are contained in a subgroup (I_5) of order 32 which includes all the operators of order four in I. Also I_5 contains 16 operators of order four and 15 of order two. As the latter are not all commutative to each other they must generate I_5. Moreover it is clear that I_5 is the direct product of its operator of order two which transforms each operator of H into its inverse and the group (I_4) of order 16 which contains just 7 operators of order two that generate it.* Since I transforms the four non-characteristic subgroups of H according to the intransitive substitution group of degree and order four, each of its non-invariant operators has just two conjugates. Hence I_5 *contains three invariant operators of order two and six pairs of conjugates.*

§ 2. *Outline of the method employed to construct the groups in question.*

If a group of order p^m is represented as a regular substitution group, every subgroup of order p^{m-1} contains p systems of intransitivity. We may therefore suppose that each of the groups under consideration contains the intransitive substitution group formed by writing H in the regular form in p distinct sets of letters and establishing a simple isomorphism between these p regular groups $H_1, H_2, \cdots, H_p$. In what follows H will represent this intransitive group of type $(m-2, 1)$ while t will be used to represent the substitution of order p which merely permutes the corresponding letters of $H_1, H_2, \cdots, H_p$. Hence t is commutative with every substitution of H. Also, I will represent the intransitive substitution group which is formed by writing the maximal subgroup of degree $p^{m-1} - a$ $(a > 0)$ of the holomorph of H_1† in the p distinct sets of letters involved in H; so that t is also commutative with each substitution of I. Finally, i will be used to represent some substitution of order p that is contained in I, while I_2, I_3, I_4, and I_5 will represent the same subgroups of I as in the preceding section.

---

*Quarterly Journal of Mathematics, vol. 28, 1896, p. 271, no. 9.

†Bulletin of the American Mathematical Society, vol. 5, 1899, p. 245.

Now H and ti clearly generate a group (G) of order p^m. All the other groups of this order which transform H in the same manner as G does, may be generated by H and sti, where s is commutative with each substitution of H; for, if such a group is generated by H and r, where r permutes the systems of H in the same way as t does, then $rt^{-1}i^{-1} = s$ must be commutative with each substitution of H, i. e., $r = sti$. Hence s is a substitution of the direct product of $H_1, H_2, \cdots, H_{p'}$. We may assume that s involves no letters except those of H_1; for if it involved the letters of $H_\alpha (p \geqq a > 1)$ but not those of $H_{\alpha+\beta} (\beta > 0)$, the transform of the group generated by H and sti with respect to the component of s which involved the letters of H_α only would be a conjugate group in which s would be replaced by a substitution involving the letters of $H_{\alpha-1}$ but not those of $H_{\alpha-1+\beta}$. Since the other conditions named would not be affected it is proved that *we may assume that s involves the letters of H_1 only.* On this account it will hereafter be denoted by s_1 and its conjugates under the group generated by t will be denoted by $s_1, s_2, \cdots, s_{p'}$.

By hypothesis $(s_1ti)^p$ is some substitution of H, but

$$(s_1ti)^p = s_1ti \cdot s_1ti \cdots = s_1is_{p'}is_{p'-1}i \cdots s_2i = s_1s'_{p'}s'_{p'-1} \cdots s'_2,$$

where s'_α is the transform of s_α with respect to some power of i. From this it follows that s_1 *and i are always commutative.* Let r_1 represent a substitution of H that is transformed by i^{-1} into itself multiplied by a substitution u_1 of order p which is commutative to i. The conjugates of r_1 and u_1 with respect to the powers of t will be represented by $r_1, r_2, \cdots, r_{p'}$; $u_1, u_2, \cdots, u_{p'}$ respectively. It is easy to verify that

$$(r_1^{p-1}r_3r_4^2 \cdots r_{p'}^{p-2})^{-1}tir_1^{p-1}r_3r_4^2 \cdots r_{p'}^{p-2} = r_1^{1-p}r_3^{-1}r_4^{-2} \cdots r_{p'}^{2-p}r_2r_3^2 \cdots r_{p'}^{p-1}u_2u_3^2 \cdots u_{p'}^{p-1}ti$$

$$= r_1^{1-p}r_2r_3 \cdots r_{p'}u_2u_3^2 \cdots u_{p'}^{p-1}ti.$$

If we transform the last substitution successively by $u_{p'}^{p-1}, u_{p'-1}^{2p-3}, \cdots, u_3^{\frac{(p+1)(p-2)}{2}}$ and multiply the result by $r_1^{-1}, r_2^{-1}, \cdots, r_{p'}^{-1}$, which is in H, we observe that the group generated by H and ti is conjugate to the one which is generated by H and $r_1^{-p}ti$; i. e., *no new group is obtained when s_1 is the pth power of a substitution of H_1 which is transformed by i into itself multiplied by a substitution of order p that is commutative with i.*

By interchanging systems of intransitivity of H, t can be transformed into any power while each of the substitutions i and s_1 is transformed into itself. Hence the $p-1$ powers of i which are prime to the order of i give rise to the same groups. It is also clear that each of the conjugates of i under I leads to the same groups. Moreover, if each of two groups generated by ti_α and ti_β respectively, contains only one subgroup that is simply isomorphic with H and if i_α is not conjugate with some power of i_β under I, then must the two groups

be distinct; for if they were simply isomorphic they could be transformed by operators of I so that the identical operators of H would correspond. The remaining operators of the group would have to transform H in the same manner; i. e., the group generated by i_α would be conjugate to the one generated by i_β. The main results of this section may be summarized as follows:

1. Only one operator from each set of subgroups of order p which are conjugate under I is to be used for i.

2. s_1 is a substitution of H_1 which is commutative with i.

3. s_1 is not the pth power of a substitution of H_1 which is either commutative with i or is transformed by i into itself multiplied by a substitution of order p that is commutative with i.

4. All the powers of s_1 which are prime to its order lead to the same group since it can be transformed into these powers by operators that are commutative with t and i.

§ 3. *Determination of the groups when p is odd.*

It was observed in § 1 that i is contained in the non-abelian group I_3 of order p^3 which contains no operator of order p^2. We shall begin with the case when i is one (i_1) of the $p-1$ operators of the invariant subgroup of order p. Hence i_1 is commutative with every operator of the non-cyclic subgroup of order p^{m-2} in H. From the equations

$$(P_1 t i_1)^p = P_1 t i_1 \cdot P_1 t i_1 \cdots p \text{ times} = P_1 i_1 \cdot P_1 i_1 \cdots t^p$$

$$= P_1 i_1 \cdot P_1 i_1^{-1} i_1^2 P_1 i_1^{-2} i_1^3 \cdots i_1^{p-1} P_1 i_1^{1-p} i_1^p = P_3 P_3^2 \cdots P_3^{p-1} P_1^p = P_1^p,$$

it follows that the product of $t i_1$ and any operator of H besides identity is of the same order as the operator of H. The group (G_1) generated by H and $t i_1$ has therefore the same number of operators of each order as the abelian group of type $(m-2, 1, 1)$. Two groups which have the same number of operators of each order may be called conformal.* Now G_1 contains just $p+1$ abelian subgroups of order p^{m-1}, p of these are of type $(m-2, 1)$ while the remaining one is of type $(m-3, 1, 1)$. As each of the former is transformed in the same manner by all the operators of G_1, we shall not arrive at G_1 when we use in place of i_1 an operator from another set of conjugates in I_3.

From § 2 it follows that $s_1 t i_1$ and H generate G_1 whenever s_1 is an operator of the characteristic cyclic subgroup of order p^{m-3} in H_1. If s_1 is the constituent of P_2 which involves the letters of H_1 only we clearly obtain a group (G_2) which contains no operator of order p besides those of H. Then G_2 contains the same number of operators of each order $> p^2$ as G_1 does. It also includes the same

*Bulletin of the American Mathematical Society, vol. 2, 1896, p. 140.

number of abelian subgroups of type $(m-2, 1)$ as G_1 does, but the group of type $(m-3, 1, 1)$ in G_1 is replaced by one of type $(m-3, 2)$ in G_2. These are the only abelian subgroups of order p^{m-1} that are contained in G_2 and all those of type $(m-2, 1)$ are transformed in the same way by all the operators of G_2. Since we obtain G_2 if we replace s_1 by any one of its $p-1$ powers multiplied into any operator of the cyclic characteristic subgroup of order p^{m-3} in H_1, i_1 gives rise to only the two groups G_1 and G_2.

We shall now consider the possible groups when i is one (i_2) of the $p(p-1)$ conjugate operators of I_2. i_2 and i_1 are commutative with the same operators of H. Just as in the preceding case we find that i_2 leads to two groups (G_3 and G_4) which are conformal with G_1 and G_2 respectively. In G_1 and G_2 all the cyclic subgroups of order p^{m-2} are invariant but in G_3 and G_4 these subgroups are conjugate in sets of p. Each of the groups G_3 and G_4 contains $p+1$ abelian subgroups of order p^{m-1}, p of these are of type $(m-2, 1)$ and are transformed by the remaining operators in the same way as H is transformed. Hence we cannot obtain any of the four groups just found by using for i an operator of I_3 which is not contained in I_2. By using for i one (i_3) of the $p(p-1)$ conjugate operators of the other invariant subgroup of I_3, when $p>3$, we clearly obtain only one additional group (G_5) that involves no operator whose order exceeds p^{m-2}. The group G_5 contains $p+1$ abelian subgroups of type $(m-2, 1)$ and it transforms each of its cyclic subgroups of order p^{m-2} into itself. It is the only group involving H that has invariant operators of order p^{m-2} without containing any operator of order p^{m-1}. It is clear that i_3 leads also to the non-abelian group (G_6) of order p^m* which involves an operator of order p^{m-1} and to no other group.

Each of the remaining $p(p-1)^2$ operators of I_3 transforms only p^{m-3} operators of H into themselves and the $p(p-1)$ subgroups which they generate are conjugate when $p>3$. Hence it is only necessary to consider one (i_4) of them when $p>3$. The group (G_7) generated by ti_4 and H contains only one abelian subgroup of order p^{m-1}. From the equation

$$(P_1 t i_4)^p = (P i_4)^p = P_1 i_4 P_1 i_4^{-1} i_4^2 P_1 i_4^{-2} i_4^3 \cdots i_4^{p-1} P_1 i_4^{1-p} = P_1^p \quad (\text{when } p>3)$$

$$\text{or } = P_3 P_1^p \quad (\text{when } p=3),$$

it follows that G_7 is conformal with G_1, G_3 and G_5. The following equations show that i_4 gives rise to only one group. In these equations r_1 represents the constituent of P_1 that is contained in H_1, while v_1 and w_1 represent the corresponding constituents of P_2 and P_3 respectively. The higher subscripts indicate the conjugates of these substitutions with respect to powers of t.

* Burnside, *Theory of groups of finite order*, 1897, p. 76.

$$(r_1^{p-1}r_3r_4^2\cdots r_p^{p-2})^{-1}ti_4r_1^{p-1}r_3r_4^2\cdots r_p^{p-2}=r_1^{1-p}r_3^{-1}r_4^{-2}\cdots r_p^{2-p}i_4r_2r_3^2\cdots r_p^{p-1}t$$

$$=r_1^{1-p}r_2r_3\cdots r_pv_2v_3^2\cdots v_p^{p-1}ti=r_1^{1-p}r_2r_3\cdots r_pw_1^{-1-3-6\cdots\frac{-(p-1)(p-2)}{2}}ti_4$$

which, when multiplied by $r^{-1}r_2^{-1}r_3^{-1}\cdots r_p^{-1}$ of H, is equal to $r_1^{-p}ti$ if $p>3$, and to $w_1^{-1}r_1^{-p}ti$ if $p=3$.

When $p=3$, I_3 contains another invariant subgroup of order p^2. If i_5 represents one of its conjugate operators we obtain an additional group G_8 by multiplying H and ti_5. It may be proved in exactly the same manner as in the preceding case that i_5 leads to only one group, and it is, moreover, clear that G_8 and G_7 are conformal. Since each contains only one abelian subgroup of order p^{m-1} and since i_4 and i_5 are not conjugate under I, G_8 and G_7 cannot be conjugate.

§ 4. *Determination of the groups when p is even.*

In § 1 it was observed that all the operators of order two in I are contained in a group of order 32 which has three invariant operators (i_1, i_2, i_3) of order 2; viz., those which transform each operator of H into its $2^{m-3}+1$, $2^{m-3}-1$, $2^{m-2}-1$ powers respectively. The group (G_1) generated by H and ti_1 is conformal with the abelian group of type $(m-2, 1, 1)$ and each of its subgroups, with exception of the four of order two which are not in H, is invariant. The group G_1 contains three abelian subgroups of order p^{m-1}, two being of type $(m-2, 1)$ and the third of type $(m-3, 1, 1)$, and it transforms all the operators of the former into their $2^{m-3}+1$ power. In exactly the same manner as in the preceding section we obtain G_2 by multiplying H and s_1ti_1, s_1 being the constituent of P_2 which is in H_1. The group G_2 contains only three operators of order 2 and each of these is the square of four operators of order 4. The operators of its two abelian subgroups of type $(m-2, 1)$ are transformed in the same way as those of the corresponding subgroups of G_1.

The group (G_3) generated by H and ti_2 contains 2^{m-2} operators of each of the orders two and four in addition to the operators of H. All the operators of order four in G_3 have the same square. Let s_1 be the constituent of P_1 which involves the same letters as H_1. From the equation $s_1^{-1}ti_2s_1=s_1^{-1}s_2^{2^{m-3}-1}ti_2$ it follows that transforms of G_3 include H and $s_1^{2^{m-3}}ti_2$. Hence there is only one additional group (G_4) which transforms H in the same way as i_2 does, viz., the group generated by H and s_1ti_2, where s_1 represents the constituent of P_2 which is in H_1. All the operators of G_4 which are not found in H are of order 4 and the squares of these operators of order 4 are the two non-characteristic operators of order two in H, each of the latter being the square of one-half of the former.

The group H and ti_3 generate a group (G_5) composed of H and 2^{m-1} operators of order 2 which are conjugate in sets of 2^{m-3}. In fact, it is clear that in all

the groups to which i_2 and i_3 lead, all the operators not contained in H are conjugate in sets of 2^{m-3}. The group (G_6) generated by H and $s_1 t i_3$, where s_1 is the component of P_3 which is found in H_1, contains $2^{m-1}+4$ operators of order four. All of these have the same square. If s_1 is the component of P_2 which belongs to H_1, the group (G_7) is conformal with G_6 but its operators of order four which are not found in H have P_2 for their square. As the other operator of H_1 which is commutative with i_3 is conjugate under I with the last s_1, the seven groups just found are all those which depend upon i_1, i_2, i_4. In each of these seven groups every cyclic subgroup whose order exceeds four is invariant. It remains to find the groups which depend upon operators that are not invariant under I.

Let i_4 be one of the two non-invariant operators of I_3 which are commutative with each operator of the non-cyclic subgroup of order 2^{m-2} in H. Since the product of $t i_4$ into any operator of H is of the same order as this operator of H, the group (G_8) generated by H and $t i_4$ is conformal with G_1. G_8 contains three abelian subgroups of types $(2^{m-2}, 1)$ and $(2^{m-3}, 1, 1)$, there being two of the former type. The four cyclic subgroups of order 2^{m-2} form two pairs of conjugates. If s_1 represents the component of P_1 which is contained in H_1 we have $s_1^{-1} t i_4 s_1 = s_1^{-1} s_2 s_2' t i_4$, s_2' being the constituent of P_2 which is in H_2. Hence a transform of G_8 contains H and $s_1' s_1^{-2} t i_4$. We therefore obtain only one additional group (G_9) by multiplying H and $s_1 t i_4$, where s_1 is a substitution of H_1. It may be convenient to assume that G_9 is generated by H and $s_1 t i_4$, where s_1 is the constituent of P_2 which is in H_1. From this we observe that G_9 is conformal with G_2. Like G_8 it contains two pairs of conjugate cyclic subgroups of order 2^{m-2} and three abelian subgroups of order 2^{m-1}. One of these is, however, of type $(2^{m-3}, 2)$ instead of type $(2^{m-3}, 1, 1)$ in G_8.

Each of the other two conjugate operators of order two in I_3 is commutative with an operator of order 2^{m-2} in H. If i_5 represents one of these operators it is clear that H and $t i_5$ generate a group (G_{10}) which is conformal with G_1. Its subgroup of order 16 which includes all the operators of orders two and four is generated by its operators of order two. Its four cyclic subgroups of order 2^{m-2} are invariant, one being generated by an invariant operator while the operators of the others are transformed into their $2^{m-3}+1$ power. The other group (G_{11}) which involves invariant operators of order 2^{m-2} is generated by H and $s_1 t i_5$, where i_5 is the constituent of P_1 which is in H_1. Hence G_{11} contains operators of order 2^{m-1} which it transforms into their $2^{m-2}+1$ power. It includes just two cyclic subgroups of order 2^{m-1}.

The group (G_{12}) generated by H and $t i_6$, where i_6 is one of the two conjugate operators of order two in I which are commutative with each operator of the cyclic subgroup of order 2^{m-3} generated by $P_1^2 P_2$ without being commutative with any other operator of H, is conformal with G_1. Like G_3, G_4, G_5, G_6,

and G_7 this group and all those which follow contain only one abelian subgroup of order 2^{m-1} while each of the other six groups $(G_1, G_2, G_8, G_9, G_{10}, G_{11})$ contains just three abelian groups of this order. From the equation $s_1^{-1}ti_6s_1 = s_2's_1^{-1}s_2ti_6$, s_1 being an operator of order 2^{m-2} in H_1, it follows that i_6 leads to one other group of order 2^m. The four cyclic subgroups of order 2^{m-2} are conjugate in pairs in G_{12}.

We have now considered all the operators of order two which are contained in I_4 as well as the two additional operators i_3 and $i_1i_3 = i_2$. It remains to consider the three operators i_3i_4, i_3i_5, and i_3i_6. We shall represent these respectively by i_7, i_8, and i_9. The group H and ti_7 generate a group (G_{13}) which contains 2^{m-2} operators of each of the orders two and four in addition to H. H and s_1ti_7, where s_1 is the constituent of P_3 which is in H_1, generate a group G_{14} which contains only operators of order four in addition to H. Neither of these two groups contains an invariant cyclic subgroup of order 2^{m-2}. From the equations $s_1^{-1}ti_7s_1 = s_1^{-1}s_2^{-1}s_2'ti_7$, s_2' being the constituent of P_2 in H_2, it follows that i_7 gives rise to no additional group.

The group (G_{15}) generated by H and ti_8 is conformal with G_{13} but all its operators of order four have the same square while in G_{13} they have two distinct squares. Moreover, its invariant operators form a cyclic group of order four while they form a non-cyclic group of this order in G_{13}. The group H and s_1ti_8, where s_1 is the H_1 component of the invariant operator of order four in G_{15}, generate a group (G_{16}) which contains only operators of order eight in addition to those of H. The two cyclic subgroups of order 2^{m-2} are clearly invariant in each one of these two groups. That these two are the only groups to which i_8 gives rise follows from the equation $s_1^{-1}ti_8s_1 = s_1^{-1}s_2^{-1+2^{m-3}}$, s_1s_2 being an operator of order 2^{m-2} in H which is not commutative with i_8.

H and ti_9 generate a group (G_{17}) which contains, besides H, 2^{m-2} operators of order eight and 2^{m-3} operators of each of the orders two and four. Like the preceding four groups it contains just four invariant operators. That this is the only group to which i_9 leads follows from the equations $s_1^{-1}ti_9s_1 = s_1^{-1}s_2^{-1}s_2'$, where s_2' is of order four if s_1 is of order 2^{m-2}, and of order one when s_1 is of order 2^{m-3} and not commutative with i_6. Hence the number of the groups of order 2^m which contain the abelian group of type $(m-2, 1)$ is twenty, three being abelian and seventeen being non-abelian. Eleven of the latter involve invariant cyclic subgroups of order 2^{m-2}.

Summary.

There are three abelian groups of order p^m that contain a subgroup of type $(m-2, 1)$, viz., those of types $(m-1, 1)$, $(m-2, 2)$, $(m-2, 1, 1)$. These exist for every prime p.

When $p > 3$ there are seven other groups. The number of their operators of each order is exhibited by the following table:

Groups	Operators				
	Order	p	p^2	p^α	p^{m-1}
G_1, G_3, G_5, G_7	Number	p^3-1	$p^3(p-1)$	$p^{\alpha+1}(p-1)$	0
G_2, G_4	"	0	p^4-1	$p^{\alpha+1}(p-1)$	0
G_6	"	p^2-1	$p^2(p-1)$	$p^\alpha(p-1)$	$p^{m-1}(p-1)$

$(p>3)$, $(2<\alpha<m-1)$.

The first four groups may be distinguished by the following properties: G_7 contains only one abelian subgroup of order p^{m-1} while each of the others contains three such subgroups; G_5 is the only one which contains invariant operators of order p^{m-2}; G_1 contains cyclic invariant subgroups of order p^{m-2} while G_3 does not; G_2 and G_4 can be distinguished by the same property as was employed to distinguish G_1 and G_3. When $p=3$ there is one more group, which is conformal with G_6. It is not simply isomorphic with G_6 since these two groups contain only one abelian subgroup of order p^{m-1} which they transform according to non-conjugate subgroups of I.

The following table exhibits the orders of the operators of the 17 non-abelian groups of order 2^m which include an abelian group of type $(m-2, 1)$:

Groups	Operators					
	Order	2	4	8	2^α	2^{m-1}
G_1, G_8, G_{10}, G_{12}	Number	7	8	16	$2^{\alpha+1}$	0
G_2, G_9	"	3	12	16	$2^{\alpha+1}$	0
G_3, G_{13}, G_{15}	"	$2^{m-2}+3$	$2^{m-2}+4$	8	2^α	0
G_4, G_6, G_7, G_{14}	"	3	$2^{m-1}+4$	8	2^α	0
G_5	"	$2^{m-1}+3$	4	8	2^α	0
G_{11}	"	3	4	8	2^α	2^{m-1}
G_{17}	"	$2^{m-3}+3$	$2^{m-3}+4$	$2^{m-2}+8$	2^α	0
G_{16}	"	3	4	$2^{m-1}+8$	2^α	0

$(3<\alpha<m-1)$.

The conformal groups may be distinguished by the following properties: G_{10} is the only one of the first four which contains invariant operators of order 2^{m-2}; G_{12} contains only one abelian group of order 2^{m-1} while G_1 and G_8 contain three such groups; in G_1 all the cyclic subgroups of order 2^{m-2} are invariant while they are conjugate in sets of two in G_8; G_2 and G_9 may be distinguished in the same way as G_1 and G_8; G_{13} contains two operators of order two which are squares of operators of order four while in G_3 and G_{15} all the

operators of order four have the same square; in G_3 the invariant operators form a non-cyclic subgroup of order four while they form a cyclic subgroup of this order in G_{15}; in G_{14} the two cyclic subgroups of order 2^{m-2} are conjugate while they are invariant in G_4, G_6, and G_7; in G_4 the operators of order 4 have three distinct squares, in G_6 they have the same square, and in G_7 they have two distinct squares.

CORNELL UNIVERSITY, *March*, 1901.

ON A FUNDAMENTAL PROPERTY OF A MINIMUM IN THE CALCULUS OF VARIATIONS AND THE PROOF OF A THEOREM OF WEIERSTRASS'S*

BY

W. F. OSGOOD

The notion of the minimum of an integral in the calculus of variations is analogous to that of the minimum of a function of a real variable. The function, $f(x)$, is thought of as continuous and it is said to have a minimum at the point $x = a$ when, for all points of the neighborhood of a with the one exception of a itself, the value of $f(x)$ is greater than $f(a)$; i. e., when the relation holds:

$$f(x) > f(a), \quad 0 < |x - a| \leqq h.$$

This is, however, only a partial description of the behavior of the function in the neighborhood of the point a, as is shown by a simple example of a discontinuous function for which the above relation is true. Let the function $\phi(x)$ be defined as follows:

$\phi(x) = 1$, when x is irrational;

$\phi(x) = 1/q$, when $x = \pm p/q$, where p/q is a positive fraction in its lowest terms (in particular, an integer).

$\phi(0) = 0$.

Then $\phi(x)$ has a minimum at the point $x = 0$, for

$$\phi(x) > \phi(0), \quad x \neq 0;$$

and yet, in the neighborhood of an arbitrarily chosen value of x, the lower limit of $\phi(x)$ is 0.

A continuous function $f(x)$ has, then, a further property in the neighborhood of a point $x = a$ at which it is a minimum, which may be expressed as follows:

Let δ be any positive quantity less than h. Then there exists a positive quantity ϵ such that the lower limit of $f(x)$ for values of x in the above neighborhood of a for which $|x - a| > \delta$ is at least as great as $f(a) + \epsilon$; or

$$f(x) \geqq f(a) + \epsilon, \quad \delta < |x - a| \leqq h.$$

* Presented to the Society April 27, 1901. Received for publication May 6, 1901.

In the calculus of variations the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

or, in the parametric form,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

is said to be made a minimum by the functions $x = \phi(t)$, $y = \psi(t)$ corresponding to a curve C if its value J for C is less than its value I for any other curve C of a specified class (K) of curves which all lie in the neighborhood T of C (Fig. 1):

$$I > J.$$

WEIERSTRASS* has given a sufficient condition that this may be the case. The example of the function $\phi(x)$ above considered raises, however, the question: *Assuming that Weierstrass's sufficient condition is fulfilled, so that no curve of the class (K) distinct from C will give the integral I so small a value as J, may we not still have a set of curves belonging to this class, $\bar{C}_1, \bar{C}_2, \cdots$, which do not cluster about C as their limit and which have the property that, if $I_1, I_2, \cdots$ denote respectively the values of the integral I formed for these curves,*

$$\lim_{n=\infty} I_n = J\ ?$$

In particular, it is conceivable that these curves might approach a curve $\bar{C}$ of the class (K) as their limit, distinct from C. For this curve, the value of I would, of course, not be $\lim_{n=\infty} I_n = J$.

This question I answer in the negative, showing in § 1 by means of the theorem of § 2 that, if $\mathfrak{T}$ denotes an arbitrarily small neighborhood of C lying wholly within T, the lower limit of I formed for all the curves of the class (K) that do not lie wholly within $\mathfrak{T}$ is greater than J by a positive quantity ϵ, so that

$$I \geqq J + \epsilon,$$

when $\bar{C}$ does not lie wholly in $\mathfrak{T}$.

By means of these results it is possible to deal with a problem of importance concerning the scope of the class of curves (K). What curves shall be admitted to this class? The larger the class (K), the more general the property of the integral which is made a minimum by C; for the integral is thus compared

* In his lectures. For a statement of the condition in the parametric case, see § 1 below. This is in substance KNESER'S form of the sufficient condition: KNESER, *Lehrbuch der Variationsrechnung*, chap. 3. Other forms of the sufficient condition, in particular, HILBERT'S form, have recently been treated in a paper by the writer: Annals of Mathematics, 2nd series, vol. 2, no. 3 (1901), p. 105.

with the curves of a more extended class. Hence it is natural to seek to enlarge the class (K') until limitations are reached which are dictated by the *nature of the problem*, not by the *exigencies of the proof*. For example, consider the integral

$$I = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2}\, dt.$$

This integral represents the length of the curve $\bar{C}$ when C has a continuous tangent. But the notion of length is not restricted to curves of this class. In fact, a curve may have a length and yet fail to have a tangent in a set of points everywhere dense along the curve. For, the length of a curve may be defined as the upper limit of the lengths of the inscribed polygons, in case a finite upper limit exists, i. e., as

$$L \sum \sqrt{\Delta x_i^2 + \Delta y_i^2},$$

where $\Delta x_i = \phi(\tau_{i+1}) - \phi(\tau_i)$, $\Delta y_i = \psi(\tau_{i+1}) - \psi(\tau_i)$, and the vertices $\tau_0 = t_0$, $\tau_1, \cdots \tau_n = t_1$ are chosen in all possible ways.* JORDAN has shown that the necessary and sufficient condition that such an upper limit exist is that the functions ϕ, ψ both belong to the class of functions with limited variation (fonctions à variation bornée).†

The above generalization of the integral that represents the length of a curve with continuous tangent suggests a corresponding generalization of the general integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt.$$

Let $\bar{C}$ be a continuous curve lying in T and connecting the points A and B. Since C will not in general have a tangent, the proper integral I will not in general have a meaning. Inscribe a polygon in $\bar{C}$, as in the case of the arc integral, and, remembering that the function F is homogeneous of the first degree in its last two arguments, form the sum

---

* This is in substance SCHEEFFER'S definition. The form, which is a particularly elegant one is due to PEANO: *Applicazioni geometriche del calcolo infinitesimale*, 1887, p. 161; *Lezioni di analisi infinitesimale*, vol. 1, 1893, p. 171; Annali di matematica, 2nd ser., vol. 23 (1895), p. 153. It is analogous to PEANO'S definition of the upper definite integral; cf. *Lezioni* (just cited), § 103; Annali, loc. cit. See also HILBERT'S definition at the close of this paper.

† JORDAN, *Cours d'analyse*, vol. 1, 2d ed., 1893, § 67. An example of a continuous curve, $y = \Phi(x)$, which has a length, but which does not have a tangent for any rational value of the abscissa can easily be given. Let $f(x) = x \sin \log x$, $x \neq 0$; $f(0) = 0$, and let $\phi(x) = f(\sin \pi x)$. Then the function $\Phi(x)$, given by the series

$$\Phi(x) = a_1\phi(x) + a_2\phi(2!\,x) + a_3\phi(3!\,x) + \cdots,$$

where the coefficients a_n are suitably chosen ($a_n = 1/n!^2$ will suffice), is a continuous function of limited variation. It is readily seen that this function, however, fails to have a derivative when x is rational. — The graph of WEIERSTRASS'S continuous function that has no derivative is an example of a continuous curve that does not have a length.

$$\sum F\left(x_i, y_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right)\Delta t_i = \sum F(x_i, y_i, \Delta x_i, \Delta y_i).$$

If this sum converges toward one and the same limit when $n = \infty$ no matter how the polygons be chosen, provided merely that their sides all converge toward 0, then this curve shall be included in the class (K) and the value of the limit of this sum shall be taken as the generalized integral I. If $\bar{C}$ has a continuous tangent, this value is the same as that of the proper integral I. Thus we have a natural limitation for the class (K) of curves $\bar{C}$ to be admitted to consideration. WEIERSTRASS, to whom the generalization here considered is due, states in his lectures the theorem that when his sufficient condition for a minimum is fulfilled, the integral will still be a minimum, even when its definition is thus generalized and its value for C is compared with its value for any curve of this class (K). I have never seen a proof of this theorem. In § 3 of the present paper a proof is given.

The theorem of § 1 is proven only for the simplest integral of the calculus of variations; but it is true for other integrals, for example, for the double integral

$$\int\int F(x, y, z, p, q)\, dx\, dy,$$

and it probably holds in general. In the case of a weak minimum, the neighborhood T of C must be replaced by the *engere Nachbarschaft* * of C.

In the proof which HILBERT has recently given of DIRICHLET'S Principle † a continuous function z of the variables x, y is defined as $\lim_{n=\infty} F_n(x, y)$ and it has to be shown that z satisfies LAPLACE'S equation at all interior points of the region. The theorem of § 1, stated for double integrals, may be employed to establish this point. A corresponding step has to be taken in the proof of the existence of a shortest line on a surface, and the theorems of this paper are applicable in this case, too.

§ 1. A CHECK FOR THE VALUE OF THE INTEGRAL I EXTENDED ALONG A CURVE $\bar{C}$ NOT LYING WHOLLY WITHIN THE NEIGHBORHOOD $\mathfrak{T}$ OF THE EXTREMAL $\bar{C}$.

1. *The quantity H, expressed in terms of the curvilinear coördinates of the field.* — Consider the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

* ZERMELO, *Untersuchungen zur Variationsrechnung*, Berlin dissertation, 1894.

† Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899) p. 184.

extended along the curve

$$C: \quad x = \phi(t), \quad y = \psi(t), \quad t_0 \leqq t \leqq t_1,$$

that connects the fixed points A, with the coördinates (x_0, y_0), $(t = t_0)$, and B, with the coördinates (x_1, y_1), $(t = t_1)$. Here, ϕ, ψ shall be continuous functions of t for the values in question, and shall have continuous first derivatives that do not vanish simultaneously for any of the above values of t. The function F shall, together with all its partial derivatives of the first and those of the second order that present themselves, be a single valued continuous function of the four arguments (x, y, x', y') regarded as independent variables, where (x, y) is any point of a certain region of the (x, y)-plane, within which the curve C must lie, and x', y' have any values except the one pair $(0, 0)$; furthermore, since the integral is to depend only on the curve C, not on the particular choice of the parameter t, the function F shall be homogeneous of the first degree in the last two arguments. Finally, we assume that F is positive for all values of the arguments considered.

A necessary condition for a minimum is that the curve C satisfy the differential equations

$$F_x - \frac{d}{dt} F_{x'} = 0, \quad F_y - \frac{d}{dt} F_{y'} = 0, \qquad \left(F_x = \frac{\partial F}{\partial x}, \text{ etc.}\right).$$

We assume that this condition is fulfilled by the extremal* C, and that C is a curve which does not cut itself between A and B; furthermore, that a field exists surrounding C. More precisely, we assume that a one-parameter family of curves exists:

$$x = \xi(t, \alpha), \quad y = \eta(t, \alpha),$$

where the functions ξ, η have the following properties:

(a) These functions, together with their first partial derivatives and the cross derivatives $\xi_{t\alpha}$, $\eta_{t\alpha}$, are continuous functions of the independent varibles (t, α) at all points of the region

$$R: \qquad \bar{t}_0 \leqq t \leqq \bar{t}_1, \quad |\alpha - \alpha_0| \leqq \kappa,$$

where $\bar{t}_0 < t_0$, $t_1 < \bar{t}_1$, and $\kappa > 0$; $\bar{t}_0$, $\bar{t}_1$, κ are constants. When $\alpha = \alpha_0$, the equations represent a curve without multiple points, which passes through A and B; and when α has any constant value, the corresponding curve is an extremal. In particular, when $\alpha = \alpha_0$, the curve is the extremal C.

* Any curve that satisfies these equations is denoted by KNESER as an extremal; *Lehrbuch der Variationsrechnung*, p. 24.

(*b*) The determinant

$$\Delta = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix} \neq 0,$$

no matter for what point of R it be formed.

Let κ be chosen small enough so that all the curves will be free from multiple points. Two curves corresponding to distinct values of a not only will not intersect each other, but, if the two values of a differ from each other by an infinitesimal α, the distance $PP' = \beta$ from a point P of the one curve to the point of intersection P' of the normal at P with the other curve will be an infinitesimal that is uniformly of the first order; in other words,

$$m\alpha \leqq \beta \leqq M\alpha,$$

where m, M are positive constants, i. e., independent of a, t, α.

The proof of these statements follows the same lines as the proof in a similar case considered in the writer's paper on *Sufficient Conditions in the Calculus of Variations.** KNESER† has given a proof by means of power series on the assumption that all the functions considered are analytic. This assumption is unfortunate, inasmuch as it introduces restrictions without being accompanied by any simplification.

By means of the family of extremals $a = \text{const.}$, a system of curvilinear coördinates of the field is introduced (KNESER, loc. cit.), $v = a$ being taken as one of the coördinates and

$$u = \int_{t'}^{t} F\big(\xi(t, a), \eta(t, a), \xi_t(t, a), \eta_t(t, a)\big)\, dt$$

being chosen as the other. The lower limit of integration, t', is so taken that the point $x = \xi(t', a)$, $y = \eta(t', a)$ lies on the curve through A which cuts all the extremals $a = \text{const.}$ transversely. In terms of the new coördinates let

$$F(x, y, dx, dy) = G(u, v, du, dv),$$

so that

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt = \int_{t_0}^{t_1} G(u, v, u', v')\, dt,$$

where I is taken along the curve C. The quantity, H, the study of which is the main object of this paragraph, is then given by the formula

$$H = I - J = \int_{t_0}^{t_1} [G(u, v, u', v') - u']\, dt.$$

*Annals of Mathematics, 2nd series, vol. 2, No. 3 (April, 1901), p. 105.

†loc. cit., chap. 3.

This quantity will be positive, and hence the integral I will be made a minimum by C, when the WEIERSTRASSIAN invariant F_1 (KNESER, p. 52) is always positive in R, and this we assume to be the case. This completes the sufficient condition for a minimum.

2. *The determination of a lower limit (Abschätzung) for the value of H for the simplest class of curves $\bar{C}$.* — Consider first the case that $u' > 0$ throughout the whole interval $t_0 \leqq t \leqq t_1$. Then u may be chosen as the parameter t and, setting

$$G(u, v, 1, s) = g(u, v, s),$$

we have

$$[G(u, v, u', v') - u']\, dt = [g(u, v, s) - 1]\, du.$$

KNESER shows that

$$g(u, v, 0) = 1, \quad g_s(u, v, 0) = 0,$$

so that

$$H = \tfrac{1}{2} \int_{u_0}^{u_1} g_{ss}(u, v, \vartheta s)\, s^2\, du,$$

the coördinates of A and B being respectively (u_0, v_0) and (u_1, v_0). The function $g_{ss}(u, v, s)$ is a continuous function of its three arguments, when s has any value whatever and (u, v) is any point of the region

$$T: \qquad \bar{u}_0 \leqq u \leqq \bar{u}_1, \qquad |v - v_0| \leqq \bar{\kappa},$$

where the constants $\bar{u}_0$, $\bar{u}_1$, $\bar{\kappa}$ are so chosen that to each point of T corresponds a point of R, while the arc AB of C lies wholly within T. This function is positive for all such values of the arguments, since the invariant F_1 is always positive. It may, however, have 0 for its lower limit when $|s| = \infty$. Let the minimum value of this function, when (u, v) is an arbitrary point of T and $|s| \leqq \mu$ (μ being a constant to be determined later) be denoted by M.* If we impose on the curve $\bar{C}$ the further restriction that s shall, at no one of its points, exceed μ in absolute value, then we have

$$H \geqq M \int_{u_0}^{u_1} s^2\, du.$$

Now mark off an arbitrary neighborhood of the arc AB of C lying within the field T and consisting of the region

$$\mathfrak{T}: \quad u_0 - L \leqq u \leqq u_1 + L, \quad |v - v_0| \leqq L; \quad L < 3\sqrt{2}(u_1 - u_0).$$

* It will be necessary to consider later the minimum value of the function $\frac{1}{2}\bar{g}_{ss}(u, v, s)$ where $G(u, v, -1, s) = \bar{g}(u, v, s)$, for the same values of the arguments, and so we will choose M once for all as the smaller of these two minimum values.

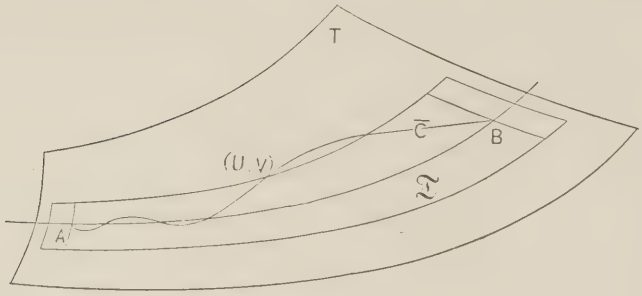


FIG. 1.

If the curve $\bar{C}$, restricted as above, does not lie wholly within $\mathfrak{T}$ and (U, V) is a point of $\bar{C}$ lying on the boundary of $\mathfrak{T}$, so that $|V| = L$, then, by the theorem of § 2

$$\int_{u_0}^{U} s^2\,du \geqq \frac{L^3}{54\,(u_1 - u_0)^2},$$

and since

$$\int_{u_0}^{u_1} s^2\,du > \int_{u_0}^{U} s^2\,du,$$

it follows that

$$(A_1) \qquad H > \frac{ML^3}{54\,(u_1 - u_0)^2}.$$

3. *Continuation; the general case.* — We have thus obtained the desired inequality, or approximate evaluation, (Abschätzung) for H for the simplest class of curves $\bar{C}$. The most general curve

$$\bar{C}: \qquad x = \phi(t), \quad y = \psi(t); \qquad t_0 \leqq t \leqq t_1,$$

connecting A and B, having at each point (extremities included) a continuous tangent, and lying in T, but not wholly in $\mathfrak{T}$, can be broken up into a finite number of pieces of the following three classes:

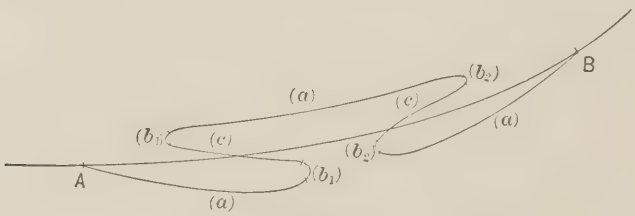


FIG. 2.

$$(a) \qquad u' > 0, \quad |s| \leqq \mu;$$

$$(b) \quad \begin{cases} (b_1) & v' > 0 \text{ and, when } u' \neq 0, \ |s| > \tfrac{1}{2}\mu; \\ (b_2) & v' < 0 \text{ and, when } u' \neq 0, \ |s| > \tfrac{1}{2}\mu; \end{cases}$$

$$(c) \qquad u' < 0, \quad |s| \leqq \mu.$$

Instead of $\frac{1}{2}\mu$, any other positive quantity less than μ would answer. — The integral H can then be decomposed in a corresponding manner into the three sums:

$$(1) \qquad H = \Big(\sum_{(a)} + \sum_{(b)} + \sum_{(c)}\Big)\int [G(u, v, u', v') - u']dt.$$

We proceed to obtain approximate evaluations for these sums, and hence for H.

Consider a term of the first sum

$$H_{a_i} = \int_{t_i}^{t_{i+1}} [G(u, v, u', v') - u']\, dt.$$

Here, $u = \phi(t)$, $u' > 0$ and hence the variable of integration, t, can be replaced by u. We will introduce, however, not u but w, where

$$w = u + a_i$$

and a_i is a constant to be determined later. We have then

$$(2) \qquad H_{a_i} = \int_{w_i}^{w_{i+1}} \tfrac{1}{2} g_{ss}(u, v, s)\, s^2\, dw \ \geqq \ M \int_{w_i}^{w_{i+1}} s^2\, dw.$$

Next consider a term of the third sum

$$H_{c_k} = \int_{t_k}^{t_{k+1}} [G(u, v, u', v') - u']dt$$

and replace t by w, where

$$w = -u + a_k$$

and a_k is a constant to be determined later. Let

$$G(u, v, -1, s) = \bar{g}(u, v, s).$$

Then, as before,

$$\bar{g}(u, v, 0) = 1, \quad \bar{g}_s(u, v, 0) = 0.$$

Hence

$$(3) \quad H_{c_k} = \int_{w_k}^{w_{k+1}} [\tfrac{1}{2}\bar{g}_{ss}(u, v, \vartheta s)\, s^2 + 2]\, dw \ \geqq \ M \int_{w_k}^{w_{k+1}} s^2\, dw + 2(w_{k+1} - w_k).$$

Finally, consider a term of the second sum for which $v' > 0$. Replace t by v, and denote the smaller of the minimum values of the two functions

$$G(u, v, r, 1), \quad G(u, v, r, -1),$$

when (u, v) is any point of T and $r = du/dv (= s^{-1}$ when $u' \neq 0$ and $v' \neq 0)$, $|r| \leqq 1$, by N. Then, since the maximum value of $|r|$ is $2/\mu$, if μ is so chosen (1) that $2/\mu \leqq 1$ and (2) that $2/\mu < N$, we obtain for H_{h_j} the following relation.

$$H_{h_j} = \int_{t_j}^{t_{j+1}} [G(u, v, u', v') - u'] dt = \int_{v_j}^{v_{j+1}} [G(u, v, r, 1) - r] dv$$

$$> \left(N - \frac{2}{\mu}\right)(v_{j+1} - v_j). \tag{4}$$

The same formula also holds for a term for which $v' < 0$, since in that case the left hand member is positive, the right hand member negative. Furthermore, the same reasoning shows that, when $v' < 0$, the quantity $v_{j+1} - v_j$ then becoming negative,

$$H_{h_j} > -\left(N - \frac{2}{\mu}\right)(v_{j+1} - v_j). \tag{4$_1$}$$

We are now ready to obtain the final approximate evaluation of the quantity H. Assume that the curve $\bar{C}$ leaves the strip $\mathfrak{T}$, and let (U, V) be a point of $\bar{C}$ on the boundary of $\mathfrak{T}$. Two cases arise:

$$u_0 \leqq U \leqq u_1, \quad |v| = L; \tag{A}$$

$$U > u_1 \quad \text{or} \quad U < u_0. \tag{B}$$

Case (A). We begin by disposing of a simple special case, namely, that in which

$$\sum_k 2(w_{k+1} - w_k),$$

the summation applying to those arcs of $\bar{C}$ which belong to class (c), exceeds an arbitrarily preassigned quantity, which we take for simplicity as L. Observe that the equation of the boundary of $\mathfrak{T}$ for which u is a maximum is $u = u_1 + L$. Here, we have at once that

$$H > L. \tag{A_2}$$

This case being disposed of, we assume that

$$\sum_k 2(w_{k+1} - w_k) \leqq L$$

and that the point (U, V) is the boundary of an interval of some one of the three classes. This condition can be fulfilled by cutting an interval in two, if necessary; both pieces belonging, then, to the same class. We now determine the constants a_i, a_k in such a manner that those intervals

$$w \leqq w \leqq w_{i+1}, \quad w_k \leqq w \leqq w_{k+1}$$

which correspond to the arcs of C lying between A and $(U,\ V)$ will follow each other without overlapping, their ends just touching; the initial value of w in the first interval shall be chosen as w_0. Denote the final value of w in the last interval by $\mathfrak{w}$. We have, then, from (2) and (3), suppressing whatever terms of the form $2(w_{k+1} - w_k)$ may arise on the right hand side of (3), — this only strengthens the inequality: —

$$\left(\sum_{(a)} + \sum_{(c)}\right)\int [G(u, v, u', v') - u']\,dt > M\int_{w_0}^{\mathfrak{w}} s^2\,dw. \tag{5}$$

The function s that here enters is a single valued function of w continuous at all points of the interval $w_0 \leqq w \leqq \mathfrak{w}$ with the exception of a finite number of points; and s approaches a limit when w approaches any one of these exceptional points from one side only. Let

$$f(w) = \int_{w_0}^{w} s\,dw.$$

Then $v = f(w)$ is a continuous curve made up (1) of the pieces of class (a) carried in the field of the curvilinear coördinates (u, v) to a new part of the field;* (2) of the pieces of class (c), first reflected in a line $u =$ constant (or $v =$ constant) and then carried like the former pieces to a new part of the field. Let $f(\mathfrak{w})$ be denoted by W; $f(w_0) = 0$. We have the relation:

$$W + \sum (v_{j+1} - v_j) = V. \tag{6}$$

Collecting and combining results we infer from (4), (4_1), (5) that†

$$H > M\int_{w_0}^{\mathfrak{w}} s^2\,dw + \left(N - \frac{2}{\mu}\right)|x|, \tag{7}$$

where

$$x = \sum (v_{j+1} - v_j).$$

Relation (7) we now proceed to transform by means of the theorem of § 2, which enables us to write the inequality

$$\int_{w_0}^{\mathfrak{w}} s^2\,dw \ \geqq\ \frac{|W|^3}{54(u_1 + L - u_0)^2},$$

provided that $|W| < 3\sqrt{2}(u_1 + L - u_0)$; a condition that will surely be satisfied when $|W| \leqq L$. We will assume this to be the case, since otherwise we could choose $0 < \mathfrak{w}_1 < \mathfrak{w}$ so that $|f(\mathfrak{w}_1)| = L$, and then we should have

* We may think of the field T as mapped on a rectangle by means of the equations $u = x$, $v = y$. The curves in question thus go over into curves lying in this rectangle and the transformation is then an ordinary euclidean rigid motion. The corresponding actual transformations in the field T now become apparent to the intuition. They are given by the formulas:

(1) $u = w - a_i,\quad v = v;$ (2) $u = -w - a_k,\quad v = v.$

† In obtaining this relation *either* (4) *or* (4_1) is to be used for all values of j.

$$\int^{w} s^2\,dw > \int_{w_0}^{w_1} s^2\,dw \geqq \frac{L^3}{54(u_1 + L - u_0)^2},$$

whence relation (A_3) would follow at once.

We have then, since $W = V - x$,

$$H > \frac{M|V-x|^3}{54(u_1+L-u_0)^2} + \left(N - \frac{2}{\mu}\right)|x|, \tag{8}$$

or, writing

$$\frac{M}{54(u_1+L-u_0)^2} = A, \quad \left(N - \frac{2}{\mu}\right) = B,$$

$$H > A|V-x|^3 + B|x|. \tag{8'}$$

A and B are both positive quantities.

When V and x have opposite signs, the inequality (8′) will only be strengthened by replacing x by 0. Since $|V| = L$,

$$H > AL^3. \tag{A_3}$$

When V and x have the same sign, we can replace (8′) by the inequality

$$H > A(L-\xi)^3 + B\xi, \tag{9_1}$$

when $|x| = \xi < L$; and by the inequality

$$H > A(\xi - L)^3 + B\xi, \tag{9_2}$$

when $\xi \geqq L$. To get the desired approximate evaluation, replace ξ in each of these formulas by a value that will make the right hand side of the inequality a minimum. The curve

$$y = A(L-\xi)^3 + B\xi$$

has its minimum value in the interval $0 \leqq \xi < L$ at the point

$$\xi = L - \sqrt{B/3A},$$

provided that this is a point of the interval, i. e., provided $L \geqq \sqrt{B/3A}$; otherwise, at the point $\xi = 0$. In the first case the minimum value is

$$B[L - \tfrac{2}{3}\sqrt{B/3A}];$$

in the second case it is AL^3.

When $\xi \geqq L$, the minimum is given by putting $\xi = L$, and is BL.

Collecting results, we have

$$(A_4) \quad \begin{cases} H > B(L - \frac{2}{3}\sqrt{B/3A}), & \text{when } 0 \leqq \xi < L, \quad \sqrt{B/3A} \leqq L; \\ H > AL^3, & \text{when } 0 \leqq \xi < L, \quad \sqrt{B/3A} > L; \\ H > BL, & \text{when } \xi \geqq L. \end{cases}$$

It remains to consider *Case* (B). It is sufficient to assume that $U > u_1$ for the case that $U < u_0$ can be dealt with in a similar manner, the points A and B merely interchanging their rôles. The curve C in this case crosses the curve $u = u_1$ (let $t = \tau_1$, be the value of t corresponding to such a point) and meets one of the three lines

$$v = L, \qquad u_1 < u \leqq u_1 + L;$$

$$v = -L, \quad u_1 < u \leqq u_1 + L;$$

$$v_0 - L \leqq v \leqq v_0 + L, \quad u = u_1 + L,$$

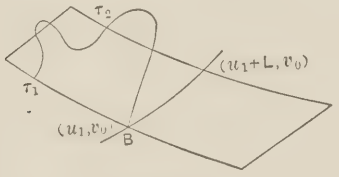


FIG. 3.

in the point (U, V) (let $t = \tau_2$ be the value of t corresponding to such a point). Let the minimum distance from the point B to the above three lines be denoted by D. Then there exists a positive constant, k, such that

$$D > kL,$$

no matter what value L may have subject to the conditions of the problem we are considering. *

We may write H in the form:

$$H = \int_{t_0}^{\tau_1} [G(u, v, u', v') - u'] \, dt + \int_{\tau_1}^{\tau_2} G(u, v, u', v') \, dt$$

$$+ \int_{\tau_2}^{t_1} G(u, v, u', v') \, dt.$$

The first of these integrals is positive or zero, since it is the difference between the integral I taken along a curve $\bar{C}$ drawn in the field from A to the curve $u = u_1$, which cuts the extremals of the field transversely, and the integral I taken from A to B along C. The second integral is also positive since G is positive. Discard these first two integrals and then introduce in the third as parameter the arc s of C. We have then

$$(A_5) \qquad H > \int_{\sigma_2}^{s_1} G(u, v, \cos \phi, \sin \phi) \, ds > k\Gamma L,$$

* For the region about A a corresponding pair of quantities D', k' must be considered, and so we will choose D, k at once each as the smaller one of the two quantities to be considered.

where $\Gamma > 0$ denotes the minimum value of $G(u, v, \cos\phi, \sin\phi)$ when (u, v) is any point of T and ϕ has any value whatever.

This completes the approximate evaluation. The final result may be stated as follows:

If L is any positive quantity so chosen that no point of the region $\mathfrak{T}$ will be exterior to the region T and that $L < 3\sqrt{2}(u_1 - u_0)$, and if E denotes the smallest of the quantities entering on the right hand sides of the inequalities for H: (A_1), (A_2), (A_3), (A_4), (A_5); then the value of the integral I extended along any curve C whatsoever that does not lie wholly within $\mathfrak{T}$ exceeds the value of this integral extended along C by a quantity greater than E; or

$$H > E$$

for such a curve C.

The curves C considered in the foregoing investigation were assumed to have a continuous tangent throughout their whole course inclusive of their extremities. It is easy to see that the above results still hold for curves C made up of a finite number of pieces, each of which enjoys the above properties. In particular, C may be made up of a finite number of pieces of straight lines.

§ 2. The Fundamental Theorem.

4. Fundamental Theorem. *Let $f(x)$ be a single valued continuous function of x in the interval $a \leqq x \leqq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let*

$$|f(l) - f(a)| = L > 0, \qquad a < l \leqq b,$$

Then

$$\int_a^l f'(x)^2\, dx \geqq \frac{L^3}{54(b-a)^2},$$

provided that

$$L < 3\sqrt{2}(b-a).$$

*The theorem still holds under the more general hypothesis that $f(x)$ is continuous throughout the interval $a \leqq x \leqq b$ and that this interval can be broken up into a finite number of subintervals throughout each of which, inclusive of its extremities, $f(x)$ has a continuous derivative.**

We may restrict ourselves to the case that $f(l) - f(a) > 0$, since the case that $f(l) - f(a) < 0$ can be referred to the former case by setting $f(x) = -\phi(x)$. We may furthermore assume that $f'(x) \geqq 0$, since, if $f'(x) < 0$ in parts of the interval, we may define a function $F(x)$ by the conditions:

* Since this theorem merely serves the purpose of a lemma for the proof of the main theorems of this paper, no attempt is made to state the theorem in more general form than is needed for present purposes.

$$F(x) = f(x) \text{ for values of } x \text{ for which } f'(x) \geqq 0;$$

$$F(x) = 0 \quad \text{for values of } x \text{ for which } f'(x) < 0,$$

and this function will have the property that

$$F(l) - F(a) = \bar{L} > L.$$

If l' be so chosen that $F(l') = L$ and $0 < l' < l$, then

$$\int_a^l f'(x)^2\,dx > \int_a^{l'} F'(x)^2\,dx > \int_a^{l'} F'(x)^2\,dx \geqq \frac{L^3}{54(b-a)^2}.$$

Thus the theorem will hold in all cases if it holds when $f(l) > f(a)$ and $f'(x) \geqq 0$, $a \leqq x \leqq b$.

Finally, we shall for simplicity confine ourselves to the proof of the theorem under the hypothesis that $f'(x)$ is continuous throughout the whole interval, for the more general case presents no principal difficulty.

5. Lemma. *If $f'(x)$ is a single-valued, continuous function of x throughout the interval $a \leqq x \leqq \beta$, and if*

$$0 \leqq f'(x) \leqq 1$$

at all points of the interval; if, furthermore,

$$f(\beta) - f(a) = \lambda > 0 \quad \text{and if} \quad \lambda < \beta - a,$$

then

$$\int_a^\beta f'(x)^2\,dx > \tfrac{1}{8}\frac{\lambda^3}{(\beta-a)^2}.$$

Let γ, τ be defined by the equations:

$$\beta - a = \tau,$$

$$\int_a^\beta f'(x)\,dx = f(\beta) - f(a) = \gamma(\beta - a),$$

so that $\gamma\tau = \lambda$, $0 < \gamma < 1$.

Let the points of the interval (a, β) be divided into two sets:

(s_1), the points in which $f'(x) \geqq \frac{1}{2}\gamma$, $\qquad \kappa'$

(s_2), the points in which $f'(x) < \frac{1}{2}\gamma$; $\qquad \tau - \kappa'$

and let κ', $\tau - \kappa'$ denote respectively the internal content of (s_1) and the external content of (s_2).* The quantity κ' varies with different functions $f'(x)$,

* Cf. Jordan, *Cours d'analyse*, vol. 1, 2d ed., 1893, p. 28, § 36.

and we wish to determine its lower limit. Consider the function $\phi(x)$ defined as follows:

$$\phi(x) = 1, \quad a \leqq x \leqq a + \kappa; \qquad \phi(x) = \tfrac{1}{2}\gamma, \quad a + \kappa < x \leqq \beta,$$

where κ is so chosen that

$$(1) \qquad \begin{cases} \int_a^\beta \phi(x)dx = \lambda, \quad \text{i. e.,} \quad \kappa + \tfrac{1}{2}\gamma(\tau - \kappa) = \lambda, \\ \kappa = \dfrac{\lambda}{2 - \gamma} > \tfrac{1}{2}\lambda. \end{cases}$$

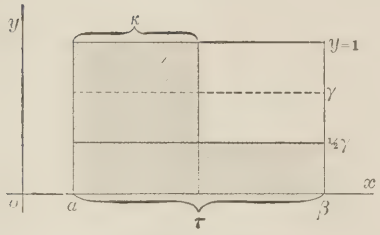


FIG. 4.

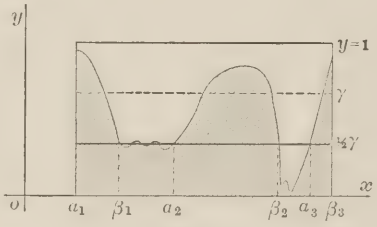


FIG. 5.

This function ϕ is, because of its discontinuity at $x = a + \kappa$, an impossible function $f'(x)$; but there exist functions $f'(x)$ that differ from $\phi(x)$ but slightly, and it seems likely that the number κ thus defined is the lower limit of all the κ's. This we will now show to be the case.

A positive number ϵ having been chosen at pleasure, it is possible to divide the interval (a, β) into a finite number of subintervals $(a_1, \beta_1), (a_2, \beta_2), \cdots, (a_n, \beta_n)$, consisting exclusively of points of (s_1) and such that

$$\kappa' - \epsilon < \sum_{i=1}^{n} (\beta_i - a_i) < \kappa'.$$

The sum of the lengths of the remaining intervals then satisfies the relation:

$$\tau - \kappa' < \sum_{i=0}^{n} (a_{i+1} - \beta_i) < \tau - \kappa' + \epsilon,$$

where $\beta_0 = a$ and $a_{n+1} = \beta$. In particular, it may happen, as in the case indicated in the figure, that $a_1 = a$, so that the first interval, $(a_1 - \beta_0)$, may be suppressed. A similar remark applies to the last interval when $\beta_n = \beta$. Furthermore, the above intervals may be so chosen that, δ being a second arbitrary positive number,

$$f'(x) < \tfrac{1}{2}\gamma + \delta,$$

at all points of each of the intervals (β_i, a_{i+1}). This follows from the uniform continuity of the function $f'(x)$.

From the formula

$$\int_\alpha^\beta f'(x)\,dx = \sum_{i=1}^{n} \int_{\alpha_i}^{\beta_i} f'(x)\,dx + \sum_{i=0}^{n} \int_{\beta_i}^{\alpha_{i+1}} f'(x)\,dx,$$

we now infer, since $f'(x) \leqq 1$ in the interval (α_i, β_i) and $f'(x) < \frac{1}{2}\gamma + \delta$ in the interval (α_{i+1}, β_i), that

$$\lambda < \kappa' + (\tfrac{1}{2}\gamma + \delta)(\tau - \kappa' + \epsilon). \tag{2}$$

From (1) and (2) it follows that

$$\kappa < \kappa' + (\epsilon, \delta),$$

where (ϵ, δ) is a quantity that can be made arbitrarily small by a suitable choice of ϵ and δ. Hence

$$\kappa' \geqq \kappa,$$

and this is the relation that we set out to establish. It is unnecessary to consider whether the lower sign can ever hold.

The proof of the lemma is now given by the relation:

$$\int_\alpha^\beta f'(x)^2\,dx > \sum_{i=1}^{n} \int_{\alpha_i}^{\beta_i} f'(x)^2\,dx > \left(\frac{\gamma}{2}\right)^2 (\kappa' - \epsilon),$$

from which it follows that

$$\int_\alpha^\beta f'(x)^2\,dx \geqq \tfrac{1}{4}\gamma^2\kappa > \tfrac{1}{8}\frac{\lambda^3}{\tau^2}, \qquad \text{q. e. d.}$$

6. *Proof of the theorem.* — To prove the theorem let us first assume that the curve $y = f'(x)$ cuts the line $y = 1$ in a finite number of points, and that, in the intervals $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$,

$$f'(x) \leqq 1, \quad a_i \leqq x \leqq b_i,$$

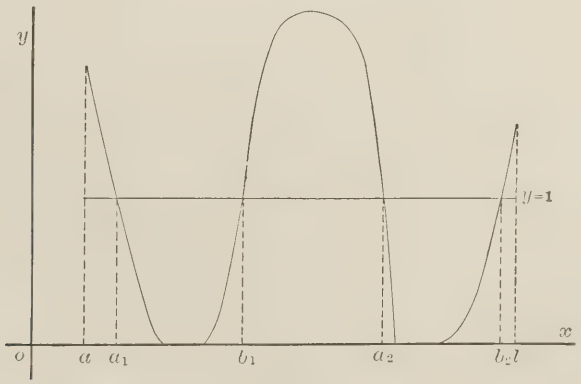


FIG. 6.

while in the remaining intervals, $(b_0, a_1), (b_1, a_2), \cdots, (b_n, a_{n+1})$, where $b_0 = a$, $a_{n+1} = l$,

$$f'(x) > 1, \quad b_i < x < a_{i+1}.$$

(In particular, the curve need not cut this line at all. The result obtained below is still true, and the modification in the proof is but formal.) Observe that we may write down the relation:

$$\sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \sum_{i=1}^{n} [f(b_i) - f(a_i)] = L. \tag{3}$$

The relation on which the proof of the theorem turns is the following:

$$\begin{cases} \int_a^l f'(x)^2\,dx = \sum_{i=0}^{n} \int_{b_i}^{a_{i+1}} f'(x)^2\,dx + \sum_{i=1}^{n} \int_{a_i}^{b_i} f'(x)^2\,dx \\ \qquad\qquad \geqq \sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \tfrac{1}{8} \sum_{i=1}^{n} \dfrac{[f(b_i) - f(a_i)]^3}{(b_i - a_i)^2}, \end{cases} \tag{4}$$

the approximate evaluation of the first sum being given by the relation $f'(x)^2 > f'(x)$, and that of the second sum by the lemma of paragraph 5. The terms that compose the last sum we divide into two classes, namely:

(A) The terms for which

$$\frac{f(b_i) - f(a_i)}{b_i - a_i} \geqq \frac{1}{\rho} L,$$

where ρ is a positive quantity to be defined later. Let the values of the index for these terms be denoted by $i_1, i_2, \cdots, i_p$;

(B) The remaining terms. For these

$$f(b_j) - f(a_j) < \frac{1}{\rho} L (b_j - a_j);$$

let the values of the index j for these terms be denoted by $j_1, j_2, \cdots, j_m$. Notice that

$$\sum_{k=1}^{m} [f(b_{j_k}) - f(a_{j_k})] < \frac{1}{\rho} L \sum_{k=1}^{m} (b_{j_k} - a_{j_k}) \leqq \frac{1}{\rho} L (b - a). \tag{5}$$

We now write down relation (4), suppressing all terms of class (B) and replacing each summand in class (A) by a smaller quantity, as follows:

$$\frac{[f(b_{i_k}) - f(a_{i_k})]^3}{(b_{i_k} - a_{i_k})^2} = \left\{ \frac{f(b_{i_k}) - f(a_{i_k})}{b_{i_k} - a_{i_k}} \right\}^2 [f(b_{i_k}) - f(a_{i_k})] \geqq \frac{L^2}{\rho^2} [f(b_{i_k}) - f(a_{i_k})].$$

The result is the relation

$$\int_a^l f'(x)^2\,dx \geqq \sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \frac{L^2}{8\rho^2} \sum_{k=1}^{p} [f(b_{i_k}) - f(a_{i_k})].$$

Next, let ρ be taken large enough so that $L^2/8\rho^2 < 1$. Then

$$\int_a^l f'(x)^2\,dx > \frac{L^2}{8\rho^2}\left\{\sum_{i=0}^{n}[f(a_{i+1})-f(b_i)]+\sum_{k=1}^{p}[f(b_{i_k})-f(a_{i_k})]\right\}.$$

The quantity in the brace we now transform by adding and subtracting the terms of class (B). The addition of these terms gives three sums whose total value is precisely L. On the other hand, the inequality will only be strengthened if, instead of

$$\sum_{k=1}^{m}[f(b_{j_k})-f(a_{j_k})]$$

we subtract the larger quantity given by (5), namely, $\rho^{-1}L(b-a)$. We thus obtain the relation:

$$\int_a^l f'(x)^2\,dx > \frac{L^2}{8\rho^2}\left\{L-\frac{b-a}{\rho}L\right\}.$$

The expression on the right hand side is seen to have its maximum value when $\rho = \frac{3}{2}(b-a)$, and hence it follows that

$$\int_a^l f'(x)^2\,dx > \frac{L^3}{54(b-a)^2},$$

provided that the above is an admissible value for ρ; i. e., if it will make $L^2/8\rho^2 < 1$. This will be the case if

$$L < 3\sqrt{2}\,(b-a),$$

and this relation was made part of the hypothesis of the theorem for this very purpose. Since we are concerned only with small values of L, this restriction is not embarrassing.

We may note in passing that we could obtain a corresponding formula for large values of L by choosing ρ so that $1 - L^2/8\rho^2$ will be an arbitrarily small positive quantity, and then replacing the factor L/ρ in the brace by the larger value $2\sqrt{2}$. The resulting formula (which, indeed, is true for all values of L) is:

$$\int_a^l f'(x)^2\,dx > \{L - 2\sqrt{2}\,(b-a)\}.$$

We have hitherto assumed that the curve $y = f(x)$ meets the line $y = 1$ only in a finite number of points. If this is not the case, then it is possible to divide the interval (a, l) into a finite number of subintervals and to separate these into two classes:

(a) Intervals within which the curve and the line do not meet. They shall meet at each extremity of such an interval, the points $x = a$ and $x = l$ excepted.

(b) Intervals within which the maximum value of $f'(x)$ is less than $1 + \epsilon$, where ϵ is an arbitrary positive quantity.

Let a function $f_1(x)$ be defined as follows:

$f_1'(x) = f'(x)$ in the intervals (a);

$f_1'(x) = f'(x)$ at those points of the intervals (b) at which $f'(x) \leqq 1$;

$f_1'(x) = 1$ at the remaining points.

Then, if

$$f_1(l) - f_1(a) = L_1, \quad L - \epsilon(l - a) < L_1 < L.$$

To the function $f_1'(x)$ all the reasoning is applicable that has been used above in the case of the function $f'(x)$, if we introduce the following modification: The curve $y = f_1(x)$ may be cut by the line $y = 1$ in an infinite number of points. The points of this set in parts of whose neighborhoods $f_1(x)$ is positive are, however, finite in number. These points play the rôle of the points of intersection in the former case; and if we denote them by $a_1, b_1, \cdots a_n, b_n$, the former proof will apply. Thus

$$\int_a^l f_1'(x)^2\, dx \geqq \frac{L_1^3}{54(b-a)^2}.$$

But

$$\int_a^l f'(x)^2\, dx > \int_a^l f_1'(x)^2\, dx,$$

and we now readily infer that

$$\int_a^l f'(x)^2\, dx \geqq \frac{L^3}{54(b-a)^2}.$$

This completes the proof of the theorem.

§ 3. The Proof of Weierstrass's Theorem.

7. Let the points A, B be joined by an arbitrary curve lying in the field T, but not lying wholly within $\mathfrak{T}$:

$$C: \qquad x = \phi(t), \quad y = \psi(t); \qquad t_0 \leqq t \leqq t_1,$$

where ϕ, ψ denote, to begin with, merely single-valued, continuous functions of

t, and let a polygon P be inscribed in C with its vertices at the points $\tau_0 = t_0$, $\tau_1, \cdots, \tau_n = t_1$, where $\tau_i < \tau_{i+1}$. Its sides may intersect one another. Form the sum

$$S = \sum_{i=0}^{n-1} F(x_i, y_i, \Delta x_i, \Delta y_i).$$

Let the curve C be now restricted to belonging to the class (K) of curves for which S approaches one and the same limit when $n = \infty$, no matter how the vertices of the polygon are chosen, provided merely that its longest side converges towards 0. Denote this limit by $\mathfrak{J}$. Then WEIERSTRASS'S theorem is established if we show that

$$\mathfrak{J} \geqq J + \frac{L^3}{54(u_1 + L - u_0)^2}, \tag{1}$$

and this we will now do.

Let ϵ and δ be two arbitrarily chosen positive quantities, and let the polygon P be so chosen that

$$|\mathfrak{J} - S| < \epsilon \tag{2}$$

and that the length of the longest side of P is less than δ. Form the (improper) integral I for the curve $\bar{C} = P$. This will be a proper integral provided we assign to the arguments x', y' definite (but arbitrary) values at the vertices of P. We have then:

$$\begin{aligned} I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt &= \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} F(x, y, x', y')\,dt \\ &= \sum_{i=0}^{n-1} F\left(\bar{x}_i, \bar{y}_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right) \Delta t_i = \sum_{i=0}^{n-1} F(\bar{x}_i, \bar{y}_i, \Delta x_i, \Delta y_i), \end{aligned} \tag{3}$$

where

$$\bar{x}_i = \phi(t_i + \vartheta_i \Delta t_i) = \phi(t_i) + \zeta_i, \qquad 0 < \vartheta_i < 1,$$

and

$$\bar{y}_i = \psi(t_i + \vartheta_i' \Delta t_i) = \psi(t_i) + \zeta_i', \qquad 0 < \vartheta_i' < 1.$$

By proper choice of δ, it is possible to make

$$|\zeta_i| < h, \quad |\zeta_i'| < h \qquad (i = 0, 1, \cdots, n-1), \tag{4}$$

where h denotes an arbitrary positive constant.

The function $F(x, y, \cos\phi, \sin\phi)$ is uniformly continuous and hence, by proper choice of h, we can insure that

$$|F(x_i + \zeta_i, y_i + \zeta_i', \cos\alpha_i, \sin\alpha_i) - F(x_i, y_i, \cos\alpha_i, \sin\alpha_i)| < \epsilon, \tag{5}$$

where $\Delta x_i / \rho_i = \cos\alpha_i$, $\Delta y_i / \rho_i = \sin\alpha_i$ and $\rho_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$. Multiplying (5)

through by p_i and adding from $i = 0$ to $i = n - 1$, we get, remembering that $F(x_i, y_i, \cos a_i, \sin a_i) p_i = F(x_i, y_i, \Delta x_i, \Delta y_i)$ etc.,

$$|I - S| < \epsilon l_n . \tag{6}$$

Since $F(x, y, \sin \phi, \cos \phi)$ is always positive and is continuous, it has a positive lower limit when x, y, ϕ vary arbitrarily, and hence it is only for curves that have a length that S can converge toward a limit, as is at once evident if the arc S of P is used as the parameter t in (3). Denoting the length of C by l, we replace (6) by the relation:

$$|I - S| < \epsilon l . \tag{7}$$

From (2) and (7) it follows that

$$|\Im - I| < \epsilon(l + 1).$$

But, from the theorem of § 1,

$$I \geqq J + \frac{L^3}{54(u_1 + L - u_0)}$$

and hence

$$\Im > J + \frac{L^3}{54(u_1 + L - u_0)^2} - \epsilon(l + 1).$$

Since ϵ is arbitrary, the truth of the relation (1) is thus established.*

* Note of July 6. HILBERT has given a definition of the integrals

$$\int F(x, y, y')\, dx \qquad \int\int F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dx\, dy,$$

which, like the Weierstrassian definition of the former of these (written in homogeneous form), is a generalization of the ordinary definition for the case that the function y or z does not have a continuous first derivative or continuous first partial derivatives; cf. E. R. HEDRICK, *Ueber den analytischen Character der Lösungen von Differentialgleichungen*, Dissertation, Göttingen, 1901, p. 69. For the integral considered in the present section, HILBERT's definition is in substance the following. Let the continuous curve C be divided as before into n segments by the points $\tau_0 = t_0, \tau_1, \cdots, \tau_n = t_1$, where $\tau_i < \tau_{i+1}$. Through these n points pass an arbitrary curve, which HILBERT assumes to be analytic; it is enough to assume that it is made up of a finite number of pieces, each of which is a continuous curve having a continuous tangent throughout its whole extent, its extremities included. From the integral $I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$ for this curve. Then when all such curves through the $n + 1$ points are considered, the various values of the integral will have a *lower limit*. In the case of the geodesic integral this lower limit is obviously the length of the inscribed polygon above considered. Next, consider the totality of values consisting of these lower limits; to each one of the infinity of modes of division of C into n parts corresponds one such value, and $n = 1, 2, \cdots$. The *upper limit* of these values HILBERT defines as the value of the integral in question.

Thus we have, in form at least, a new generalization of the integral I for curves C for which this upper limit is finite; and the totality of such curves form a class (K'). *This class coincides, however, with the class* (K) *and the value of the generalized integral for a curve of the*

new class coincides with the value of the integral generalized according to Weierstrass's definition for the same curve. The proof of this theorem depends on the fact that the integral I can be made a minimum by a curve c connecting any two points of T, provided that the distance between these points does not exceed a suitably chosen positive constant h, and that these curves have continuous curvature which lies (numerically) uniformly below a suitably chosen positive constant G. Thus, when n is large and the maximum distance between two successive points τ_i, τ_{i+1} is small, the lower limit of the integral will be obtained by employing the curve made up of the n curves connecting the $n+1$ points, each curve being so chosen as to make I a minimum over against all other curves having the same extremities. These curves differ but slightly from right lines in all of their properties that are essential for the above proof.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., *April*, 1901.

CONCERNING HARNACK'S THEORY OF IMPROPER DEFINITE INTEGRALS*

BY

ELIAKIM HASTINGS MOORE

Introduction.

In this paper I consider the improper simple definite integrals of Harnack (1883, 1884). In the introduction I wish to characterize somewhat clearly the theories of the improper simple and multiple integrals recently given by Jordan (1894) and Stolz (1898, 1899), and in this introductory paragraph I summarize the contents of the whole introduction. These theories for the simple integrals have intimate relations with the Harnack theory. The definition adopted for the multiple integrals is more exacting than that for the simple integrals. The multiple integrals converge or exist (as limits) only absolutely. For the simple integrals we have then two theories, on the one hand, of the integrals with the milder definition, and, on the other hand, of the integrals with the stronger definition and so with a larger body of properties. The first class of integrals includes the second class of integrals. The Harnack theory relates to the first and general class of integrals; this theory has not received systematic development; however, for the theory of the absolutely convergent Harnack integrals this is not true, and these integrals constitute the second and special class of integrals. I discuss both classes of simple integrals simultaneously and by uniform process; this is made possible by suitable determinations of the definitions; the absolute convergence of the integrals of the second class appears only at the conclusion, and hence it is desirable to introduce terms of discrimination connoting the two definitions, the milder and the stronger; the terms chosen, "narrow," "broad," connote the geometric form of the definitions, and likewise the fact that the class of narrow integrals has a less extensive body of properties than the (included) class of broad integrals. There has been a tendency to do away with the non-absolutely convergent Harnack integrals; I hope to show that this tendency rests upon misconceptions.—The theory of de la Vallée Poussin (initiated in 1892) is in form distinct from the Harnack theory and

* Presented to the Society (Chicago) December 28, 1900. Received for publication June 12, 1901.

relates primarily to absolutely convergent integrals. It is not involved in the present paper.

HARNACK, in papers* published in 1883 and 1884, in volumes 21 and 24 of the Mathematische Annalen, first gave a definition for the notion of an improper definite integral from a to b of a real function $F(x)$ having these two properties:

(1) The function $F(x)$ assumes values indefinitely great in the neighborhood of points ζ constituting a point-set † Z lying on the interval ab and of content zero; and

(2) The function $F(x)$ is properly integrable in the neighborhood of every other point x of ab, or what is equivalent, it is properly integrable from a' to b' where $a'b'$ is any interval of ab, which contains no point ζ.

HARNACK's definition was formally a definition not so much of the definite integral

$$\int_a^b F(x)\,dx \tag{1}$$

itself as of the definite integral function

$$J(X) = \int_a^X F(x)\,dx \qquad (X \text{ of } ab).$$

Indeed his definition, in common with many limit-definitions of that period and of earlier periods, was not expressed in explicit form. From the context one may infer that HARNACK was so desirous of passing to the less immediate applications of the notion that he was unwilling to attend to the systematic exposition of the fundamental elements of the theory. Thus he was led into error,—with respect to the general theory, notably in the theorem ‡:

> If the set Z of singularities of the function $F(x)$ is reducible (and so of content zero), and a continuous function $\phi(x)$ of x on ab exists, for which
>
> $$\phi(x'') - \phi(x') = \int_{x'}^{x''} F(x)\,dx$$
>
> for every interval $x'x''$ of ab, containing no singularity ζ, then the integral

**Anwendung der Fourier'schen Reihe auf die Theorie der Functionen einer complexen Veränderlichen;* vol. 21 (1883), pp. 305–326; cf. pp. 324–326. *Die allgemeinen Sätze über den Zusammenhang der Functionen einer reellen Variabelen mit ihren Ableitungen, II. Theil;* vol. 24 (1884), pp. 217–252; cf. p. 220 fg.

† It is convenient to use *set* as the equivalent of *Menge* and *ensemble*.—For CANTOR's *Punktmenge* OSGOOD has used the English term *Cantor's set*.

‡ Loc. cit., vol. 24, p. 222, theorem 3. This theorem depends on theorem 2, which is likewise erroneous (cf. § 5 6°).

$$\int_a^x F(x)\,dx \qquad (x \text{ of } ab)$$

exists * or converges and has the value

$$\phi(x) - \phi(a).$$

This theorem was designed to show that the improper definite integrals defined by HÖLDER (Mathematische Annalen, vol. 24 (1884), p. 190 fg.) are special cases of the HARNACK integrals, the set Z being reducible. And this does follow, by the elucidations of HÖLDER as to the uniqueness of his definite integrals, whenever the corresponding HARNACK integrals exist. Thus the HARNACK integrals with reducible sets of singularities are HÖLDER integrals, and the HÖLDER integrals for which the corresponding HARNACK integrals (with reducible sets of singularities) exist are these HARNACK integrals. By examples in § 5 (4°, 9°) I show that each system of integrals is more extensive than their common subsystem.

I accept the sharp formulation of the limit-definition of the integral (1) given by STOLZ † (1898) as doubtless a correct expression of the content of HARNACK's meaning.

As SCHOENFLIES ‡ has remarked, the generalized simple definite integrals of JORDAN (*Cours d'analyse*, ed. 2, vol. 2 (1894), pp. 46, 50 fg.) are § HARNACK's improper integrals, although there is no reference to HARNACK and the form of exhibition is different. JORDAN (loc. cit., p. 46) selects two properties of the

---

* I think of the integral as a limit, that is, as a certain number obtained by a certain limiting process, and prefer to say in general that the limit exists rather than that it converges ; the limit- and expression converges to the number in question as its limit.

† The full reference is given below.

‡ Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8: *Bericht* (1900), p. 176.—SCHOENFLIES (loc. cit., p. 186) gives suggestively but not explicitly a definition of the HARNACK integrals in terms involving the set of non-overlapping intervals of ab which enclose no point ζ and every point not-ζ enclosed by ab. One may form three explicit definitions in these terms. The third definition agrees with the definition given by STOLZ, and the second is equivalent to it, while the first, which alone fully expresses the implications to me of the language used by SCHOENFLIES, is a definition considerably milder.

§ At least if the content of the set Γ of points c (loc. cit., p. 50) is zero. SCHOENFLIES suggests that a condition to this effect may have been unintentionally omitted. But JORDAN defines (loc. cit., p. 76) double integrals without the insertion of the analogous condition ; for (contrary to a statement of SCHOENFLIES) he speaks merely of the *interior* content of the region of integration. However, in the case of the simple integrals, if this condition is omitted, the theorem cited in the text will fail, whenever the condition for $\int_a^b F(x)\,dx$ is not satisfied, even for all cases in which $G(x)$ is a non-zero constant. It may be noted here that JORDAN's set Γ is the set of all points c in whose neighborhood the function $F(x)$ is not capable of proper integration. Thus, the set Γ includes HARNACK's set Z. But we have the theorem that the set Γ is HARNACK's set Z, in case Γ has content zero and the improper definite integral exists. Cf. theorem VIIA of § 3.

proper integrals which serve to suggest the explicit definition of the generalized integrals. But the theorems of the sequel and the reference theorem implied (p. 226, ll. 9, 10) in the proof of the second mean value theorem for the generalized integrals, certainly need fuller proofs; indeed, the theorem* that $\int_a^b \big(F(x)+G(x)\big)\,dx$ exists and

$$\int_a^b \big(F(x)+G(x)\big)\,dx = \int_a^b F(x)\,dx + \int_a^b G(x)\,dx$$

in case the integrals on the right exist, is not in general true. In § 5 8° I give a simple example in which the integral in question does not exist.

Quite recently HARNACK's theory (without reference to JORDAN's theory) has been considered systematically and critically by STOLZ, first in 1898 in a paper† entitled "*Zur Erklärung der absolut convergenten uneigentlichen Integrale*," and then in 1899 (with but slight modifications) in appendix III to volume 3 of his *Grundzüge der Differential- und Integralrechnung*, and further immediately thereafter, in 1899, in a paper entitled "*Über die absolute Convergenz der uneigentlichen bestimmten Integrale*."†

STOLZ gives‡ an explicit definition of the HARNACK integral (1) as a certain limit, and then, considering the theory of these integrals, he affirms‡ that it fails to justify for the limit in question the notation and designation *definite integral* —that one has not even the property that if $\int_a^b F(x)\,dx$ exists, so do all the integrals $\int_a^c F(x)\,dx$ $(a<c<b)$. Contenting himself at this point with an affirmation, STOLZ turns to the theory of the absolutely convergent integrals (1), that is, those for which also the corresponding integral $\int_a^b |F(x)|\,dx$ exists, and finds§ that these integrals deserve the name, since they possess certain four fundamental properties.

The definition given by STOLZ has application with respect to any point-set Ξ lying on the interval ab and of content zero, which contains the set Z used by HARNACK, and in so far his definition is formally more comprehensive than HARNACK's definition. This extension was necessary for the formulation of the third and fourth of the four fundamental propositions which are as follows:

Under the hypothesis that $F(x)$ is absolutely integrable from a to b with respect to a point-set Ξ of content zero:

I. $F(x)$ is likewise integrable from a to c and from c to b, where c is any point of ab, and

---

* Theorem 3, § 58, p. 56.—This theorem would follow directly from HARNACK's erroneous theorem 2 cited above.

† Wiener Berichte, vol. 107_{IIa}, pp. 207–224; vol. 108_{IIa}, pp. 1234–1238.

‡ Wiener Berichte, vol. 107, pp. 207, 211. *Grundzüge*, vol. 3, p. 277.

§ Wiener Berichte, vol. 107, p. 211, § 2, and pp. 216–221, § 5. *Grundzüge*, vol. 3, pp. 277, 279–284.

$$\int_a^b F(x)\,dx = \int_a^c F(x)\,dx + \int_c^b F(x)\,dx.$$

II. For every positive ϵ there is a positive δ_ϵ such that for any set of a finite number (n) of non-overlapping intervals $a_k b_k$ $(a_k < b_k;\ k = 1, \cdots, n)$ of the interval ab of total length less than δ_ϵ the sum of the n integrals $\int_{a_k}^{b_k} F(x)\,dx$ is in absolute value less than ϵ.

III. $F(x)$ is absolutely integrable from a to b with respect to the point-set $\Xi + \mathrm{H}$ obtained by extending Ξ by any set H of content zero, and the integrals with respect to Ξ and to $\Xi + \mathrm{H}$ are equal.

IV. If $G(x)$ is absolutely integrable from a to b with respect to the point-set H of content zero, then $F(x) + G(x)$ is likewise integrable with respect to the aggregate set $\Xi + \mathrm{H}$, and

$$\int_a^b \big(F(x) + G(x)\big)\,dx = \int_a^b F(x)\,dx + \int_a^b G(x)\,dx.$$

As to these theorems it is to be noted that the third is an immediate consequence of the first two, and that the fourth is an immediate consequence of the third.

As to the corresponding theorems for the HARNACK integrals in general I shall show (§ 3 V, VIII) that the first is in fact true, notwithstanding the statement* of STOLZ to the contrary; the second is the erroneous theorem 2 of HARNACK and the fourth is the erroneous theorem 3 of JORDAN. Indeed, the validity of the second theorem for a HARNACK integral implies its absolute convergence (cf. § 2, def. 1, note 1, footnote, and § 4 III). The second theorem is in effect a generalization of the uniform continuity of the definite integral function $J(x)$; for the non-absolutely convergent HARNACK integrals the uniform continuity of $J(x)$ holds (§ 3 IX, § 5 6°).

The limit-definition of the integral (1) relates to the various interval-sets I of a finite number of intervals enclosing all points ζ or ξ of the set Z or Ξ; in notation $I = I(\mathrm{Z})$ or $I(\Xi)$. According to the implication of HARNACK and of STOLZ and the explicit statement of JORDAN (loc. cit., p. 51, top) every interval of I encloses (at least) one point ζ or ξ.

JORDAN'S definitions† for the improper simple and multiple integrals relate to closed measurable regions D lying within the bounded or limited region E

* This statement of STOLZ is reaffirmed by PRINGSHEIM (Münchener Berichte, 1900, p. 220) in his recent paper on the second mean value of the integral calculus, and it is the basis of PRINGSHEIM'S rejection of HARNACK'S extension (Annalen, vol. 21, p. 326) of that theorem to the general improper integrals; the theorem as extended is, however, correct, and the following development of the general theory makes HARNACK'S elegant proof effective.

† loc. cit., pp. 51, 76, and 87. The theory of the improper upper and lower multiple integrals is first developed.

of integration and containing no singularity c. For the simple integrals the complementary region $E - D$ is the interval-set $I(\Xi) = I(\Gamma)$ each of whose intervals encloses a point c. For the multiple integrals the corresponding condition is not imposed, and accordingly the condition for the existence of the improper integral is stronger, and JORDAN proves* that the multiple integrals as defined exist only absolutely. For the simple integrals the corresponding theorem does not hold, and JORDAN (p. 87) remarks that the loss of the theorem is due to the fact that the definition for the simple integrals is less exacting, although he expresses himself somewhat obscurely, in terms of one dimension not capable of immediate extension to more dimensions.

STOLZ† develops the JORDAN definition of improper double integrals for the case‡ of a region E with ordinary boundaries on which all the singularities c lie. The one-dimensional analogue of this case is that of a finite interval E with singular extremities, and for the simple integrals D is a sub-interval of E. In his last paper§ STOLZ, following WIRTINGER, notices that if (as in two dimensions) the D is allowed to be an interval-set the improper integrals exist only absolutely. (This, we have seen, was the implication of the remark of JORDAN.) And, further, he likewise notices (as a generalization for the special case of the theorem of JORDAN) that the double integrals would still exist only absolutely even if they were given a less exacting definition, the region D being required to be (as in the case of the simple integrals) connected‖ or of one piece.

To revert to the general case of multiple integrals, it is now apparent that for a definition formally less exacting than JORDAN's we may impose on the region D which converges to E the conditions: (1) the region $E - D$ consists of one or more regions, each being of one piece and each enclosing a singular point c; (2) the region D consists of one or more regions, each being of one piece and no two being capable of union as parts of an including connected region D; (3) the region D consists of a finite number of connected regions. And these conditions on the region D may be imposed or not imposed independently of one another.

In this paper I confine attention to the simple integrals over a finite interval $E = ab$; the conditions (1) and (2) then become identical; we impose the condition (3), and thus have the theory of the narrow or of the broad simple inte-

* loc. cit., pp. 80, 87.

† *Grundzüge*, vol. 3 (1899), p. 122 fg.

‡ This case is hardly equivalent to the general case, although STOLZ considers that it is (cf. loc. cit., p. 122). The proof (p. 141 fg.) of the theorem of the absolute existence of the improper double integrals I find inconclusive.

§ Wiener Berichte, vol. 108 (1899), pp. 1234–1238.

‖ SCHOENFLIES (loc. cit., pp. 203–205) in his report on this paper of STOLZ seems to confuse definitional conditions for the existence of an integral and properties resulting from its existence.

grals, according as the interval-set $E - D$ is not or is required to have on its every interval a singularity c. By developing the two theories simultaneously we obtain a new insight into the varying properties of the general and the absolutely convergent HARNACK integrals.

In my judgment the general integrals—even the non-absolutely convergent integrals—deserve to be classed with the proper definite integrals under the designation definite integrals. To be sure they lack in general many fundamental properties of the absolutely convergent integrals. We may however look at these matters in such a way that the properties of the different classes are more nearly equivalent. Let us think not of the proper or improper definite integral but, more generally, of the definite integral with respect to a point-set Ξ of content zero and thus write for instance $\int_{a\,\Xi}^{b} f(x)\,dx$, and speak of the Ξ-integrals. Then for every Ξ there is a theory of the narrow and of the broad Ξ-integrals, and for the various sets Ξ these theories, in so far as they relate to a single set Ξ or to a single set Ξ and its subsets Ξ_0 including the set Z of singularities of the integral function $F(x)$, are to a large extent the same. (I remark in passing that the desirable theorem (§ 2 V): if $F(x)$ is Ξ-integrable from a to b and the set Z is non-existent, then $F(x)$ is properly integrable from a to b and $\int_a^b F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx$, holds only if the content of the set Ξ is zero. And in this fact I see one of the strongest reasons for considering only such sets Ξ.)

In the sequel the theories of the narrow and the broad HARNACK integrals are developed from this point of view and otherwise essentially in the spirit of the original HARNACK papers. The necessary fixed hypotheses, etc., being introduced in § 1, I give in § 2 for the Ξ-integrals the definitions and a number of fundamental theorems, and then in § 3 I develop the properties of a function $F(x)$ Ξ-integrable from a to b. I bound the inquiry of § 3 in effect by the form of the four fundamental theorems of STOLZ. Then in § 4 the questions of absolute convergence are introduced. And in § 5 (added July 1, 1901) after exhibiting an important condition necessary and sufficient for the existence of the general or narrow HARNACK integral, I construct for the general closed point-set Ξ on ab of content zero a Ξ-integral from a to b for which the set Ξ is the set Z of singularities, and which is near every point $x = \xi$ essentially narrow or non-absolutely convergent; and in connection with the simplest case, $\Xi = (b)$, I exhibit the examples needed to show the error of various statements already referred to.

§ 1.

FUNDAMENTAL HYPOTHESES, DEFINITIONS, AND NOTATIONS.

1°. We consider the finite interval ab of values of the real variable x. In the proofs of the propositions we shall consider the case $a < b$, to which the case $a > b$ is immediately reducible.

2°. We consider further a point-set Ξ of points ξ which is of content zero. As explained in § 2 the set Ξ will later be supposed to be closed; this supposition involves no restriction of generality. The aggregate or sum $\Xi_1 + \Xi_2$ of two sets Ξ_1, Ξ_2, each closed and of content zero is again a set of the same kind.

3°. On the interval ab the real function $F(x)$ is supposed to be single valued wherever defined, and it is supposed to be defined certainly at all points x not of Ξ. Moreover it is supposed that $F(x)$ is capable of proper definite integration from a' to b' where the interval $a'b'$ is any interval lying on the interval ab and containing no point ξ and no limit-point ξ' of Ξ.

4°. With respect to $F(x)$ it is convenient to separate points $x = x_0$ into two classes. The point $x = x_0$ is *regular* if near x_0 (i. e., on some interval $x_0 - \delta \cdots x_0 + \delta$) $F(x)$ is everywhere defined, single valued, and limited; and otherwise it is *singular*. A singular point $x = \zeta$ may be both progressively and regressively singular or merely progressively ($\zeta = \zeta_+$) or regressively ($\zeta = \zeta_-$) singular. We speak of the singularities ζ and of the singular point-set Z of all points ζ. The set Z is closed. Obviously every singular point ζ is a point ξ or a limit-point ξ' of Ξ; if the set Ξ is closed, the set Z is a subset of it.

5°. A finite number of intervals such that no two have a common point is called an interval-set. Denoting by I an interval-set we denote its length, the sum of the lengths of its intervals, by D_I.

6°. Two intervals $i_1 i_2$ having a common inner point determine a definite interval $i_{12} = i_{21}$ common to i_1 and i_2. Two interval-sets $I_1 I_2$ having a common inner point determine a definite interval-set $I_{12} = I_{21}$ which may be called the set of intervals common to $I_1 I_2$, that is, every i_{12} of I_{12} is the interval common to certain two intervals i_1 of I_1, i_2 of I_2. The set common to a set I and an interval $x_1 x_2$ is denoted by $I_{x_1}^{x_2}$.

For brevity, especially in partitioning interval-sets, it is sometimes convenient to use these notations, even when the interval-sets in question do not exist; for example, an interval-set I is, by an interval $x_1 x_2$ separated into two interval-sets $I_{x_1}^{x_2}$, I' (of which one may not exist), and we write $I = I_{x_1}^{x_2} + I'$.

7°. An interval-set I is said to contain any point-set S, all of whose points are points of I; and in this case S is said to be of or to lie on I. Further an interval-set is said to enclose a point-set S if every point s and likewise every limit-point s' of S lies within some interval of I; and in this case S is said to be an inner set of or to lie within I. The interval-set I is said to contain or to enclose the point-set S narrowly, and S to lie respectively on or within I narrowly, if furthermore every interval of I contains or encloses at least one point s of S. It is convenient to denote by $I(S)$ an interval-set which encloses S narrowly, and by $I\{S\}$ one which encloses S not necessarily narrowly, or, say, one which encloses S broadly. Thus the broad is the generic enclosure, and the narrow is a specific enclosure.

8°. The set Ξ being by hypothesis of content zero, for any positive number ϵ there an interval-set I_ϵ enclosing Ξ narrowly and of length $D_{I_\epsilon} < \epsilon$.

9°. On the fixed interval ab in connection with any interval-set I (not necessarily lying on the interval ab) we introduce a function $F_I(x)$ by the following stipulation: according as the point x of ab lies or does not lie on I, $F_I(x)$ has the value 0 or $F(x)$, with the understanding that, if I is the symbol of a non-existent interval-set, $F_I(x) = F(x)$ for every x of ab. It is to be noticed that if I encloses Ξ, then $F_I(x)$ is on the interval ab everywhere defined, single valued, limited, and capable of proper definite integration from a to b.

§2.

The definite integrals narrow and broad

$$\int_{a\,(\Xi)}^{b} F(x)\,dx, \qquad \int_{a\,\{\Xi\}}^{b} F(x)\,dx$$

of the function $F(x)$ from a to b with respect to the set Ξ: definitions, conditions necessary and sufficient for existence, and fundamental theorems.

We consider simultaneously the two cases:

(1) the narrow Ξ-integral, $\displaystyle\int_{a\,(\Xi)}^{b} F(x)\,dx$;

(2) the broad Ξ-integral, $\displaystyle\int_{a\,\{\Xi\}}^{b} F(x)\,dx$.

When these integrals are considered simultaneously and disjunctively (as in §§ 2, 3) we speak simply of

(3) the Ξ-integral: $\displaystyle\int_{a\,\Xi}^{b} F(x)\,dx$.

In case I denotes an interval-set enclosing Ξ narrowly or broadly the function $F_I(x)$ is capable of proper definite integration from a to b. We consider the proper definite integrals:

(4) $$\int_a^b F_I(x)\,dx,$$

for the various interval-sets I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$, and denote by the respective limit-notations:

(5) $$\mathop{\mathrm{L}}_{D_I=0}^{I\;I\,(\Xi)} \int_a^b F_I(x)\,dx; \quad \mathop{\mathrm{L}}_{D_I=0}^{I\;I\,\{\Xi\}} \int_a^b F_I(x)\,dx,$$

certain finite constant limits, if existent. Then we give those limits the specific Ξ-integral notations:

$$\int_{a\,(\Xi)}^{b} F(x)dx; \quad \int_{a\,\{\Xi\}}^{b} F(x)\,dx. \tag{6}$$

Thus we have the explicit

Definition * 1. *The (existent)* $\overset{narrow}{broad}$ Ξ*-integral:*

$$\int_{a(\Xi)}^{b} F(x)\,dx = \underset{D_I=0}{\overset{I\,(I(\Xi)}{\mathrm{L}}} \int_a^b F_I(x)\,dx;$$

$$\int_{a\,\{\Xi\}}^{b} F(x)\,dx = \underset{D_I=0}{\overset{I\,(I\,\{\Xi\}}{\mathrm{L}}} \int_a^b F_I(x)\,dx; \tag{7}$$

is a certain finite constant such that for every positive number ϵ *there exists a positive number*† δ_ϵ *such that*

$$\left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi} F(x)\,dx \right| < \epsilon \tag{8}$$

for every interval-set‡ I *enclosing* Ξ $\overset{narrowly}{broadly}$ *and of length* $D_I < \delta_\epsilon$.

Note 1. The broad Ξ-integrals constitute a special type of the narrow Ξ-integrals §. In this connection the adjectives *narrow* and *broad* may connote the fact that the body of properties of the narrow integrals is less extensive than and included in the body of properties of the broad integrals. The essentially narrow integrals are the not-broad narrow integrals.

---

* This definition (of the narrow Ξ-integral) seems to express (for $\Xi = Z$) exactly HARNACK'S meaning in form as well as in content.

HARNACK defines the improper definite integral over a certain interval as a certain limit of the proper definite integral over the same interval of a modified function. There are certain advantages (cf. the remark of § 3 VI) in this type of definition. A different definition of this type is that of DE LA VALLÉE POUSSIN for the absolutely convergent generalized definite integrals. (With respect to this matter SCHOENFLIES (loc. cit., pp. 186, 187) has erred in setting in contrast the two limiting processes in question.)

† To the various δ's related in this paper to the arbitrary positive ε I give the notations $\delta_\epsilon, \delta'_\epsilon, \delta^1_\epsilon, \delta^2_\epsilon$, etc., where the superscripts are discriminating affixes and not exponents.

‡ This fundamental definitional property of a function $F(x)$ Ξ-integrable from a to b is (with modification of the δ_ϵ) considerably extended in theorem XIV of § 3.

§ That the narrow Ξ-integral (1) be a broad Ξ-integral (2) it is obviously necessary and sufficient that for every ε a δ^3_ϵ exist such that, in the notation (11),

$$\left| \int_J F(x)\,dx \right| < \varepsilon$$

for every interval-set J lying on ab and containing no point ξ and of length $D_J < \delta^3_\epsilon$. Indeed this condition (cf. JORDAN, loc. cit., § 74, p. 77) is necessary and sufficient for the existence of the broad integral (2). The reader will compare this remark with § 3 XIII′.

Note 2. From the definition it is apparent that the Ξ-integral and the $(\Xi + \Xi')$-integral, where Ξ' is the set of all limit-points ξ' of Ξ, coexist and are equal; that is, if either exists, so does the other with the same value.—The set $\Xi + \Xi'$ is closed and of content zero.—*We suppose hereafter that the set* Ξ *is closed.* This supposition involves no essential loss of generality and it facilitates the phrasing of many proofs.

If this Ξ-integral exists we say that $F(x)$ is Ξ-integrable from a to b.

Definition 2. *The function* $F(x)$ *is progressively* $\overset{narrowly}{broadly}$ Ξ*-integrable at* $x = x_0$ *if there is an interval* $x_0 \cdots x_0 + \epsilon (\epsilon > 0)$ *such that the* $\overset{narrow}{broad}$ Ξ*-integral from* x_0 *to* $x_0 + \epsilon$ *exists.—Regressive* Ξ*-integrability at a point is similarly defined.*

Definition 3. *The function* $F(x)$ *is* $\overset{narrowly}{broadly}$ Ξ*-integrable on* ab $(a < b)$ *if at* a *it is progressively, at* b *it is regressively, and at every other point of* ab *it is both progressively and regressively* $\overset{narrowly}{broadly}$ Ξ*-integrable.*

Remark. It will appear (§ 3, theorem V, corollary) that if $F(x)$ is Ξ-integrable on ab it is Ξ-integrable from a to b.

One proves by the usual limit-considerations the following two theorems:

I. *The* Ξ*-integral if existent is uniquely existent.*

II. *For the existence of the definite* $\overset{narrow}{broad}$ Ξ*-integral*

$$\int_{a\,\Xi}^{b} F(x)dx \tag{9}$$

it is necessary and sufficient that for every ϵ *a* δ_ϵ^1 *exists such that*

$$\left| \int_a^b F_{I_1}(x)dx - \int_a^b F_{I_2}(x)dx \right| < \epsilon \tag{10}$$

for every pair of interval-sets $I_1 I_2$ *each enclosing* Ξ $\overset{narrowly}{broadly}$ *and of length less than* δ_ϵ^1.

Remark. If the Ξ-integral (9) exists, the number $\delta_{\epsilon/2}$ is effective as a number δ_ϵ^1 of the necessary condition (10). And if the sufficient condition (10) is fulfilled the number $\delta_{\epsilon/2}^1$ is effective as a number δ_ϵ connected with the conclusion that the Ξ-integral (9) exists.

III. *If the function* $F(x)$ *is* Ξ*-integrable from* a *to* c *and from* c *to* b, *where* c *lies between* a *and* b, *then the function* $F(x)$ *is* Ξ*-integrable from* a *to* b, *and*

$$\int_{a\,\Xi}^{b} F(x)dx = \int_{a\,\Xi}^{c} F(x)dx + \int_{c\,\Xi}^{b} F(x)dx.$$

One proves this theorem by use of the corresponding theorem for proper definite integrals together with theorem I. The converse of this theorem is theorem VIII of § 3.

Notation. For a function $f(x)$ properly integrable on the various intervals $a_l b_l$ $(a_l < b_l;\ l = 1, 2, \cdots, n)$ of an interval-set J we introduce the notation

$$\int_J f(x)\,dx, \tag{11}$$

by the definitional equation:

$$\int_J f(x)\,dx = \sum_{l=1}^{n} \int_{a_l}^{b_l} f(x)\,dx; \tag{12}$$

and, in case J is the symbol of a non-existent interval-set, it is convenient to understand by the notation (11) the constant zero.

Lemma I. If the function $f(x)$ is properly integrable from a to b and J is any interval-set of the interval ab, then

$$\left|\int_J f(x)\,dx\right| < D_J\, C,$$

where C is any positive number greater than $|f(x)|$ for every point x of the interval ab.

This is a fundamental theorem of the theory of proper definite integrals.

Lemma II. If the function $f(x)$ is properly integrable from a to b, then for every ϵ there is a δ'_ϵ such that

$$\left|\int_J f(x)\,dx\right| < \epsilon$$

for every interval-set J lying on ab and of length $D_J < \delta'_\epsilon$.

Lemma II follows from lemma I immediately. Lemma II is a generalization of the theorem of the uniform continuity on the X-interval ab of the definite integral function $\int_a^X f(x)\,dx$.

IV. *If the function $F(x)$ is properly integrable from a to b, then with respect to every set Ξ of content zero the function $F(x)$ is Ξ-integrable from a to b, and*

$$\int_{a\,\Xi}^{b} F(x)\,dx = \int_a^b F(x)\,dx.$$

*Corollary.** If the function $F(x)$ is properly integrable from a to b and a function $G(x)$ is defined and equal to $F(x)$ on the interval ab, except perhaps at points of a set Ξ of content zero, then $G(x)$ is Ξ-integrable from a to b, and

$$\int_{a\,\Xi}^{b} G(x)\,dx = \int_a^b F(x)\,dx.$$

* Compare the remark of STOLZ, *Grundzüge*, vol. 3 (1899), pp. 283, 284.

Taking a definite set Ξ we denote by I any interval-set enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$; by J the interval-set I_a^b, and by K the interval-set making up with J the interval ab. Either J or K may be non-existent, but one at least is existent. In these notations we have evidently

$$\int_a^b F(x)\,dx = \int_J F(x)\,dx + \int_K F(x)\,dx,$$

$$\int_a^b F_I(x)\,dx = \int_K F(x)\,dx,$$

and so

$$\int_a^b F_I(x)\,dx - \int_a^b F(x)\,dx = -\int_J F(x)\,dx.$$

From this equality, since $D_J \leqq D_I$, the truth of theorem IV appears by the use of lemma II, the definition of the Ξ-integral, and theorem I.

V. *If the function $F(x)$ exists on the interval ab as a single valued limited function of x, and if it is with respect to a certain set Ξ of content zero Ξ-integrable from a to b, then it is likewise properly integrable from a to b, and hence, by theorem IV,*

$$\int_a^b F(x)\,dx = \int_{a\,\Xi}^b F(x)\,dx.$$

This theorem is a corollary of the known

Lemma III. A necessary and sufficient condition for the proper integrability from a to b of a function $F(x)$ existent on the interval ab as a single valued limited function is this, that, with respect to a closed point set S of content zero, the function $F(x)$ be properly integrable from a' to b' where the interval $a'b'$ is any interval of ab containing no point s of the set S.

The theorem of lemma III was developed by PASCH* for the proper definite integral as defined by RIEMANN, and later independently by STOLZ† for the integral as defined by PEANO.‡ The two definitions are equivalent in content.

The lemma-theorem is a corollary of a theorem of DINI (1878: DINI-LÜROTH, § 185), which has close relations to the PEANO definition, and which is likewise given by STOLZ§ independently, but with reference to another theorem of DINI (DINI-LÜROTH, § 184), a particular case of the one here referred to.

VI. *If $F(x)$ is Ξ-integrable from a to b and if Ξ_0 is any subset of Ξ, such that for every interval-set I_0 enclosing Ξ_0 the function $F_{I_0}(x)$ is properly integrable from a to b, then $F(x)$ is Ξ_0-integrable from a to b, and*

---

* Mathematische Annalen, vol. 30 (1887), p. 147.

† Wiener Berichte, vol. 107 (1898), p. 736; *Grundzüge*, vol. 3 (1899), p. 272.

‡ *Lezioni di Analisi infinitesimale* (1893), § 103; Annali di Matematica, ser. 2, vol. 23 (1895), p. 157.

§ Monatshefte für Mathematik und Physik, vol. 7 (1896), p. 293.

$$\int_{a\,\Xi_0}^{b} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx. \tag{13}$$

Remark. The set Ξ_0 includes the subset Z_{ab} lying on ab of the singular set Z of the function $F(x)$ (cf. § 1 4°), and it will appear by theorem VIIA of § 3 that Ξ_0 may be any such subset of Ξ.

We shall prove that

$$\left| \int_{a}^{b} F_{I_0}(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \epsilon \tag{14}$$

for every interval-set I_0 enclosing Ξ_0 $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length $D_{I_0} < \frac{1}{2}\delta_{\epsilon/2}$.

We extend I_0 by an interval-set J so that the interval-set I, $I = I_0 + J$, encloses Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$. Since Ξ has content zero and in accordance with the present hypothesis, in view of an obvious extension of lemma II from an interval ab, to an interval-set (in the present application, to the interval-set extending I'_{0a} to the interval ab), we are able to choose J so that $D_J < \frac{1}{2}\delta_{\epsilon/2}$ and

$$\left| \int_{J\,a}^{b} F(x)\,dx \right| < \tfrac{1}{2}\epsilon. \tag{15}$$

Then $D_I < \delta^1_{\epsilon/2}$, and hence

$$\left| \int_{a}^{b} F_I(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \tfrac{1}{2}\epsilon. \tag{16}$$

Then the desired inequality (14) follows from (15), (16) since

$$\int_{a}^{b} F_I(x)\,dx + \int_{J\,a}^{b} F(x)\,dx = \int_{a}^{b} F_{I_0}(x)\,dx.$$

§ 3.

FUNDAMENTAL PROPERTIES OF A FUNCTION $F(x)$ Ξ-INTEGRABLE FROM a TO b.

Fixed hypothesis. The following theorems concerning the function $F(x)$ involve the fixed hypothesis that the $\begin{smallmatrix}\text{narrow}\\ \text{broad}\end{smallmatrix}$ Ξ-integral

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists.

I. *The function $F(x)$ is Ξ-integrable from b to a, and*

$$\int_{b\,\Xi}^{a} F(x)\,dx = -\int_{a\,\Xi}^{b} F(x)\,dx.$$

II. *The function $cF(x)$, where c is any constant, is Ξ-integrable from a to b, and*

$$\int_{a\,\Xi}^{b} cF(x)\,dx = c\int_{a\,\Xi}^{b} F(x)\,dx.$$

These two theorems follow immediately from the corresponding theorems concerning proper definite integrals together with the definition of the Ξ-integral.

III. *For every ϵ there is a δ_ϵ^1 such that the inequality:*

$$\left|\int_a^b F_{I_1}(x)\,dx - \int_a^b F_{I_2}(x)\,dx\right| < \epsilon,$$

holds for every pair of integral-sets $I_1 I_2$, each enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$ and of length less than δ_ϵ^1.

This theorem is the necessary condition given in theorem II of § 2.

IV. *For every ϵ there is a δ_ϵ^2 such that*

$$\left|\int_{x_1}^{x_2} F_{I_1}(x)\,dx - \int_{x_1}^{x_2} F_{I_2}(x)\,dx\right| < \epsilon \tag{1}$$

for every two points $x_1 x_2$ of the interval ab and every pair of interval sets I_1, I_2 each enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$ and of length less than δ_ϵ^2.

This theorem, whose proof will be given in connection with theorem X, of which it is a particular case, affirms that with respect to any two points x_1, x_2 of the interval ab the condition sufficient for the existence* of the Ξ-integral $\int_{x_1\,\Xi}^{x_2} F(x)\,dx$ is satisfied, and we obtain immediately theorems V and VI.

V. *The definite integral*

$$\int_{x_1\,\Xi}^{x_2} F(x)\,dx$$

exists where $x_1 x_2$ are any two points of the interval ab, and indeed uniformly† on the set of all such point-pairs $x_1 x_2$.

Corollary.‡ If the function $F(x)$ is Ξ-integrable on ab, then it is Ξ-integrable from a to b.

VI. *For every ϵ and every two points $x_1 x_2$ of the interval ab*

$$\left|\int_{x_1}^{x_2} F_I(x)\,dx - \int_{x_1\,\Xi}^{x_2} F(x)\,dx\right| \leqq \varepsilon \tag{2}$$

*At this point one observes that our definition of the integral $\int_{a\,\Xi}^{b} F(x)\,dx$ is more convenient than a definition applicable merely to cases in which the point-set Ξ lies on the interval ab.

† In that δ_ϵ^2 is independent of $x_1 x_2$.

‡ Cf. definition 3 and theorem III of § 2. The proof is indirect and quite analogous to the proof of the corresponding theorem as to proper definite integrals.

for every interval-set I enclosing Ξ narrowly broadly and of length less than δ_ϵ^2.

Remark. Theorem VI is useful especially in case $x_1 = a$, $x_2 = X$, when it establishes a relation on the X-interval ab between the various definite integral functions:

$$\int_{a\,\Xi}^{X} F(x)\,dx, \qquad \int_{a}^{X} F_I(x)\,dx,$$

where I denotes an interval-set enclosing Ξ narrowly broadly.

VII A. *If $F(x)$ is Ξ-integrable from a to b and if Ξ_0 is any subset of Ξ which contains the subset Z_{ab} lying on ab of the singular set Z, then $F(x)$ is Ξ_0-integrable from a to b, and*

$$\int_{a\,\Xi_0}^{b} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx = \int_{a\,Z}^{b} F(x)\,dx.$$

VII B. *If the function $F(x)$ is integrable from a to b with respect to each of two point-sets Ξ_1, Ξ_2 of content zero, then it is likewise with respect to their common subset Ξ_{12} and with respect to their aggregate set $\Xi_1 + \Xi_2$, and the four integrals are equal:*

$$\int_{a\,\Xi_1+\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_1}^{b} F(x)\,dx = \int_{a\,\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_{12}}^{b} F(x)\,dx.$$

The theorem VII A is a generalization of theorem VI of § 2, and, in the light of § 3 V and § 2 V, it is a corollary of that theorem, the function $F_{I_0}(x)$ for any interval-set I_0 enclosing Ξ_0 being on ab a single valued limited function of x, as one proves from the definition of the singular set Z (§ 1, 4°) indirectly by the usual interval-halving process.

As to the theorem VII B, the sets Ξ_1, Ξ_2, Ξ_{12} have the set Z of essential singularities as a common subset. Hence by VII A the Ξ_{12}-integral exists and we have:

$$\int_{a\,\Xi_1}^{b} F(x)\,dx = \int_{a\,\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_{12}}^{b} F(x)\,dx.$$

Denoting for the moment by δ_ϵ the least of the δ_ϵ's related to these three integrals, we consider an interval-set I enclosing $\Xi_1 + \Xi_2$ narrowly broadly and of length $D_I < \delta_{\epsilon/3}$. Then, setting in a definite way:

$$I = I_{12} + I_1 + I_2,$$

where I_{12} encloses Ξ_{12} narrowly, and $I_{12} + I_1$ encloses Ξ_1 narrowly broadly, and $I_{12} + I_2$ encloses Ξ_2 narrowly broadly, we have obviously, in view of the equality of the three integrals, the three inequalities:

$$\left| \int_{a}^{b} F_J(x)\,dx - \int_{a\,\Xi_{12}}^{b} F(x)\,dx \right| < \tfrac{1}{3}\epsilon \quad (J = I_{12},\ I_{12} + I_1,\ I_{12} + I_2),$$

from which, in view of the fact that $F(x)$ is properly integrable on I_1 and I_2, we deduce readily the inequality:

$$\left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi_{12}}^{b} F(x)\,dx \right| < \epsilon,$$

needed to complete the proof of the theorem.

VIII. *The relation:*

$$\int_{x_1\,\Xi}^{x_2} F(x)\,dx + \int_{x_2\,\Xi}^{x_3} F(x)\,dx = \int_{x_1\,\Xi}^{x_3} F(x)\,dx, \tag{3}$$

holds for every three points $x_1 x_2 x_3$ of the interval ab.

This follows easily from the corresponding theorem concerning proper definite integrals. We have for every interval-set I enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$

$$\int_{x_1}^{x_2} F_I(x)\,dx + \int_{x_2}^{x_3} F_I(x)\,dx = \int_{x_1}^{x_3} F_I(x)\,dx. \tag{4}$$

For an interval-set I of length $D_I < \delta_\epsilon^2$ the two sides of the equality (4) differ respectively from the two corresponding sides of (3) by less than 2ϵ. Hence the two sides of (3) differ by less than 4ϵ and so are indeed equal.

IX. *The definite integral function:*

$$\phi(X) = \int_{a\,\Xi}^{X} F(x)\,dx,$$

is a continuous function of the variable upper limit X on the X-interval ab.

Introducing with respect to an ϵ an interval-set I enclosing Ξ narrowly and of length $D_I < \delta_{\epsilon/2}^2$, we have by the use of theorems VIII and VI the relation:

$$|\phi(X_1) - \phi(X_2)| < \epsilon, \tag{5}$$

where X_1 and X_2 are any two points of the X-interval ab such that

$$|X_1 - X_2| < \delta'_{\epsilon/2},$$

the $\delta'_{\frac{1}{2}\epsilon}$ being the number related to the properly integrable function $F_I(x)$ quâ $f(x)$ by lemma II of § 2. And this relation establishes the uniform continuity of $\phi(X)$ on the X-interval ab. This theorem is generalized in theorems XIII′ and XIII″.

Remark. Up to this point of the present development the theories of the narrow and of the broad Ξ-integrals are equivalent. From this point on they diverge; I secure for the narrow Ξ-integrals conclusions analogous to the conclusions for the broad Ξ-integrals, by inserting an additional hypothesis in the first case of every theorem.

X. *For every* ϵ *there is a* δ_ϵ^2 *such that for every interval-set* J *lying on the interval* ab

$$\left| \int_J F_{I_1}(x)\,dx - \int_J F_{I_2}(x)\,dx \right| < \epsilon \tag{6}$$

for every pair $I_1 I_2$ *of interval-sets satisfying the following conditions:*

(1) *Each interval-set* $I_1 I_2$ *encloses* Ξ $^{narrowly}_{broadly}$;

(2) *Each length* $D_{I_1} D_{I_2}$ *is less than* δ_ϵ^2;

and, moreover, in the first case, whenever J *consists of more than a single interval,*

(3) *Every interval* i *of* I_1 *or* I_2 *which encloses an interval* l *joining two consecutive intervals of* J *has with respect to every such enclosed interval* l *the following property: the interval* i *contains either on the interval* l *a point* ξ *or on each of the adjoining intervals of* J *a point* ξ.

Remark. In case J consists of a single interval, theorem X becomes theorem IV; and in case J consists of more than a single interval, the third condition is fulfilled in particular if there is imposed on J an obvious restriction, and likewise, if there are imposed on I_1 and I_2 obvious restrictions, which have reference to the interval-set J in question. In the theorem as here stated, the uniformity with respect to the set of all interval-sets I_1, I_2, J of the nature specified is especially important for the sequel.

We use the determination

$$\delta_\epsilon^2 = \tfrac{1}{2}\delta_\epsilon^1, \tag{7}$$

and have from theorem III the inequality:

$$\left| \int_a^b F_{I_1}(x)\,dx - \int_a^b F_{I_2}(x)\,dx \right| < \epsilon, \tag{8}$$

valid for every ϵ and for every pair of interval-sets $I_1 I_2$ each enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$ and of length less than $2\delta_\epsilon^2$.

For the purposes of an indirect proof of theorem X, we suppose that the inequality:

$$\left| \int_J F_{I_1}(x)\,dx - \int_J F_{I_2}(x)\,dx \right| \geqq \epsilon, \tag{9}$$

holds for a certain ϵ, a certain interval-set J lying on the interval ab, and a certain pair of interval-sets $I_1 I_2$ each enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$ and of length less than δ_ϵ^2, and, in the first case, satisfying the condition (3) of X, and proceed to exhibit for this ϵ two interval-sets $I_3 I_4$ each enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$ and of length less than $2\delta_\epsilon^2$ for which, in contradiction with (8), the inequality:

$$\left| \int_a^b F_{I_3}(x)\,dx - \int_a^b F_{I_4}(x)\,dx \right| \geqq \epsilon, \tag{10}$$

has validity.

It is easy to see how the inequality (9) may be transformed into an inequality of the general form (10). For denoting by K the interval-set obtained by excising J from the interval ab, we have for every pair of sets $I_3 I_4$ enclosing Ξ, the equation:

$$(11)\quad \int_a^b F_{I_3}(x)\,dx - \int_a^b F_{I_4}(x)\,dx = \begin{cases} \left(\int_J F_{I_3}(x)\,dx - \int_J F_{I_4}(x)\,dx\right) \\ + \left(\int_K F_{I_3}(x)\,dx - \int_K F_{I_4}(x)\,dx\right), \end{cases}$$

since the functions $F_{I_3}(x)$, $F_{I_4}(x)$ are properly integrable from a to b. Hence from (9) we obtain the relation (10) for every pair of sets $I_3 I_4$ each enclosing Ξ and such that

$$(12)\quad \begin{aligned} F_{I_3}(x) &= F_{I_1}(x), \qquad F_{I_4}(x) = F_{I_2}(x) && (x \text{ within } J), \\ F_{I_3}(x) &= F_{I_4}(x) && (x \text{ within } K), \end{aligned}$$

the various proper definite integrals having values independent of the values of the respective integrand functions at the extremities of the intervals of J and K.

Thus we seek to find two sets $I_3 I_4$ each enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$ and of length less than $2\delta_\epsilon^2$, and, moreover, such that they agree on J with $I_1 I_2$ respectively and on K with each other.

We denote by L the complete x-axis with the omission of the inner points of J. Thus L contains K. And we denote by $I_J I_L$ respectively the J-section and the L-section of an interval-set I, so that

$$(13)\quad I = I_J + I_L.$$

With respect to J and L an interval i may lie on J or L and so be itself an interval i_J or i_L. Otherwise the interval i is by J and L partitioned into a sequence of two or more parts i_J, i_L, two adjoining parts being a part i_J and a part i_L. The two extreme parts may or may not be undivided intervals j or l. The intermediate parts are undivided intervals j or l.

We must for an interval-set I enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$ distribute the intervals i_L of I_L into two complementary classes.

An interval i_L is an interval i_L'' if it contains no point ξ and is either an extreme part of the interval i of I from which it is derived or an intermediate part each of whose adjoining parts i_J contains a point ξ. An interval i_L is an interval i_L' if it either contains a point ξ or is of its interval i an intermediate part one (at least) of whose adjoining intervals i_J contains no point ξ.

Thus we have two interval-sets $I_L' I_L''$ such that $I_L = I_L' + I_L''$.

In case J encloses Ξ narrowly we notice two facts which are of use in the immediate sequel. An interval i which lies on J or L is an interval i_J or i'_L. An interval i undergoing partition with respect to J and L contains at least one part i_J and the sum of its parts i_J, i'_L is an interval-set every interval of which contains a point ξ. These intervals of i are the intervals arising by the excision from i of its various parts i''_L. The interval i itself contains a point ξ. If it contains no part i''_L it itself is the interval in question. If, however, the interval i contains one or more parts i''_L, we consider any such part i''_L. If i''_L is an extreme part the adjoining part i_J contains a point ξ or has in progressive consecution an adjoining part i'_L, since an adjoining part i''_L would be an extreme part, and the interval i consisting of these three parts would have no point ξ. If this third part has no point ξ, there is a fourth part, i_J. If this part i_J has no point ξ, there is a fifth part i'_L. Thus the parts i_J, i'_L enter in alternation and the interval consisting of this complete and unbroken sequence of parts i_J, i'_L contains a point ξ. If on the other hand i''_L is an intermediate part each adjoining part i_J contains a point ξ.

In these notations we take $I_3 I_4$ as follows:*

$$I_3 = I_{1J} + I'_{1L} + I'_{2L}, \quad I_4 = I_{2J} + I'_{1L} + I'_{2L}. \tag{14}$$

And these sets satisfy the prescribed conditions:

(1) Each set is of length less than $2\delta_\epsilon^2$;

(2) $$I_{3J} = I_{1J}, \quad I_{4J} = I_{2J};$$

(3) $$I_{3L} = I_{4L} = I'_{1L} + I'_{2L};$$

(4) Each set encloses Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$.

We need to prove that I_3 and so I_4 encloses Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$. The set Ξ is by hypothesis closed.

We consider first any particular point ξ. If ξ lies within J or L it is enclosed by an interval of I_{1J} or I'_{1L}, and so by an interval of I_3. If, however, ξ is common to J and L it is an extremity of an interval of each set; the point ξ lies within a certain interval of I_1, and it is the point of junction of two intervals of $I_{1J} I'_{1L}$ respectively, and in $I_3 = I_{1J} + I'_{1L} + I'_{2L}$ these two intervals perhaps with others form an interval i_3 enclosing the point ξ. Thus the interval-set I_3 encloses Ξ at least broadly and theorem X is proved for the second case.

For the first case we consider any particular interval i_3 of I_3 and now need to prove merely that it contains a point ξ. The interval i_3 arises by the union of certain intervals of $I_{1J} I'_{1L} I'_{2L}$. In the introduction of the notation I'_L

* In the second case, one may determine suitable interval-sets I_3, I_4 more simply as follows:

$$I_3 = I_{1J} + I''', \quad I_4 = I_{2J} + I''',$$

where I''' is the interval-set common to I_{1L} and I_{2L}.

for an interval-set $I = I(\Xi)$ it was noted that every interval i_J or i'_L forms part of a sequence of intervals $i_J i'_L$ constituting an interval containing a point ξ. Thus i_3 contains a point ξ or it consists of a single interval i''_{2L} which contains no point ξ. But this latter case is impossible. For an interval i''_{2L} containing no point ξ is an intermediate part of an interval i_2 one of whose adjoining parts i_{2J} contains no point ξ. But this is impossible, by virtue of the third hypothesis of theorem X.

Thus, indeed, every interval i_3 contains a point ξ. And now theorem X is proved also for the first case, and, hence, completely.

XI. *For every interval-set J lying on the interval ab there exists a constant, in notation,*

$$\int_{J\Xi} F(x)\,dx, \tag{15}$$

the integral on the interval-set J with respect to the point-set Ξ of the function $F(x)$, such that for every ϵ the relation:

$$\left| \int_J F_I(x)\,dx - \int_{J\Xi} F(x)\,dx \right| \leqq \epsilon, \tag{16}$$

holds for every interval-set I satisfying the following conditions:

(1) *The set I encloses Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$;*

(2) *The length D_I is less than δ_ϵ^2;*

and, moreover, in the first case, whenever J consists of more than a single interval,

(3) *Every interval i of I which encloses an interval l joining two consecutive intervals of J has with respect to its every such enclosed interval l the following property: the interval i contains either on the interval l a point ξ or on each of the adjoining intervals of J a point ξ.*

This theorem follows directly from the preceding theorem by the usual limit-considerations, wherein it needs to be noticed that interval-sets $I_1 I_2$ satisfying the conditions 1, 2, 3 of that theorem actually exist for every ϵ and interval-set J of the interval ab.

Remark. If J is a single interval $a'b'(a' < b')$ then obviously

$$\int_{J\Xi} F(x)\,dx = \int_{a'\Xi}^{b'} F(x)\,dx. \tag{17}$$

This remark is generalized in the following theorem:

XII. *For every interval-set J of the interval ab,*

$$\int_{J\Xi} F(x)\,dx = \sum_{l=1}^{n} \int_{a_l\Xi}^{b_l} F(x)\,dx, \tag{18}$$

where the intervals $a_l b_l (a_l < b_l;\ l = 1, \cdots, n)$ are the intervals of J.

For every interval-set I enclosing Ξ we have, by the definition (11) of § 2,

$$\int_J F_I(x)\,dx = \sum_{l=1}^{n} \int_{a_l}^{b_l} F_I(x)\,dx. \tag{19}$$

We introduce a set I which satisfies with respect to J and so with respect to every interval $a_l b_l$ of J the conditions of theorem XI for $\epsilon = \epsilon'/(n+1)$ where ϵ' is any particular positive constant. Then by theorem XI we have

$$\left| \int_J F_I(x)\,dx - \int_{J\,\Xi} F(x)\,dx \right| \leqq \frac{\epsilon'}{n+1}, \tag{20}$$

$$\left| \int_{a_l}^{b_l} F_I(x)\,dx - \int_{a_l\,\Xi}^{b_l} F(x)\,dx \right| \leqq \frac{\epsilon'}{n+1} \qquad (l = 1, 2, \cdots, n), \tag{21}$$

and then from (19), (20), (21) we have

$$\left| \int_{J\,\Xi} F(x)\,dx - \sum_{l=1}^{n} \int_{a_l\,\Xi}^{b_l} F(x)\,dx \right| \leqq \epsilon', \tag{22}$$

from which the theorem follows at once.

Remark. Theorem XI is to be understood as applying also to the case in which the symbol J denotes a non-existent interval-set, the constant in question having the value 0. Then in view of theorem XII it is evident that with respect to an interval-set J lying on the interval ab and a function $F(x)$ Ξ-integrable from a to b a definition of the symbol $\int_{J\,\Xi} F(x)\,dx$ might be given similar to that given in § 2 for the symbol $\int_J f(x)\,dx$ with respect to a function $f(x)$ properly integrable from a to b.

XIII′. *For every ϵ there exists a δ_ϵ^3 such that*

$$\left| \int_{J\,\Xi} F(x)\,dx \right| < \epsilon \tag{23}$$

for every interval-set J satisfying the following conditions:

(1) *The interval-set J lies on the interval ab;*

(2) *The length D_J is less than δ_ϵ^3;*

and, moreover, in the first case, whenever J consists of more than a single interval,

(3′) *The interval-set J has with respect to every interval l joining two consecutive intervals of J the following property: either the interval l contains a point ξ or each of the adjoining intervals of J contains a point ξ.*

XIII″. *In the first case: For every two positive numbers $\epsilon\epsilon'$ there exists a $\delta_{\epsilon\epsilon'}^3$ such that*

$$\left| \int_{J\,\Xi} F(x)\,dx \right| < \epsilon \tag{23}$$

for every interval-set J satisfying the following conditions:

(1) *The interval-set J lies on the interval ab;*

(2) *The length D_J is less than $\delta^3_{\epsilon\epsilon'}$;*

(3″) *Whenever J consists of more than a single interval, and has an interval l joining two consecutive intervals of J, and lying within the ϵ'-neighborhood of a point ξ, it has with respect to every such interval l the following property: either the interval l contains a point ξ or each of the adjoining intervals of J contains a point ξ.*

Note 1. The condition 3′ of XIII′ is satisfied in particular by interval-sets J satisfying the condition:

(3‴) *Every interval of J contains a point ξ.*

Note 2. For an interval-set I enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$ and of length less than δ^3_ϵ the inequality

$$\left| \int_{I^b_a \Xi} F(x)\,dx \right| < \epsilon$$

holds. This is a particular case of XIII′. With δ^3_ϵ replaced by δ_ϵ, it follows from the definition of the Ξ-integral from a to b, in view of the relation:

$$\int_{I^b_a \Xi} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx - \int_a^b F_I(x)\,dx,$$

derived from theorem XII, together with § 2 IV.

Theorem XIII is for functions Ξ-integrable a generalization* of the second lemma of § 2 for functions properly integrable and it is proved by means of that lemma and theorem XI.

We take a particular interval-set I enclosing Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$ and of length D_I less than $\delta^2_{\epsilon/2}$, and moreover, in connection with theorem XIII″, of intervals each of length at most ϵ'. The function $F_I(x)$ is properly integrable from a to b.

An effective number δ^3_ϵ, or, for the second theorem of the first case, $\delta^3_{\epsilon\epsilon'}$, is the smaller of certain numbers: viz., the number $\delta'_{\epsilon/2}$ connected with this function $F_I(x)$ quâ $f(x)$ by the second lemma of § 2, and if I^b_a consists of more than a single interval, the various lengths of the intervals joining two consecutive intervals of I^b_a. Thus an interval j of an interval-set J of length $D_J < \delta^3_\xi$ or $\delta^3_{\epsilon\epsilon'}$ has points in common with at most one interval i of I.

This number δ^3_ϵ or $\delta^3_{\epsilon\epsilon'}$ is indeed effective. For the inequality (23) follows at once from the simultaneous inequalities:

$$\left| \int_J F_I(x)\,dx \right| < \tfrac{1}{2}\epsilon, \tag{24}$$

* The inequality (23) of XIII holds if on the right ϵ be replaced by 2ϵ and on the left the absolute value of the Ξ-integral over the interval-set J be replaced by the sum of the absolute values of the integrals over the constituent intervals of J. This is seen by considering separately the intervals yielding positive integrals and those yielding negative intergals. Cf. JORDAN, loc. cit., p. 226, ll. 9, 10; pp. 224–5.

$$\left| \int_J F_I(x)\,dx - \int_{J\Xi} F(x)\,dx \right| \leqq \tfrac{1}{2}\epsilon, \tag{25}$$

where J is an interval-set satisfying the conditions of theorem XIII. The inequality (24) is by virtue of the second lemma of § 2, since $D_J < \delta'_{\epsilon/2}$. The inequality (25) is by virtue of theorem XI.

And theorem XI is, indeed, applicable. For J lies on the interval ab, and I encloses Ξ $\begin{matrix}\text{narrowly}\\ \text{broadly}\end{matrix}$ and is of length D_I less than $\delta^2_{\epsilon/2}$. Thus the double theorem XIII is proved, in the second case, and also in the first case, if J consists of a single interval.

In the first case, if J consists of more than a single interval, we have further to prove that condition (3) of the hypothesis of theorem XI is satisfied. We consider the interval-sets J and I with respect to an interval l joining two consecutive intervals j', j'' of J. If l contains a point ξ or if j' and j'' each contain a point ξ, then an interval i of I enclosing l, if one there be, has with respect to l the property of condition (3) of theorem XI, that is, the interval i contains either on the interval l a point ξ or on each interval j', j'' a point ξ. For each point ξ lies in some interval of I, and since j' and j'' intersect i neither intersects a second interval of I. If, however, l contains no point ξ and either j' or j'' contains no point ξ, then no interval i encloses l; for we have now to do with XIII″, and by hypothesis (3″), the interval l lies within the ϵ'-neighborhood of no point ξ, while every interval i contains a point ξ and is of length at most ϵ'.

XIV. *The function $F_I(x)$ is properly integrable from a to b and the inequality:*

$$\left| \int_a^b F_I(x)\,dx - \int_{a\Xi}^b F(x)\,dx \right| < \epsilon, \tag{26}$$

holds for every interval-set I whose length D_I is less than δ^3_ϵ or $\delta^3_{\epsilon\epsilon'}$, and moreover, in the first case, whose interval-set I^b_a, quâ interval-set J satisfies the condition* (3′) or (3″) of theorem* XIII, *provided moreover (in each case) that the interval-set I contains the singular point-set Z of the function $F(x)$ in such a way that if a singular point ζ is an extremity of an interval i of I it is singular merely towards the interior of i.*

Note. This theorem is, in accordance with note 2 of theorem XIII, *of the nature of an extension of the definitional property of the function $F(x)$ Ξ-integrable from a to b.*—In particular, we see that *the relation* (26) *holds for every interval-set I of length $D_I < \delta^3_\epsilon$ whose corresponding function $F_I(x)$ is properly integrable from a to b, which contains Ξ, and, in the first case, whose every*

* The $\delta^3_{\epsilon\epsilon'}$ enters only in the first case and then only if the hypothesis imposes (merely) the condition (3″) of theorem XIII″.

interval contains a point ξ of Ξ. Such an interval-set I obviously may perhaps not enclose Ξ, for an interval i may have a limit-point ξ' of Ξ as an extremity.

We proceed to prove theorem XIV. The singular point-set Z was defined in §1 4°. From the final condition as to the interval-set I it follows by the usual process of indirect proof that the function $F_I(x)$ is on the interval ab everywhere defined, single valued, and limited. It is moreover Ξ-integrable from a to b, by virtue of theorems III and IV of §2, and V and VIII of §3. Thus by theorem V of §2 it is properly integrable from a to b. Further by these theorems and theorem XII we have the equality:

$$\int_a^b F_I(x)\,dx = \int_{a\,\Xi}^{b} F_I(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx - \int_{I_a^b\,\Xi} F(x)\,dx, \tag{27}$$

from which the theorem follows by virtue of theorem XIII.

XV. *If the function $F(x)$ is integrable from a to b with respect to a closed point-set Ξ of content zero, and if* H *is any point-set of content zero, such that $\Xi + \mathrm{H}$ is closed, and which, moreover, in the first case, satisfies the condition specified below, then the function $F(x)$ is $(\Xi + \mathrm{H})$-integrable from a to b, and*

$$\int_{a\,\Xi+\mathrm{H}}^{b} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx. \tag{28}$$

The set H *in the first case has the following property: there exist on the interval ab at most a finite number of intervals $a'b'$ enclosing no point ξ and no point η and having for extremities points η which are not points ξ.*

In the second case this theorem is evident, since the interval-sets I enclosing $\Xi + \mathrm{H}$ broadly are amongst those enclosing Ξ broadly.

We proceed to prove the theorem in the first case. An interval-set K enclosing $\Xi + \mathrm{H}$ narrowly is with respect to Ξ separated into an interval-set I enclosing Ξ narrowly and a complementary interval set J perhaps not existent.

With respect to a number ϵ we shall prove that

$$\left| \int_a^b F_K(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \epsilon \tag{29}$$

for every such interval-set K of length $D_K < \delta_\epsilon^{\iota}$, where δ_ϵ^{ι} is the least of the three numbers $\delta_{\epsilon/2}$, ϵ', and $\delta^{\iota}_{\epsilon/2\,\epsilon'}$, where ϵ' is a certain positive number to be specified in the sequel.

Indeed the inequality (29) follows at once from the fact that the relations:

$$\int_a^b F_K(x)\,dx = \int_a^b F_I(x)\,dx - \int_{J_a^b\,\Xi} F(x)\,dx, \tag{30}$$

$$\left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \tfrac{1}{2}\epsilon, \tag{31}$$

$$\left| \int_{J_a^b\,\Xi} F(x)\,dx \right| < \tfrac{1}{2}\epsilon, \tag{32}$$

hold simultaneously. Of these relations the first and second are clear, and the third follows from theorem XIII′′, in connection with the stipulations $D_{J_a^b} \leqq D_K < \delta_\epsilon^1 < \epsilon'$, $\delta^3_{\epsilon;2\,\epsilon'}$. In order to secure this application of theorem XIII′′ we are to determine the positive number ϵ' so that every interval l joining two consecutive intervals of J_a^b either contains a point ξ or lies within the ϵ'-neighborhood of no point ξ. This determination is made by means of the final condition of the theorem.

We denote by H* the set of points $\eta = \eta^*$ which lie on the interval ab and which are not points ξ; by H_* the set obtained by adding to H* the points a and b; and for brevity by J_0 the set J_a^b.

An interval j of J contains a point η and no point ξ. An interval j_0 of J_0 contains a point η_* and no point ξ. Two consecutive intervals j_0', j_0'' of J determine an interval $a'b'$ where a' is that point η_* of j_0' nearest to j_0'', and where similarly b' is that point η_* of j_0'' nearest to j_0'. These points $a'b'$ are definite points, for the sets H_* on the two intervals $j_0' j_0''$ are closed sets. This interval $a'b'$ encloses no ξ and no η,—except perhaps within the interval l joining j_0' and j_0''. Now if l encloses no point ξ it encloses no interval i of I, and hence, since $j_0' j_0''$ are consecutive intervals of J_0, it encloses no point η. Thus the two consecutive intervals j_0', j_0'' of J_0 determine a joining-interval l containing a point ξ or else an interval $a'b'$ enclosing no point ξ and no point η and having for extremities points η_*.

Now a enters as an η_*-extremity of such an interval at most once, and likewise b enters at most once. And by the present hypothesis at most a finite number of such intervals $a'b'$ with η^*-extremities exist.

We denote by $3\epsilon'$ the least of the lengths of the various intervals $a'b'$ with η_*-extremities, setting $3\epsilon' = 1$, if no such interval exists. Thus ϵ' is a definite positive number.

Then, since $D_K < \epsilon'$, the intervals $j_0' j_0''$ are each of length less than ϵ', while their interval $a'b'$, whenever it contains no point ξ, is of length at least $3\epsilon'$ and hence their joining-interval l, whenever it contains no point ξ, is of length greater than ϵ' and so lies within the ϵ'-neighborhood of no point ξ.—Thus theorem XIII′′ is available for the completion of the proof of theorem XVII.

XVI. *In the first case: If the function $F(x)$ is Ξ-integrable from a to b, and hence* * Z*-integrable, if however it is not* (Z + H)*-integrable, and hence* *

* Theorem VIIA.

not $(\Xi + \mathrm{H})$*-integrable, where* H *is a closed point-set of content zero, then there is at least one point* $\zeta = \zeta_0$ *of* Z *on* ab *at which* $F(x)$ *fails either of progressive or of regressive* $(\mathrm{Z} + \mathrm{H})$*- and so of* $(\Xi + \mathrm{H})$*-integrability. Hence, by theorem* XV, *in every progressive or in every regressive neighborhood of such a point* ζ_0 *there is an infinitude of intervals* $a'b'$ *enclosing no point* ξ *and no point* η *and having for extremities points* η *which are not points* ξ.

For if there were no such point ζ_0, then $F(x)$ would be $(\mathrm{Z} + \mathrm{H})$-integrable on ab and hence from a to b. We need to recall the statements of 4° of § 1, the definitions 2 and 3 and theorem IV of § 2, and the corollary of theorem V of § 3.

Remark. According to theorem XV the hypotheses of theorem XVI imply that the Ξ-integral of $F(x)$ is essentially narrow (that is, not broad). It has not been proved however that, in this case, there exists a point-set H for which the narrow $(\Xi + \mathrm{H})$-integral of $F(x)$ is non-existent.

XVII. *If with respect to two closed point-sets* Ξ, H *of content zero two functions* $F(x)$, $G(x)$ *are respectively integrable from* a *to* b, *and if, in the first case,* Ξ, H *are so related* (*for instance, as indicated in theorem* XV) *that each function is* $(\Xi + \mathrm{H})$*-integrable from* a *to* b, *then* $F(x) + G(x)$ *is* $(\Xi + \mathrm{H})$*-integrable from* a *to* b, *and*

$$\int_{a\,\Xi}^{b} F(x)\,dx + \int_{a\,\mathrm{H}}^{b} G(x)\,dx = \int_{a\,\Xi+\mathrm{H}}^{b} \big(F(x) + G(x)\big)\,dx.$$

This theorem is easily seen to be true.

§ 4.

Concerning the absolutely convergent $\overset{\text{narrow}}{\text{broad}}$ Ξ-integrals.

Definition. The $\overset{\text{narrowly}}{\text{broadly}}$ existent integral $\int_{a\,\Xi}^{b} F(x)\,dx$ is said to converge or exist (as a limit) *absolutely* in case the integral $\int_{a\,\Xi}^{b} |\,F(x)\,|\,dx$ exists $\overset{\text{narrowly}}{\text{broadly}}$.

Making suitable use of the references to the work of Jordan and Stolz given in the introduction, the reader will readily construct the proofs of the following theorems.

I. *The upper limits*

$$\underset{I\,|\,I(\Xi)}{\mathrm{L}} \int_{a}^{b} F_I(x)\,dx, \quad \underset{I\,|\,I\{\Xi\}}{\mathrm{L}} \int_{a}^{b} F_I(x)\,dx$$

with respect to a non-negative function $F(x)$ *on the set of all the interval-sets* I *enclosing* Ξ *respectively narrowly or broadly are equal. If they are finite, the* Ξ*-integral*

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists broadly and is equal to the common value of the upper limits. Conversely, if the Ξ*-integral exists (even) narrowly, then the upper limits are finite, and the* Ξ*-integral exists broadly.*

II. *If the integral*

$$\int_{a\,\Xi}^{b} |F(x)|\,dx$$

exists (if narrowly, then broadly), then the integral

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists broadly, and hence as a broad integral it converges absolutely. Conversely, if the latter integral exists absolutely (even) narrowly, it exists also absolutely broadly.

III. *The broad and absolutely convergent* Ξ*-integrals are in fact identical; hence, the essentially narrow and the non-absolutely convergent* Ξ*-integrals are in fact identical.*

§ 5.*

Concerning the general or narrow Ξ-integrals

$$\int_{a\,(\Xi)}^{b} F(x)\,dx.$$

1°. The following determination of the general point-set Ξ lying on ab closed and of content zero is well known.

Every finite or numerably infinite set H of intervals $h_\nu = a_\nu b_\nu (\nu = 1, 2, \cdots, n$ or $\nu = 1, 2, 3, \cdots)$ lying everywhere densely on ab and no two having a common inner point determines a closed nowhere dense point-set Ξ; and, conversely, every closed nowhere dense point-set Ξ of ab is so determinable by its "point-free" intervals h_ν. Every inner point x of ab not a point ξ lies lies within a definite interval h_ν. The points ξ together with a and b are the extremities $a_\nu b_\nu$ of the intervals h_ν and the limit-points of such extremities.— For later use I introduce subintervals $h'_\nu = a'_\nu b'_\nu$ of the respective intervals $h_\nu = a_\nu b_\nu$, with the understanding that $a'_\nu = a_\nu$ if $a_\nu = a =$ not-ξ; $b'_\nu = b_\nu$ if $b_\nu = b =$ not-ξ; and otherwise $a_\nu < a'_\nu < b'_\nu < b_\nu$.

The content of the set Ξ so determined is $|a - b| - \sum_\nu |a_\nu - b_\nu|$, so that for our purposes

$$\sum_\nu |a_\nu - b_\nu| = |a - b|. \tag{1}$$

2°. We consider any such set Ξ and have the following important theorem:

* Addition of July 1, 1901.

THEOREM. *For the existence of the narrow Ξ-integral*

$$\int_{a(\Xi)}^{b} F(x)\,dx \tag{2}$$

a necessary and sufficient (twofold) condition is this, that

(α) *The narrow Ξ_ν-integrals*

$$\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \qquad (\Xi_\nu = (a_\nu, b_\nu)); \tag{3}$$

exist;

and, if Ξ contains an infinitude of points,

(β) *The infinite series*

$$\sum_{\nu=1}^{\infty} O_\nu \tag{4}$$

converges, where for every O_ν denotes the oscillation on the X-interval $a_\nu b_\nu$ of the continuous definite integral function

$$J_\nu(X) = \int_{a_\nu(\Xi_\nu)}^{X} F(x)\,dx. \tag{5}$$

This sufficient twofold condition being satisfied, the integral (2) has the value

$$\int_{a(\Xi)}^{b} F(x)\,dx = \sum_{\nu} \int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx. \tag{6}$$

As to the final statement, it is obvious that the terms of the series on the right of (6) are by (α) definite numbers, and that the series (6) if infinite is convergent by (β).

The twofold condition is necessary. In fact, § 3 V gives (α) at once, while (β) follows indirectly * easily from § 3 XIII′. For the oscillation O_ν may be expressed in the form

$$O_\nu = \int_{X_\nu''(\Xi_\nu)}^{X_\nu'} F(x)\,dx = \int_{X_\nu''(\Xi)}^{X_\nu'} F(x)\,dx, \tag{7}$$

where $X_\nu' X_\nu''$ are points of $a_\nu b_\nu$ at which $J_\nu(X)$ has its maximum and minimum values respectively. Now we suppose that the infinite series (4) diverges, and we consider any two positive numbers $\epsilon\,\delta$. In view of (1) there is an integer ν_δ such that

$$\sum_{\nu > \nu_\delta} |a_\nu - b_\nu| < \delta. \tag{8}$$

* The direct proof is equally simple.

Then in connection with the divergent series

$$\sum_{\nu>\nu_\delta} O_\nu \tag{9}$$

there is an integer $\nu_{\delta\,\epsilon} > \nu_\delta$ such that

$$\sum_{\nu=\nu_\delta+1}^{\nu=\nu_{\delta\epsilon}} O_\nu > 2\epsilon. \tag{10}$$

The terms O_ν of (10) separate into two sets according as $X'_\nu > X''_\nu$ or $X'_\nu < X''_\nu$, and thereby determine two interval-sets J_1, J_2 of intervals $X''_\nu X'_\nu$, $X'_\nu X''_\nu$, each lying on ab and of length less than δ, and each having the property that every interval l joining two of its intervals contains a point ξ, and furthermore either J_1 or J_2 quâ J having the property that

$$\left| \int_{J(\Xi)} F(x)\,dx \right| > \epsilon. \tag{11}$$

And this result is in contradiction with §3 XIII′.

It remains to prove that the twofold condition is sufficient. For a finite set Ξ this follows at once from §2 III. We consider* then an infinite set Ξ, so that $\nu = 1, 2, \cdots$. For every ϵ a δ_ϵ must be exhibited such that for every interval-set I enclosing Ξ narrowly and of length $D_I < \delta_\epsilon$ we have

$$\left| \int_a^b F_I(x)\,dx - \sum_\nu \int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \right| < \epsilon. \tag{12}$$

We denote by J the complementary interval-set on ab of I''_a; J consists then of say m intervals $a'_{\nu'}, b'_{\nu'}$ $(\nu' = \nu_l;\ l = 1, 2, \cdots, m)$; thus we have

$$\int_a^b F_I(x)\,dx = \int_J F(x)\,dx = \sum_{\nu'} \int_{a'_{\nu'}}^{b'_{\nu'}} F(x)\,dx. \tag{13}$$

Here we have denoted certain m indices ν by ν'. Denoting the remaining indices ν by ν'', we have the desired inequality (12) in the more convenient form

$$\left| \sum_{\nu'} \left(\int_{a_{\nu'}(\Xi_{\nu'})}^{b_{\nu'}} - \int_{a'_{\nu'}}^{b'_{\nu'}} \right) F(x)\,dx + \sum_{\nu''} \int_{a_{\nu''}(\Xi_{\nu''})}^{b_{\nu''}} F(x)\,dx \right| < \epsilon. \tag{14}$$

The series (4) converges, and there is an integer ν_ϵ such that

$$\sum_{\nu>\nu_\epsilon} O_\nu < \frac{\epsilon}{4}. \tag{15}$$

* By slight modifications the proof of the sequel may be made to cover also the case of the finite set Ξ.

The ν_ϵ integrals

$$\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \qquad (\nu=1, 2, \cdots, \nu_\epsilon)$$

exist, and there is a number δ_ϵ such that

$$(16) \qquad \delta_\epsilon < |a_\nu - b_\nu|, \quad \left|\left(\int_{a_\nu(\Xi_\nu)}^{b_\nu} - \int_{a'_\nu}^{b'_\nu}\right) F(x)\,dx\right| < \frac{\epsilon}{2\nu_\epsilon} \qquad (\nu=1, 2, \cdots, \nu_\epsilon)$$

if throughout

$$(17) \qquad |a_\nu - b_\nu| - |a'_\nu - b'_\nu| < \delta_\epsilon.$$

This number δ_ϵ is the number desired: if I is any interval-set enclosing Ξ narrowly and of length $D_I < \delta_\epsilon$ the inequality (14) will hold. For

$$(18) \qquad D_I = \sum_{\nu'} (|a_{\nu'} - b_{\nu'}| - |a'_{\nu'} - b'_{\nu'}|) + \sum_{\nu''} |a_{\nu''} - b_{\nu''}|,$$

and so

$$(19) \qquad |a_{\nu'} - b_{\nu'}| - |a'_{\nu'} - b'_{\nu'}| < \delta_\epsilon, \quad |a_{\nu''} - b_{\nu''}| < \delta_\epsilon.$$

By comparison of (16) and (19) we see that the indices ν' include the first ν_ϵ indices ν, $\nu = 1, 2, \cdots, \nu_\epsilon$. Denoting the left member of (14) by L, we have

$$L \leqq \sum_{\nu=1}^{\nu_\epsilon} \left|\left(\int_{a_\nu(\Xi_\nu)}^{b_\nu} - \int_{a'_\nu}^{b'_\nu}\right) F(x)\,dx\right| + \sum_{\nu>\nu_\epsilon} \left|\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx\right|$$

$$+ \sum_{\nu'>\nu_\epsilon} \left|\int_{a'_{\nu'}}^{b'_{\nu'}} F(x)\,dx\right|,$$

so that by (15) and (16) in view of (19) we have indeed the desired inequality (14)

$$L < \epsilon.$$

3°. The preceding theorem reduces the problem of construction of all narrow Ξ-integrals essentially to the corresponding problem for the case $\Xi = (b)$, and this remark holds likewise if the integrand function $F(x)$ is required to be on ab continuous except at points $x = \xi$.

In 5° I exhibit by a known process an essentially narrow or non-absolutely convergent integral

$$(20) \qquad \int_{a\,((b))}^{b} F(x)\,dx,$$

and in connection with it develop examples designed to show the error of certain statements referred to in the introduction.

In 4° I use such an integral (20), for the case $ab = 01$, as the element for a simple construction according to the conditions of 2° of a Ξ-integral for the general set Ξ of ab, non-absolutely convergent in the neighborhood of every point ξ.

4°. We denote by $F(x) = \psi_{01}(x)$ a function on the x-interval 01 continuous except at $x = 1$ for which the integral

$$\int_{0\,((1))}^{1} F(x)\,dx$$

converges non-absolutely. A function $F(x) = \psi_{ab}(x)$ of similar character on the interval ab $(a \gtrless b)$ is given by the transformation

$$\psi_{ab}(x') = \psi_{01}(x), \qquad x' = (b - a)x + a.$$

Denoting the middle point of ab by c, we denote by $\chi_{ab}(x)$ the function on ab everywhere continuous except at a and b which on ac is $\psi_{ca}(x)$ and on cb is $\psi_{cb}(x)$. The functions $\chi_{ab}(x)$ are derivable from $\chi_{01}(x)$ by the transformation used above. The integral

$$\int_{a\,((a\,,\,b))}^{b} \chi_{ab}(x)\,dx$$

converges, non-absolutely near a and b.

Then with respect to any set Ξ (1°) we construct a function $F(x)$ everywhere continuous on ab except at its singularities ξ and such that the integral

$$\int_{a\,(\Xi)}^{b} F(x)\,dx$$

converges non-absolutely near every point ξ. The function $F(x)$ is on $a_\nu b_\nu$ the function $\chi_{a_\nu b_\nu}(x)$, or ψ_{ab_ν} if $a_\nu = a =$ not-ξ, or ψ_{ba_ν} if $b_\nu = b =$ not-ξ.

The effectiveness of this construction follows easily from the theorem of 2° since the oscillations O_{ab}, O_{01}, O'_{ab}, of the integrals

$$\int_{a\,((b))}^{X} \psi_{ab}(x)\,dx, \quad \int_{0\,((1))}^{X} \psi_{01}(x)\,dx, \quad \int_{a\,(a\,,\,b)}^{X} \chi_{ab}(x)\,dx$$

on the respective intervals ab, 01, ab have the relations

$$O_{ab} = O_{01}|a - b|, \quad O'_{ab} \leqq O_{ca} + O_{cb} = O_{ab}.$$

5°. *Construction of a function $F(x)$ everywhere continuous on ab except at b, for which the integral*

$$\int_{a\,((b))}^{b} F(x)\,dx \tag{20}$$

converges non-absolutely.

We introduce any non-absolutely convergent series

$$\sum_{\nu=1}^{\infty} u_\nu, \tag{21}$$

—for instance, the modified harmonic series $(u_\nu = -(-1)^\nu/\nu)$,—and further any infinite sequence H of intervals $h_\nu = a_\nu b_\nu (\nu = 1, 2, 3, \cdots)$ lying progressively on ab, two consecutive intervals having no common inner point, and b being the limit-point of each of the sequences $(a_1, a_2, \cdots), (b_1, b_2, \cdots)$.

Then we determine the function $F(x)$ as follows. On the interval h_ν $F(x)$ is everywhere continuous, does not change sign, and vanishes at the extremities $a_\nu b_\nu$, and further

$$\int_{a_\nu}^{b_\nu} F(x)\,dx = u_\nu, \tag{22}$$

—for instance, $y = F(x)$ is given graphically by the two sides of the isosceles triangle of (signed) height $u_\nu/2(b_\nu - a_\nu)$ on the base $a_\nu b_\nu$. At points x of ab but of no interval $a_\nu b_\nu$ $F(x)$ vanishes.

Then indeed $F(x)$ is everywhere continuous on ab except at $x = b$. In view of (22) $|F(x)|$ has on h_ν an upper limit greater than $|u_\nu|/|a_\nu - b_\nu|$, and hence, since the series (21) is non-absolutely convergent, $F(x)$ has values indefinitely great positive and negative in every neighborhood of $x = b$.

The narrow (b)-integral (20) is by definition (§ 2)

$$\underset{X=b}{\overset{X \text{ on } ab}{\mathrm{L}}} \int_a^X F(x)\,dx. \tag{23}$$

Now we have

$$\int_a^X F(x)\,dx = \sum_{\nu=1}^{n} u_\nu + \theta u_{n+1}, \tag{24}$$

where* X lies on the interval $b_n b_{n+1}$ and θ is some number, $0 \leqq \theta \leqq 1$. Hence it follows easily that

$$\int_{a\ ((b))}^{b} F(x)\,dx = \sum_{\nu=1}^{\infty} u_\nu, \tag{25}$$

and similarly that $|F(x)|$ is not $((b))$-integrable from a to b, since the series $\sum_\nu |u_\nu|$ diverges.

Thus, the integral (20) is non-absolutely convergent, and so (§ 4 III) it is an essentially narrow (b)-integral.

---

* We set $b_0 = a$ and $\sum_{\nu=1}^{0} u_\nu = 0$.

6°. *Exhibition for the* (b)*-integral* (20) *with respect to any two positive numbers* $\epsilon\,\delta$ *of an interval-set* J *not containing the point* b, *for which*

$$D_J < \delta, \quad \left| \int_J F(x)\,dx \right| > \epsilon. — \tag{26}$$

An example contravening HARNACK's *theorem* 2.

In accordance with the footnote to note 1 of the first definition of § 2 this exhibition furnishes a more direct proof that the (b)-integral (20) is essentially narrow. And it evidently serves to disprove HARNACK's theorem 2.

In view of the convergence for $\nu = \infty$ of the intervals h_ν to b there is an integer ν_0 such that

$$\sum_{\nu > \nu_0} |a_\nu - b_\nu| < \delta. \tag{27}$$

The series

$$\sum_{\nu > \nu_0} u_\nu$$

converges non-absolutely, and so contains a finite number (m) of positive terms $u_{\nu'} (\nu' = \nu_l > \nu_0;\ l = 1, 2, \cdots, m)$ such that

$$\sum_{\nu} u_{\nu'} > \epsilon. \tag{28}$$

For the interval-set J consisting of the m intervals $h_{\nu'}$ the inequalities (26) follow from (27) and (28).

7°. *Exhibition for the* (Ξ)*-integral* (20) $\big(\Xi = (b)\big)$ *of a set* H *closed and of content zero such that the corresponding* $(\Xi + \mathrm{H})$*-integral does not exist.*

The set H of all points $a_\nu b_\nu$ $(\nu = 1, 2, 3, \cdots)$ and the point b has the desired property. This follows directly from the exhibition of 6° in view of § 3 XIII′.

This example serves to show the presence of errors in certain statements of HARNACK* and BRODÉN* concerning the possibility of modifying the functional values of the integrand function at points of a set of content zero without altering the determination of the integral.

8°. *An example contravening* JORDAN's *theorem* 3.

We have by 7° for the function $F(x)$ of (20) the (Ξ)-integral existent and the $(\Xi + \mathrm{H})$-integral non-existent, and furthermore $\Xi + \mathrm{H} = \mathrm{H}$.

Now we take by 4° a function $G(x)$ for which the $(\mathrm{H}) = (\Xi + \mathrm{H})$-integral exists. Then the $(\Xi + \mathrm{H})$-integral of $F(x) + G(x)$ is non-existent; for its existence would by § 3 II and XVII imply the existence of the $(\Xi + \mathrm{H})$-integral of $F(x)$.

9°. *An example contravening* HARNACK's *theorem* 3.

The example of 8° is effective. For, if we set

---

*Mathematische Annalen, vol. 24 (1884), p. 220, and vol. 52 (1899), p. 222.

$$J(x) = \int_{a\,(\Xi)}^{x} F(x)\,dx, \quad K(x) = \int_{a\,(\mathrm{H})}^{x} G(x)\,dx, \tag{29}$$

$$L(x) = J(x) + K(x), \tag{30}$$

the functions $J(x)$, $K(x)$, $L(x)$ are everywhere continuous on ab; further

$$L(x'') - L(x') = \int_{x'}^{x''} \big(F(x) + G(x)\big)\,dx$$

for every interval $x'x''$ of ab containing no singularity ζ of $F(x) + G(x)$; and the set of singularities $\mathrm{Z} = \Xi + \mathrm{H}$ is reducible; and yet by 8° the HARNACK integral

$$\int_{a\,(\mathrm{Z})}^{b} \big(F(x) + G(x)\big)\,dx \tag{31}$$

does not exist.

The HÖLDER integral (31) however exists and its value is $L(b)$.

The non-existent integral

$$\int_{a\,(\Xi+\mathrm{H})}^{b} F(x)\,dx \tag{32}$$

would afford a simpler example from the point of view of this paper. However from the original HARNACK point of view one would object that the set of singularities of $F(x)$ is Ξ and not $\Xi + \mathrm{H}$.

In the example given the integrand function $F(x) + G(x)$ is everywhere continuous on ab except at its singularities ζ.

THE UNIVERSITY OF CHICAGO,

June 12, 1901.

ZUR LINEAREN TRANSFORMATION DER ϑ-REIHEN*

VON

F. MERTENS

§ 1.

Es sei ω eine complexe Grösse, in welcher der Coefficient von i positiv ist, und zur Abkürzung

$$q = e^{i\pi\omega}.$$

Definirt man die ϑ-Reihen durch die Gleichungen:

$$\vartheta_0(x, \omega) = \sum (-1)^n q^{n^2} e^{2n\pi ix},$$

$$\vartheta_1(x, \omega) = -i \sum (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi ix},$$

$$\vartheta_2(x, \omega) = \sum q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi ix},$$

$$\vartheta_3(x, \omega) = \sum q^{n^2} e^{2n\pi ix},$$

$$(n = 0, \pm 1, \pm 2, \pm 3, \cdots)$$

so besteht die lineare Transformation dieser Reihen in der Ersetzung von ω durch $(\gamma + \delta\omega)/(\alpha + \beta\omega)$, wo α, β, γ, δ ganze der Gleichung

$$\alpha\delta - \beta\gamma = 1$$

genügende Zahlen sind.

Wird zur Abkürzung

$$\alpha + \beta\omega = N, \quad \gamma + \delta\omega = M$$

gesetzt, so kann mit Hilfe der elementaren Transformationsformeln:

$$\vartheta_0(x, 1 + \omega) = \vartheta_3(x, \omega),$$

$$\vartheta_1(x, 1 + \omega) = e^{\frac{i\pi}{4}} \vartheta_1(x, \omega),$$

$$\vartheta_2(x, 1 + \omega) = e^{\frac{i\pi}{4}} \vartheta_2(x, \omega),$$

$$\vartheta_3(x, 1 + \omega) = \vartheta_0(x, \omega),$$

$$\vartheta_0\left(\frac{x}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\, e^{\frac{i\pi x^2}{\omega}} \vartheta_2(x, \omega),$$

* Presented to the Society (Chicago) April 6, 1901. Received for publication March 5, 1901.

$$\vartheta_1\left(\frac{x}{\omega}, -\frac{1}{\omega}\right) = -i\sqrt{-i\omega}\, e^{\frac{i\pi x^2}{\omega}}\, \vartheta_1(x, \omega),$$

$$\vartheta_2\left(\frac{x}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\, e^{\frac{i\pi x^2}{\omega}}\, \vartheta_0(x, \omega),$$

$$\vartheta_3\left(\frac{x}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\, e^{\frac{i\pi x^2}{\omega}}\, \vartheta_3(x, \omega),$$

leicht gezeigt werden, dass $\vartheta_1(x/N, M/N)$ die Gestalt $a e^{bx^2} \vartheta_1(x, \omega)$ hat, wo a, b Constanten bezeichnen.

Ist nämlich β nicht $= 0$, so sei β' der kleinste nicht negative Rest von δ in Bezug auf den Theiler β und

$$\delta = g\beta + \beta', \qquad \gamma = ga + a',$$
$$\gamma' = -a, \qquad \delta' = -\beta,$$
$$a' + \beta'\omega = N', \qquad \gamma' + \delta'\omega = M'.$$

Es ist dann

$$a'\delta' - \beta'\gamma' = 1, \quad |\beta'| < |\beta|,$$

$$\vartheta_1\left(\frac{x}{N}, \frac{M}{N}\right) = \vartheta_1\left(\frac{-x}{M'}, g - \frac{N'}{M'}\right) = -e^{\frac{g\pi i}{4}}\, \vartheta_1\left(\frac{x}{M'}, \frac{-N'}{M'}\right)$$
$$= ie^{\frac{g\pi i}{4}} \sqrt{\frac{-iM'}{N'}}\; e^{\frac{i\pi x^2}{M'N'}}\, \vartheta_1\left(\frac{x}{N'}, \frac{M'}{N'}\right);$$

$\vartheta_1(x/N, M/N)$ wird also die behauptete Gestalt haben, wenn $\vartheta_1(x/N', M'/N')$ sie hat. Die Ausdrücke M', N' sind aber insofern einfacher wie M, N, als $|\beta'| < |\beta|$ ist.

Ist β' noch nicht $= 0$, so können ebenso Ausdrücke

$$M'' = \gamma'' + \delta''\omega, \qquad N'' = a'' + \beta''\omega$$

angegeben werden, in welchen $|\beta''| < |\beta'|$ ist, und $\vartheta_1(x/N, M/N)$ hat zugleich mit $\vartheta_1(x/N'', M''/N'')$ die behauptete Gestalt.

Durch Fortsetzung dieser Schlüsse gelangt man zu einer linearen Transformation $(\gamma^{(m)} + \delta^{(m)}\omega)/(a^{(m)} + \beta^{(m)}\omega)$, in welcher $\beta^{(m)} = 0$ ist. Es genügt daher die behauptete Gestalt von $\vartheta_1(x/N, M/N)$ für den Fall $\beta = 0$ nachzuweisen.

Der Gleichung

$$a\delta = 1$$

zufolge ist dann $a = \delta = \pm 1$ und daher

$$\vartheta_1\left(\frac{x}{N}, \frac{M}{N}\right) = \vartheta_1(ax, a\gamma + \omega) = a\, e^{\frac{a\gamma\pi i}{4}}\, \vartheta_1(x, \omega).$$

Um die Constante b zu bestimmen, ersetze man x durch $x + N$. Es wird

$$\vartheta_1\left(\frac{x+N}{N}, \frac{M}{N}\right) = -\vartheta_1\left(\frac{x}{N}, \frac{M}{N}\right) = a\, e^{b(x+N)^2}\vartheta_1(x+N, \omega)$$

$$= a\, e^{bx^2+2bNx+bN^2}(-1)^{\alpha+\beta}\frac{\vartheta_1(x, \omega)}{q^{\beta^2}e^{2\beta\pi i x}}$$

und nach Forthebung von $a\, e^{bx^2}\vartheta_1(x, \omega)$

$$-1 = e^{2(bN-i\beta\pi)x}e^{bN^2}q^{-\beta^2}(-1)^{\alpha+\beta}.$$

Es muss also

$$bN - \beta\pi i = 0, \quad b = \frac{\beta\pi i}{N}$$

sein.

Die grösste Schwierigkeit bereitet die Bestimmung der Constante a. Differentiirt man die Gleichung

$$\vartheta_1\left(\frac{x}{N}, \frac{M}{N}\right) = a\, e^{bx^2}\vartheta_1(x, \omega)$$

nach x und setzt hierauf $x = 0$, so ergiebt sich

$$\frac{1}{N}\vartheta_1'\left(0, \frac{M}{N}\right) = a\,\vartheta_1'(0, \omega).$$

Es ist aber, wenn mit Herrn DEDEKIND

$$\eta(\omega) = q^{\frac{1}{12}}\Pi(1-q^{2n}) \qquad (n = 1, 2, 3, \cdots)$$

gesetzt wird,

$$\vartheta_1'(0, \omega) = 2\pi\eta^3(\omega)$$

und daher

$$a = \frac{1}{N}\left(\frac{\eta(M/N)}{\eta(\omega)}\right)^3.$$

Die Bestimmung von a ist hierdurch auf die von $\eta(M/N)$ zurückgeführt.

In dem Folgenden soll der Ausdruck $\eta(M/N)$ mit Hilfe der Formeln

$$\eta(1+\omega) = e^{\frac{i\pi}{12}}\eta(\omega),$$

$$\eta\left(-\frac{1}{\omega}\right) = \sqrt{-i\omega}\,\eta(\omega)$$

ermittelt werden.*

---

* HERMITE, Liouville's Journal, Ser. II., T. III.

DEDEKIND, *Erläuterungen zum XXVII Fragment in Riemann's Werken.*

H. WEBER, *Elliptische Funktionen und algebraische Zahlen* und *Zur Theorie der elliptischen Funktionen*, Acta Mathematica, Bd. VI.

§ 2.

Hilfssatz. Es seien A, B von Null verschiedene complexe Grössen, deren reelle Bestandtheile nicht negativ und nicht beide $= 0$ sind. Zieht man die Quadratwurzeln $\surd A$, $\surd B$ so aus, dass ihre reellen Bestandtheile positiv sind, so hat auch das Product $\surd A \surd B$ einen positiven reellen Bestandtheil und ist daher derjenige Werth der Quadratwurzel $\surd AB$, welcher dieselbe Eigenschaft hat.

Setzt man

$$\surd A = a + ia', \quad \surd B = b + ib',$$

so ist

$$A = a^2 - a'^2 + 2iaa', \quad B = b^2 - b'^2 + 2ibb',$$

$$\surd A \surd B = ab - a'b' + i(ab' + ba'),$$

und nach der Annahme a, b positiv, eine der Grössen $a^2 - a'^2$, $b^2 - b'^2$ positiv, die andere nicht negativ. Ist etwa $a^2 - a'^2$ positiv, so sind $a + a'$, $a - a'$ positiv. Da ferner die Grössen $b + b'$, $b - b'$ nicht entgegengesetzte Vorzeichen haben und ihre Summe positiv ist, so ist eine derselben positiv, die andere nicht negativ. Daher ist auch eine der Grössen

$$(a + a')(b - b'), \quad (a - a')(b + b')$$

positiv, die andere nicht negativ und ihre Summe $2(ab - a'b')$ somit positiv.

§ 3.

Es sei

$$a + \beta\omega = N, \quad \gamma + \delta\omega = M,$$

wo a, β, γ, δ ganze, der Gleichung

$$a\delta - \beta\gamma = 1$$

genügende, Zahlen sind, und es handele sich um die Bestimmung des Ausdrucks $\eta(M/N)$.

Es sind zwei Fälle zu unterscheiden, je nachdem $a\delta$ oder $\beta\gamma$ ungerade ist.

Sind a, δ ungerade, so darf a positiv angenommen werden, da man eintretendenfalls die Vorzeichen aller vier Zahlen a, β, γ, δ umkehren könnte, ohne den Quotienten M/N zu ändern.

Es sei zunächst β gerade und vom Null verschieden.

Der absolut kleinste Rest δ_1 der ungeraden Zahl $\delta - \beta(\gamma + 1)$ in Bezug auf den Theiler 2β ist ungerade und liegt demzufolge, vom Vorzeichen abgesehen, unter β. Setzt man

$$\delta = g\beta + \delta_1, \quad \gamma = ga + \gamma_1,$$

so ist

$$a\delta_1 - \beta\gamma_1 = 1$$

und γ_1 den Congruenzen

$$g \equiv \gamma + 1 \pmod 2,$$

$$\gamma_1 \equiv \gamma - g \pmod 2$$

zufolge ungerade. Es sei ferner β_1 der absolut kleinste Rest von β in Bezug auf den Theiler $2\delta_1$ und

$$\beta = h\delta_1 + \beta_1, \quad \alpha = h\gamma_1 + \alpha_1,$$

so dass h, β_1 gerade, α_1 ungerade sind.

Bezeichnet $\mathfrak{a}$ die Einheit mit dem Verzeichen von α_1 und setzt man

$$\alpha_1 = \mathfrak{a}\alpha', \quad \beta_1 = \mathfrak{a}\beta', \quad \gamma_1 = \mathfrak{a}\gamma', \quad \delta_1 = \mathfrak{a}\delta',$$

$$\alpha' + \beta'\omega = N', \quad \gamma' + \delta'\omega' = M',$$

so ist

$$\alpha'\delta' - \beta'\gamma' = 1,$$

α' positiv und die Bestimmung von $\eta(M/N)$ auf die von $\eta(M'/N')$ zurückführbar.

Man hat, wenn zur Abkürzung

$$\alpha_1 + \beta_1\omega = N_1, \quad \gamma_1 + \delta_1\omega = M_1,$$

$$e^{12}{}^{i\pi} = \rho$$

gesetzt wird,

$$\eta\left(\frac{M}{N}\right) = \eta\left(g + \frac{M_1}{N}\right) = \rho^g \eta\left(\frac{M_1}{N}\right),$$

$$\eta\left(\frac{M_1}{N}\right) = \sqrt{\frac{iN}{M_1}}\,\eta\left(-\frac{N}{M_1}\right),$$

$$\eta\left(\frac{-N}{M_1}\right) = \eta\left(-h - \frac{N_1}{M_1}\right) = \rho^{-h}\eta\left(\frac{-N_1}{M_1}\right),$$

$$\eta\left(\frac{-N_1}{M_1}\right) = \sqrt{\frac{-iM_1}{N_1}}\,\eta\left(\frac{M_1}{N_1}\right) = \sqrt{\frac{-iM_1}{N_1}}\,\eta\left(\frac{M'}{N'}\right),$$

wo die Quadratwurzeln, wie überall in dem Folgenden, mit positivem reellen Bestandtheil auszuziehen sind. Hiernach wird

$$\eta\left(\frac{M}{N}\right) = \rho^{g-h}\sqrt{\frac{iN}{M_1}}\sqrt{\frac{-iM_1}{N_1}}\,\eta\left(\frac{M'}{N'}\right).$$

Bezeichnen $\mathfrak{c}$, $\mathfrak{d}$ die Einheit mit dem Vorzeichen von γ_1 beziehungsweise δ_1, so hat β der Gleichung $\beta = (\alpha\delta_1 - 1)/\gamma_1$ zufolge das Vorzeichen von $\mathfrak{cd}$ und jede

der Grössen $iN/M_1, -i\mathfrak{d}M_1, -ic\mathfrak{d}N$ einen positiven reellen Bestandtheil, daher ist

$$\sqrt{\frac{iN}{M_1}}\,\sqrt{-i\mathfrak{d}M_1} = \sqrt{\mathfrak{d}N},$$

$$\sqrt{-ic\mathfrak{d}N}\,\sqrt{\mathfrak{c}i} = \sqrt{\mathfrak{d}N}, \quad \sqrt{-ic\mathfrak{d}N}\,\sqrt{\mathfrak{c}\mathfrak{d}i} = \sqrt{N}$$

und sonach

$$\sqrt{\frac{iN}{M_1}} = \frac{\sqrt{N}}{\sqrt{-i\mathfrak{d}M_1}}\,\frac{\sqrt{\mathfrak{c}i}}{\sqrt{\mathfrak{c}\mathfrak{d}i}}.$$

Ist β_1 von Null verschieden, so hat es, der Gleichung $\beta_1 = (\alpha_1\delta_1 - 1)/\gamma_1$ zufolge, das Vorzeichen von $\alpha_1\delta_1/\gamma_1$ oder $\mathfrak{acd}$ und die reellen Bestandtheile der Grössen $-iM_1/N_1, -1/i\mathfrak{d}M_1, -ic\mathfrak{d}N'$ sind positiv. Sonach ist

$$\sqrt{\frac{-iM_1}{N_1}}\,\sqrt{\frac{1}{-i\mathfrak{d}M_1}} = \frac{1}{\sqrt{\mathfrak{d}N_1}} = \frac{1}{\sqrt{\mathfrak{ad}N'}},$$

$$\sqrt{-ic\mathfrak{d}N'}\,\sqrt{\mathfrak{a}ci} = \sqrt{\mathfrak{ad}N'}, \quad \sqrt{-ic\mathfrak{d}N'}\,\sqrt{\mathfrak{c}\mathfrak{d}i} = \sqrt{N'},$$

und demzufolge

$$\sqrt{\frac{-iM_1}{N_1}} = \frac{\sqrt{-i\mathfrak{d}M_1}}{\sqrt{N'}}\,\frac{\sqrt{\mathfrak{c}\mathfrak{d}i}}{\sqrt{\mathfrak{a}ci}}.$$

Diese Gleichung ist aber auch noch richtig, wann $\beta_1 = 0$ ist, weil dann $\alpha_1\delta_1 = 1$, also

$$\alpha_1 = \delta_1 = \mathfrak{a} = \mathfrak{d}, \quad N' = 1,$$

$$\sqrt{\frac{-iM_1}{N_1}} = \sqrt{-i\mathfrak{d}M_1}, \quad \sqrt{i\mathfrak{ac}} = \sqrt{\mathfrak{c}\mathfrak{d}i}$$

ist.

Die Einsetzung der gefundenen Werthe von $\sqrt{iN/M_1}$, $\sqrt{-iM_1/N_1}$ ergiebt

$$\eta\left(\frac{M}{N}\right) = \frac{\rho^{g-h}\sqrt{\mathfrak{c}i}\,\sqrt{N}}{\sqrt{\mathfrak{a}ci}\,\sqrt{N'}}\,\eta\left(\frac{M'}{N'}\right);$$

da aber

$$\mathfrak{c}i = i^{\mathfrak{c}}, \quad \mathfrak{a}ci = i^{\mathfrak{ac}}$$

ist, so wird

$$\sqrt{\mathfrak{c}i} = e^{\frac{\mathfrak{c}\pi i}{4}}, \quad \sqrt{\mathfrak{a}ci} = e^{\frac{\mathfrak{ac}\pi i}{4}},$$

$$\frac{\sqrt{\mathfrak{c}i}}{\sqrt{\mathfrak{a}ci}} = e^{\frac{i\pi}{4}\mathfrak{c}(1-\mathfrak{a})} = i^{\frac{\mathfrak{c}(1-\mathfrak{a})}{2}}$$

und demnach

$$\frac{1}{\sqrt{N}}\,\eta\left(\frac{M}{N}\right) = \rho^{g-h}\,i^{\frac{\mathfrak{c}(1-\mathfrak{a})}{2}}\,\frac{1}{\sqrt{N'}}\,\eta\left(\frac{M'}{N'}\right).$$

Die Potenz

$$i^{\frac{c(1-\alpha)}{2}}$$

lässt sich mit Hilfe des LEGENDRE-JACOBI'schen Zeichens vereinfachen.

Die Identität

$$\begin{aligned} c(1-\alpha) &= (1-\alpha)a'\gamma_1 + (\alpha-1)(a'\gamma_1 - c) \\ &= (a'-a)\gamma_1 + (a-a_1)\gamma_1 + c(\alpha-1)(a'c\gamma_1 - 1) \end{aligned}$$

ergiebt nach Ersetzung von $a - a_1$ durch $h\gamma_1$

$$c(1-\alpha) = h\gamma_1^2 + (a'-a)\gamma_1 + c(\alpha-1)(a'c\gamma_1 - 1),$$

und es wird daher, weil γ_1 ungerade ist,

$$c(1-\alpha) \equiv h + (a'-a)\gamma_1 + (\alpha-1)(a'c\gamma_1 - 1) \pmod 8,$$

$$i^{\frac{c(1-\alpha)}{2}} = i^{\frac{h+(a'-a)\gamma_1}{2}} (-1)^{\frac{\alpha-1}{2}\frac{a'c\gamma_1-1}{2}}.$$

Es ist aber

$$(-1)^{\frac{\alpha-1}{2}\frac{a'c\gamma_1-1}{2}} = \left(\frac{\alpha}{a'c\gamma_1}\right) = \left(\frac{\alpha}{a'}\right)\left(\frac{\alpha}{\gamma_1}\right),$$

$$i^{\frac{(a'-a)\gamma_1}{2}} = i^{\frac{a-a'}{2}} (-1)^{\frac{a'-1}{2}\frac{\gamma_1+1}{2}} (-1)^{\frac{\alpha-1}{2}\frac{\gamma_1+1}{2}}$$

und nach dem quadratischen Reciprocitätssatze,

$$(-1)^{\frac{\alpha-1}{2}\frac{\gamma_1+1}{2}} = \left(\frac{a'}{-\gamma_1}\right)\left(\frac{-\gamma_1}{a'}\right) = \left(\frac{a'}{\gamma_1}\right)\left(\frac{-\gamma_1}{a'}\right),$$

$$(-1)^{\frac{\alpha-1}{2}\frac{\gamma_1+1}{2}} = \left(\frac{a}{-\gamma_1}\right)\left(\frac{-\gamma_1}{a}\right) \quad \left(\frac{a}{\gamma_1}\right)\left(\frac{-\gamma_1}{a}\right).$$

Aus den Gleichungen

$$a = h\gamma_1 + a_1,$$

$$-\beta\gamma_1 = 1 - a\delta_1, \quad -\beta_1\gamma_1 = 1 - a_1\delta_1$$

folgt überdies

$$\left(\frac{a}{\gamma_1}\right) = \left(\frac{a_1}{\gamma_1}\right) = \left(\frac{\alpha a'}{\gamma_1}\right) = \left(\frac{\alpha}{\gamma_1}\right)\left(\frac{a'}{\gamma_1}\right),$$

$$\left(\frac{-\beta\gamma_1}{a}\right) = \left(\frac{-\gamma_1}{a}\right)\left(\frac{\beta}{a}\right) = 1, \quad \left(\frac{-\gamma_1}{a}\right) = \left(\frac{\beta}{a}\right),$$

$$\left(\frac{-\beta_1\gamma_1}{a'}\right) = \left(\frac{-\gamma_1}{a'}\right)\left(\frac{\beta_1}{a'}\right) = 1, \quad \left(\frac{-\gamma_1}{a'}\right) = \left(\frac{\beta_1}{a'}\right) = \left(\frac{\alpha}{a'}\right)\left(\frac{\beta'}{a'}\right).$$

Somit wird

$$i^{\frac{(1-a)}{2}} = i^{\frac{h+a-a'}{2}} \left(\frac{\beta}{a}\right)\left(\frac{\beta'}{a'}\right).$$

Nach Ersetzung von $i^{h/2}$ durch ρ^{3h} ergiebt sich daher

$$\left(\frac{\beta}{a}\right)\frac{i^{\frac{1-a}{2}}}{\sqrt{N}}\,\eta\left(\frac{M}{N}\right) = \rho^{g+2h}\left(\frac{\beta'}{a'}\right)\frac{i^{\frac{1-a'}{2}}}{\sqrt{N'}}\,\eta\left(\frac{M'}{N'}\right).$$

Der Factor ρ^{g+2h} kann in zwei andere zerlegt werden, welche beziehungsweise nur von a, β, γ, δ und a', β', γ', δ' abhängen.

Der einfachste Weg zu dieser Zerlegung beruht auf dem Umstande, dass die Produkte $(a^2-1)(\beta^2-1)$, $(\gamma_1^2-1)(\delta_1^2-1)$ Vielfache von 24 sind. Denn einer der Factoren a^2-1, β^2-1 muss nach dem FERMAT'schen Satze ein Vielfaches von 3 sein, weil a, β nicht beide durch 3 theilbar sein können, und a^2-1 ist ein Vielfaches von 8. Dieselben Gründe gelten für $(\gamma_1^2-1)(\delta_1^2-1)$. Es ist daher

$$\begin{aligned} 1 &\equiv a^2+\beta^2-a^2\beta^2 \\ 1 &\equiv \gamma_1^2+\delta_1^2-\gamma_1^2\delta_1^2 \end{aligned} \pmod{24},$$

und die Multiplication mit g, h ergiebt

$$\begin{aligned} g &\equiv a\,ga+\beta\,g\beta-a^2\beta\,g\beta \\ h &\equiv \gamma_1\,h\gamma_1+\delta_1\,h\delta_1-h\gamma_1^2\delta_1^2 \end{aligned} \pmod{24}.$$

Nach Ersetzung von ga, $g\beta$, $h\gamma_1$, $h\delta_1$, durch $\gamma-\gamma_1$, $\delta-\delta_1$, $a-a_1$, $\beta-\beta_1$ folgt hieraus

$$\begin{aligned} g &\equiv a(\gamma-\gamma_1)+\beta(\delta-\delta_1)-a^2\beta(\delta-\delta_1) \\ h &\equiv \gamma_1(a-a_1)+\delta_1(\beta-\beta_1)-h\gamma_1^2\delta_1^2 \end{aligned} \pmod{24},$$

und man erhält durch Addition

$$\begin{aligned} g+h \equiv\ & a\gamma-a_1\gamma_1+\beta\delta-\beta_1\delta_1-a^2\beta\delta+a_1^2\beta_1\delta_1 \\ & +\delta_1(a^2\beta-a_1^2\beta_1)-h\gamma_1^2\delta_1^2 \pmod{24}. \end{aligned}$$

Ersetzt man aber in den Identitäten

$$\begin{aligned} a^2\beta-a_1^2\beta_1 &= (a^2+aa_1+a_1^2)(\beta-\beta_1)+(a+a_1)(a\beta_1-a_1\beta), \\ a\beta-a_1\beta_1 &= (a+a_1)(\beta-\beta_1)+a\beta_1-a_1\beta_1 \end{aligned}$$

$\beta-\beta_1$ durch $h\delta_1$, $a\beta_1-a_1\beta$ durch $-h$, so wird

$$\begin{aligned} \delta_1(a^2\beta-a_1^2\beta_1) &= h\delta_1^2(a^2+aa_1+a_1^2)-h\delta_1(a+a_1), \\ a\beta-a_1\beta_1 &= h\delta_1(a+a_1)-h, \end{aligned}$$

und die Addition ergiebt

$$\delta_1(a^2\beta - a_1^2\beta_1) = -a\beta + a_1\beta_1 - h + h\delta_1^2(a^2 + aa_1 + a_1^2).$$

Mit Hilfe dieser Gleichung nimmt die vorstehende Congruenz für $g + h$, wenn zur Abkürzung

$$A = a\gamma + \beta\delta - a^2\beta\delta - a\beta,$$

$$A' = a'\gamma' + \beta'\delta' - a'^2\beta'\delta' - a'\beta',$$

$$C = h(a^2 + aa_1 + a_1^2 - \gamma_1^2)$$

gesetzt wird, die Gestalt

$$g + 2h \equiv A - A' + \delta_1^2 C \pmod{24}$$

an.

Nach Ersetzung von a durch $h\gamma_1 + a_1$ wird ferner

$$C = (h^3 - h)\gamma_1^2 + 3h^2a_1\gamma_1 + 3ha_1^2.$$

Es ist also einerseits, weil $h^3 - h$ nach dem FERMAT'schen Satze ein Vielfaches von 3 ist,

$$C \equiv 0 \pmod 3$$

und anderseits, weil h gerade, a_1 und γ_1 ungerade sind,

$$C \equiv -h + h^2 + 3h = 4\frac{h}{2}\left(\frac{h}{2} + 1\right) \pmod 8$$

$$\equiv 0 \pmod 8.$$

Da hiernach C ein Vielfaches von 24 ist, so folgt

$$g + 2h \equiv A - A' \pmod{24},$$

$$\rho^{g+2h} = \rho^{A-A'},$$

und man hat

$$\left(\frac{\beta}{a}\right)\frac{i^{\frac{1-a}{2}}\rho^{-A}}{\sqrt{N}}\,\eta\left(\frac{M}{N}\right) = \left(\frac{\beta'}{a'}\right)\frac{i^{\frac{1-a'}{2}}\rho^{-A'}}{\sqrt{N'}}\,\eta\left(\frac{M'}{N'}\right).$$

Diese Gleichung zeigt, dass der Ausdruck

$$\left(\frac{\beta}{a}\right)\frac{i^{\frac{1-a}{2}}\rho^{-A}}{\sqrt{N}}\,\eta\left(\frac{M}{N}\right)$$

ungeändert bleibt, wenn man von den Zahlen a, β, γ, δ zu a', β', γ', δ' übergeht. Letztere Zahlen besitzen aber dieselben Eigenschaften, welche bei den

ursprünglichen Zahlen a, β, γ, δ vorausgesetzt werden und sind insofern einfacher, als $|\beta'| < |\beta|$ ist. Denn es ist

$$a'\delta' - \beta'\gamma' = 1,$$

a', δ' ungerade, β' gerade, a' positiv und $|\beta'| < |\delta_1| < |\beta|$.

Ist β' noch nicht $= 0$, so kann eine neue Reihe von vier Zahlen $a'', \beta'', \gamma'', \delta''$ ermittelt werden, für welche der vorstehende Ausdruck denselben Werth hat wie für a, β, γ, δ. Da die Zahlen $|\beta|, |\beta'|, \cdots$ eine abnehmende Reihe bilden, so führt dieses Verfahren nach einigen Schritten zu vier Zahlen $a^{(n)}, \beta^{(n)}, \gamma^{(n)}, \delta^{(n)}$, deren zweite $= 0$ ist.

Hiermit ist die Aufgabe in allen Fällen auf die Bestimmung des vorstehenden Ausdruckes in dem Falle $\beta = 0$ zurückgeführt. Dann ist aber

$$a = \delta = 1, \quad A = \gamma, \quad \left(\frac{\beta}{a}\right) = \left(\frac{0}{1}\right) = 1,$$

und der fragliche Ausdruck wird

$$= \rho^{-\gamma}\, \eta(\gamma + \omega) = \eta(\omega).$$

Somit ist bei geradem β

$$\eta\left(\frac{M}{N}\right) = \left(\frac{\beta}{a}\right) i^{\frac{a-1}{2}} \rho^A \sqrt{N}\, \eta(\omega).$$

Ist β ungerade, so ist $\beta - a$ gerade, $\delta - \gamma$ ungerade,

$$a(\delta - \gamma) - (\beta - a)\gamma = 1$$

und daher nach dem vorhergehenden Falle

$$\eta\left(\frac{M}{N}\right) = \eta\left\{\frac{\gamma + \dfrac{(\delta - \gamma)\omega}{1 + \omega}}{a + \dfrac{(\beta - a)\omega}{1 + \omega}}\right\} = \left(\frac{\beta - a}{a}\right) i^{\frac{a-1}{2}} \rho^C \sqrt{\frac{N}{1 + \omega}}\, \eta\left(\frac{\omega}{1 + \omega}\right),$$

wo

$$C = a\gamma + (\beta - a)(\delta - \gamma)(1 - a^2) - a(\beta - a).$$

Es ist aber

$$\left(\frac{\beta - a}{a}\right) = \left(\frac{\beta}{a}\right),$$

$$\begin{aligned} C - A - 1 &= (1 - a^2)(a\gamma - a\delta - \beta\gamma - 1) \\ &= (a^3 - a)(2\delta - \gamma) \\ &\equiv 0 \pmod{24}, \end{aligned}$$

weil $a^3 - a$ durch 3 und 8 theilbar ist, und

$$\eta\left(\frac{\omega}{1+\omega}\right) = \sqrt{\frac{i(1+\omega)}{\omega}}\,\eta\left(-1-\frac{1}{\omega}\right)$$

$$= \rho^{-1}\sqrt{\frac{i(1+\omega)}{\omega}}\,\eta\left(-\frac{1}{\omega}\right)$$

$$= \rho^{-1}\sqrt{\frac{i(1+\omega)}{\omega}}\,\sqrt{-i\omega}\,\eta(\omega).$$

Da überdies die Grössen $iN\omega$, $-i\omega(1+\omega)$, $-i\omega$ positive reelle Bestandtheile haben, so wird

$$\sqrt{\frac{iN}{\omega}}\sqrt{\frac{-i\omega}{1+\omega}} = \sqrt{\frac{N}{1+\omega}}, \quad \sqrt{\frac{iN}{\omega}}\sqrt{-i\omega} = \sqrt{N}$$

und demnach

$$\sqrt{\frac{N}{1+\omega}} = \frac{\sqrt{N}}{\sqrt{-i\omega}}\sqrt{\frac{-i\omega}{1+\omega}}.$$

Daher ist, wie bei geradem β,

$$\eta\left(\frac{M}{N}\right) = \left(\frac{\beta}{a}\right) i^{\frac{a-1}{2}}\rho^{A}\sqrt{N}\,\eta(\omega).$$

§ 4.

Es sei $\beta\gamma$ ungerade. Man darf β positiv annehmen.
Man hat

$$\eta\left(\frac{M}{N}\right) = \eta\left\{\frac{\delta+\gamma\frac{1}{\omega}}{\beta+a\frac{1}{\omega}}\right\}$$

und daher auf Grund des bereits erledigten Falles

$$\eta\left(\frac{M}{N}\right) = \left(\frac{-a}{\beta}\right) i^{\frac{\beta-1}{2}}\rho^{B}\sqrt{\frac{N}{\omega}}\,\eta\left(\frac{-1}{\omega}\right),$$

wo

$$B = \beta\delta + a\gamma - a\gamma\beta^2 + a\beta$$

ist. Es ist aber

$$\left(\frac{-a}{\beta}\right) = \left(\frac{-1}{\beta}\right)\left(\frac{a}{\beta}\right) = (-1)^{\frac{\beta-1}{2}}\left(\frac{a}{\beta}\right),$$

$$\eta\left(\frac{-1}{\omega}\right) = \sqrt{-i\omega}\,\eta(\omega),$$

$$\sqrt{-iN}\,\frac{1}{\sqrt{-i\omega}} = \sqrt{\frac{N}{\omega}}$$

und sonach

$$\eta\left(\frac{M}{N}\right) = \left(\frac{a}{\beta}\right) i^{\frac{1-\beta}{2}}\,\rho^{B}\sqrt{-iN}\,\eta(\omega).$$

LIBRARY
OF THE
UNIVERSITY of ILLINOIS.

GEOMETRY OF A SIMULTANEOUS SYSTEM OF TWO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

BY

E. J. WILCZYNSKI

In a former paper† I have laid the foundation for a general theory of invariants of a system of linear homogeneous differential equations. I have actually determined the invariants for the special case of two equations, each of the second order, i. e., for the system

$$\begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned} \tag{1}$$

the independent variable being x. The transformations, which were considered, were the most general which could convert (1) into another system of the same form and order, viz: the transformations of the group G

$$x = f(\xi), \quad y = \alpha(\xi)\eta + \beta(\xi)\zeta, \quad z = \gamma(\xi)\eta + \delta(\xi)\zeta, \tag{2}$$

where f, α, β, γ, δ are arbitrary functions of ξ, subject only to the condition that

$$\alpha\delta - \beta\gamma$$

must not vanish identically.

The present paper, besides deducing some new theorems, will be mainly concerned with geometrical interpretations. We shall again confine ourselves to the special case of equations (1) for two reasons. In the first place this will enable us to make use of the concrete results of our former paper, and in the second place we can thus avoid the consideration of configurations in hyperspace. It will not be difficult to generalize our considerations so as to include the general case, if only a space of the proper number of dimensions be employed.

* Presented to the Society April 27, 1901. Received for publication April 9, 1901.

† E. J. WILCZYNSKI, *Invariants of systems of linear differential equations*, Transactions of the American Mathematical Society, vol. 2 (1901), no. 1.

Geometrical considerations of a similar nature have been applied to the theory of a single linear differential equation by HALPHEN and FANO.*

§ 1. *Definition of the general solutions, and of a simultaneous fundamental system of solutions.*

According to the fundamental theorem of the theory of differential equations, the equations (1) define two functions of x, which are analytic if the coefficients are analytic, and which can be made to satisfy the further conditions that y, z, dy/dx, dz/dx assume arbitrarily prescribed values for the particular value of $x = x_0$, provided that the coefficients p_{ik}, q_{ik}, are holomorphic in the vicinity of $x = x_0$.

Such a system of functions, y and z, is said to constitute a system of general solutions of (1).

Now let (y_i, z_i) for $i = 1, 2, 3, 4$, be four systems of simultaneous solutions of the given system (1), so that

$$\begin{aligned} y_i'' + p_{11}y_i' + p_{12}z_i' + q_{11}y_i + q_{12}z_i &= 0, \\ z_i'' + p_{21}y_i' + p_{22}z_i' + q_{21}y_i + q_{22}z_i &= 0 \end{aligned} \qquad (i=1, 2, 3, 4). \tag{3}$$

Then, denoting by c_1, c_2, c_3, c_4 four arbitrary constants,

$$y = \sum_{i=1}^{4} c_i y_i, \qquad z = \sum_{i=1}^{4} c_i z_i, \tag{4}$$

will also form a simultaneous system of solutions. Moreover, they constitute a system of general solutions, if, and only if, the determinant

$$D = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1 & z_2 & z_3 & z_4 \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix} \tag{5}$$

does not vanish identically. For then, if x_0 is a value of x for which $D \neq 0$, $c_1 \cdots c_4$ can be so determined as to make y, z, dy/dx, dz/dx equal to arbitrarily assigned constants for $x = x_0$.

If (y_i, z_i) are such functions of x which satisfy (1) and do not make D identically zero, we say of them that they constitute a *fundamental system* of simultaneous solutions. Equations (4) then furnish the general integrals of system (1).

We can express the condition

$$D \neq 0$$

* For the history of this subject Mr. FANO's excellent paper in Mathematische Annalen, vol. 53, may be consulted.

in another way. If $D = 0$, it is possible to find four functions of x, λ, μ, ν, ρ, so that the four equations

$$\lambda y_k + \mu y_k' + \nu z_k + \rho z_k' = 0 \qquad (k = 1, 2, 3, 4), \tag{6}$$

may be verified.

If (y_k, z_k) form a fundamental system of simultaneous solutions, it must therefore be impossible to find functions λ, μ, ν, ρ so as to satisfy (6), or what amounts to the same thing, it must be impossible to find functions α, β, γ, δ which satisfy the system of equations

$$\begin{aligned} &ay_1 + \beta y_2 + \gamma y_3 + \delta y_4 = 0, \\ &ay_1' + \beta y_2' + \gamma y_3' + \delta y_4' = 0, \\ &az_1 + \beta z_2 + \gamma z_3 + \delta z_4 = 0, \\ &az_1' + \beta z_2' + \gamma z_3' + \delta z_4' = 0, \end{aligned} \tag{7}$$

which, in particular, prevents $y_1 \cdots y_4$ and $z_1 \cdots z_4$ from verifying simultaneous relations of the form

$$\sum_{i=1}^{4} c_i y_i = 0, \quad \sum_{i=1}^{4} c_i z_i = 0, \tag{8}$$

where $c_1 \cdots c_4$ are constants.

Suppose now that four pairs of functions (y_i, z_i) which verify no relations of the form (6) or (7) are given. Then we can consider them as constituting a fundamental system of simultaneous solutions of a system of differential equations of the form (1). The relations between the coefficients p_{ik}, q_{ik} and the functions (y_k, z_k) follow from (3). They are

$$\begin{aligned} Dp_{11} &= -D(y_k'', z_k', y_k, z_k), \quad Dp_{12} = -D(y_k', y_k'', y_k, z_k), \\ Dp_{21} &= -D(z_k'', z_k', y_k, z_k), \quad Dp_{22} = -D(y_k', z_k'', y_k, z_k), \\ Dq_{11} &= -D(y_k', z_k', y_k'', z_k), \quad Dq_{12} = -D(y_k', z_k', y_k, y_k''), \\ Dq_{21} &= -D(y_k', z_k', z_k'', z_k), \quad Dq_{22} = -D(y_k', z_k', y_k, z_k''), \\ D &= D(y_k', z_k', y_k, z_k), \end{aligned} \tag{9}$$

where we have adopted the notation

$$D(a_k, b_k, c_k, d_k) = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \tag{10}$$

We have moreover

$$p_{11} + p_{22} = -\frac{1}{D}\frac{dD}{dx}, \tag{11}$$

whence

$$D = Ce^{-\int (p_{11}+p_{22})dx}, \tag{12}$$

where C is a non-vanishing constant.

All of these theorems are simple generalizations of well-known facts, but it is necessary for the purposes of this paper to formulate them explicitly.

If we subject the general solutions of our system (1) to a transformation of the form

$$\eta = ay + \beta z, \quad \zeta = \gamma y + \delta z, \tag{13}$$

where α, β, γ, δ are functions of x, then η and ζ will be the general solutions of a system of differential equations of the same form as (1). Moreover, if we put

$$\begin{aligned} \eta_i &= ay_i + \beta z_i, \\ \zeta_i &= \gamma y_i + \delta z_i \end{aligned} \qquad (i = 1, 2, 3, 4), \tag{13a}$$

the four pairs of functions (η_i, ζ_i) will form a definite fundamental system of the new system of differential equations, and its general solutions will be

$$\eta = \sum_{i=1}^{4} c_i \eta_i, \quad \zeta = \sum_{i=1}^{4} c_i \zeta_i.$$

Thus, if we consider instead of a pair of general solutions of system (1), four pairs of solutions which form a fundamental system, these are transformed cogrediently with each other, and with the pair of general solutions.

§ 2. *Geometrical interpretation.*

Let us interpret (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) as the homogeneous coördinates of two points in space. If (1) is integrated, we shall have all of these quantities expressed as functions of x:

$$y_k = f_k(x), \quad z_k = g_k(x) \qquad (k = 1, 2, 3, 4), \tag{14}$$

so that we can say that the system (1) defines two curves in space C_y and C_z, whose points moreover are put into a definite correspondence with one another, namely, those being corresponding points that belong to the same value of x.

But there is a restriction on these curves, owing to the condition that (y_i, z_i) are to be the members of a fundamental system, so that equations (6) must not be verified. Let us write (6) as follows

$$\lambda y_k + \mu y_k' = -(\nu z_k + \rho z_k') \qquad (k = 1, 2, 3, 4). \tag{6a}$$

Now the quantities $\lambda y_k + \mu y_k'$ for each value of x denote the homogeneous coördinates of some point on the tangent to the curve C_y constructed at the point whose parameter is x, and $\nu z_k + \rho z_k'$ denote the coördinates of a point on the

tangent to C_z at the corresponding point of this curve. If equation $(6a)$ is satisfied these two tangents intersect for all pairs of corresponding points.

In order, then, that the curves C_y and C_z may be the integral curves of a system of form (1), *it is necessary and sufficient that the tangents of these curves at all corresponding points do not intersect.*

In particular the curves may be plane curves, but they must not be in the same plane.

What is the geometrical meaning of transformations of form (13)?

Let us mark on the curves C_y and C_z the points P_y and P_z corresponding to the same value of x, and let us join them by a straight line L_{yz}. Then it is clear that the transformations (13), or more properly $(13a)$, convert the points P_y and P_z of the line L_{yz} into two other points P_η and P_ζ of the same line. Moreover, since $\alpha, \beta, \gamma, \delta$ are arbitrary functions of x, it becomes possible to convert P_y and P_z into *any* other two points of this line. The line L_{yz} then is invariant for all such transformations.

If we make this construction for all values of x, we obtain a ruled surface S, whose generators are the straight lines L_{yz}. A change of the independent variable

$$x = f(\xi),$$

where $f(\xi)$ is an arbitrary function, interchanges the generators in the most general way.

Thus, there belongs to every system of two linear homogeneous differential equations of the second order a ruled surface, which we shall call the integrating ruled surface, whose generators are the lines joining corresponding points of the two integral curves. This ruled surface is the same for all such systems which can be transformed into each other by a transformation of the form

$$\begin{aligned} \eta &= \alpha y + \beta z, \\ \zeta &= \gamma y + \delta z, \end{aligned} \qquad \xi = f(x), \tag{15}$$

where $\alpha, \beta, \gamma, \delta, f$ are arbitrary functions of x.

There is one important restriction, however, viz.: *the ruled surface cannot be a developable surface.* For, in the case of a developable surface, and only in that case, would the corresponding tangents of the two integral curves be coplanar, and therefore intersect.

This consideration also teaches us the meaning of the singular values of the independent variable for which

$$D = 0.$$

They are the values of x for which two consecutive generators of the ruled surface intersect, i. e., they correspond to the generators upon which are situated singular points of the surface.

It should be remarked that the ruled surface S will be different for different fundamental systems of solutions of (1), if we regard the tetrahedron of reference as fixed. All of the ruled surfaces belonging to the different possible fundamental systems of the same system of differential equations are obtained from one of them by projective transformation.

By means of equations (13a) we associated with each fundamental system of the original system of equations, a definite fundamental system of the transformed system of equations, and it is only for such associated fundamental systems of the two systems of equations that it is true that they have the same integrating ruled surface. In general the two integrating ruled surfaces will only be projective transformations of each other. If we call two systems of differential equations of form (1) *equivalent*, when they can be transformed into each other by a transformation of the form (15), we can state our theorem more precisely as follows:

If two systems of differential equations of form (1) *are equivalent, their integrating ruled surfaces are projective transformations of each other. Moreover if the fundamental systems of solutions be properly selected, the ruled surfaces coincide.*

Conversely, if the ruled surfaces of two such systems coincide, the systems are equivalent.

If the ruled surface is not of the second order, this converse is clear at once. For the arbitrary functions α, β, γ, δ in the transformation (13) can be chosen so as to convert any pair of curves on the surface, which are not generating straight lines, into any other pair, for instance the pair of curves corresponding to the first system into that corresponding to the second. Moreover this ensures the equivalence of the two systems, if the surface is not of the second order. But if it is of the second order it may be generated by straight lines in two different ways. But even then the theorem holds, since we can transform one set of generators into the other by a projective transformation.

Suppose that any non-developable ruled surface S is given. There corresponds to it a class of mutually equivalent systems of linear differential equations. To find a representative of this class, we trace any two curves, not generating straight lines, on the surface and express the coördinates of their points as functions of a parameter x, in such a way that to the same value of x correspond points of the two curves which are situated upon the same generator of the surface. The system of differential equations whose integral curves are these, parametrically represented in this manner, is the required representative system.

We see further that it is always possible, by a transformation of form (15) to convert any system of differential equations of the kind here considered into another with plane integral curves. We can even take the planes of these curves parallel, and moreover this can be done in an infinity of ways.

If the ruled surface is algebraic, the system can be transformed into one whose integral curves are algebraic space or plane curves.

Any equation or system of equations between p_{ik}, q_{ik}, *etc., which remains invariant for all of the transformations of the form* (15), *expresses a projective property of the integrating ruled surface.* For, it expresses a property common to all possible pairs of curves on this surface, and p_{ik}, q_{ik}, p'_{ik}, $\cdots$ are differential invariants of the general projective group of space. *Conversely any projective property of the integrating ruled surface can be expressed by an invariant system of equations.*

§ 3. *Expression of the integrating ruled surface in line coördinates.*

The surface S is generated by the motion of the line L_{yz} which joins the points $y = (y_1, y_2, y_3, y_4)$ of the curve C_y and $z = (z_1, z_2, z_3, z_4)$ of the curve C_z.

The Plückerian line-coördinates of L_{yz} are the six determinants of the second order in the matrix

$$\begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

Put

$$\omega_{ik} = y_i z_k - y_k z_i \qquad (i, k = 1, 2, 3, 4), \tag{16}$$

then the identical relation will hold

$$\omega_{12}\omega_{34} + \omega_{23}\omega_{14} + \omega_{31}\omega_{24} = 0. \tag{17}$$

Corresponding to the transformations of the infinite group G, we shall have for ω_{ik} the transformations

$$\omega_{ik} = (\alpha\delta - \beta\gamma)\bar{\omega}_{ik}, \quad x = f(\xi).$$

But the six line-coördinates will verify a linear homogeneous differential equation, in general of the 6th order, say

$$P_0\omega^{(6)} + P_1\omega^{(5)} + P_2\omega^{(4)} + P_3\omega^{(3)} + P_4\omega^{(2)} + P_5\omega^{(1)} + P_6\omega = 0. \tag{18}$$

Now the most general transformations which convert such an equation into another of the same form and order are those of the infinite group H,

$$\omega = \phi(x)\bar{\omega}, \quad x = f(\xi),$$

where f and ϕ are arbitrary functions. Such functions of $P_1, \cdots, P_6$, $P'_1, \cdots, P'_6$, etc., which remain unaltered by all transformations of the group H are called invariants of (18).

It is clear, then, that the invariants of the linear differential equation of the sixth order, which the line-coördinates of a generator of the integrating ruled

surface satisfy, are also invariants of the original system of differential equations.

We proceed to set up this linear differential equation of the sixth order in a normal form. We have shown, in a former paper, that every system of form (1) can be reduced to a particular form, which we have called the *semi-canonical* form, for which $p_{ik} = 0$.* Let (y, z) and (η, ζ) be any two simultaneous solutions of a system of differential equations in the semi-canonical form, so that we shall have

$$\begin{aligned} y'' &= -q_{11}y - q_{12}z, & \eta'' &= -q_{11}\eta - q_{12}\zeta, \\ z'' &= -q_{21}y - q_{22}z, & \zeta'' &= -q_{21}\eta - q_{22}\zeta. \end{aligned} \tag{19}$$

Let us put

$$\omega = y\zeta - z\eta = (y\zeta). \tag{20}$$

We wish to find the differential equation satisfied by ω. To do this, let us first notice the relations

$$(\eta y'') = +q_{12}\omega, \quad (\eta z'') = +q_{22}\omega, \quad (\zeta y'') = -q_{11}\omega, \quad (\zeta z'') = -q_{21}\omega, \tag{21}$$

which are obtained from (19). Then, by successive differentiation, and using the abbreviations

$$\begin{aligned} 2v &= \omega'' + (q_{11} + q_{22})\omega, \\ w &= v'' - 2(q_{11}q_{22} - q_{12}q_{21})\omega + (q_{11} + q_{22})v, \\ t &= w' - (q_{11}q_{22}' + q_{22}q_{11}' - q_{12}q_{21}' - q_{21}q_{12}')\omega + (q_{11}' + q_{22}')v, \\ \rho &= t' - (q_{11}q_{22}'' + q_{22}q_{11}'' - q_{12}q_{21}'' - q_{21}q_{12}'')\omega + (q_{11}'' + q_{22}'')v, \end{aligned} \tag{22}$$

we obtain the system of equations

$$\begin{aligned} \omega' &= -(\eta z') + (\zeta y'), \\ v' &= q_{11}(\eta z') + q_{12}(\zeta z') - q_{21}(\eta y') - q_{22}(\zeta y'), \\ w &= q_{11}'(\eta z') + q_{12}'(\zeta z') - q_{21}'(\eta y') - q_{22}'(\zeta y'), \\ t &= q_{11}''(\eta z') + q_{12}''(\zeta z') - q_{21}''(\eta y') - q_{22}''(\zeta y'), \\ \rho &= q_{11}^{(3)}(\eta z') + q_{12}^{(3)}(\zeta z') - q_{21}^{(3)}(\eta y') - q_{22}^{(3)}(\zeta y'). \end{aligned} \tag{23}$$

Eliminating the determinants $(\eta z')$, etc., we find the required differential equation of the 6th order for ω, viz:

$$\begin{vmatrix} \rho, & q_{11}^{(3)}, & q_{12}^{(3)}, & -q_{21}^{(3)}, & -q_{22}^{(3)} \\ t, & q_{11}'', & q_{12}'', & -q_{21}'', & -q_{22}'' \\ w, & q_{11}', & q_{12}', & -q_{21}', & -q_{22}' \\ v', & q_{11}, & q_{12}, & -q_{21}, & -q_{22} \\ \omega', & -1, & 0, & 0, & +1 \end{vmatrix} = 0. \tag{24}$$

*Transactions of the American Mathematical Society, vol. 2 (1901), no. 1, p. 19.

This differential equation reduces to the 5th order if the minor of ρ in this determinant is zero, i. e., it

$$\begin{vmatrix} q''_{11} - q''_{22}, & q''_{12}, & q''_{21} \\ q'_{11} - q'_{22}, & q'_{12}, & q'_{21} \\ q_{11} - q_{22}, & q_{12}, & q_{21} \end{vmatrix} = 0,$$

or, when the equation is not written in the semi-canonical form, if

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \\ w_{11} - w_{22}, & w_{12}, & w_{21} \end{vmatrix} = 0. \tag{25}$$

It is clear from this, that Δ must be an invariant, obviously of weight nine. This may also be verified analytically. Δ, being isobaric of weight 9, cannot be rationally expressed as a function of the seminvariants involved in it, for it can be shown that no such rational invariant of weight 9 exists. But there is a simple syzygy between Δ and the invariant of weight 18.* By expressing both of these quantities in terms of u_{ik}, v_{ik}, w_{ik}, it will be found that

$$\Delta^2 + 4\theta_{18} = 0. \tag{26}$$

We have seen that the differential equation for ω reduces to the 5th order if and only if $\Delta = 0$.

Therefore, the equation

$$\Delta = 0 \tag{27}$$

means, that the generators of the integrating ruled surface belong to a linear complex; for, the six line coördinates ω_{ik} must, in this case, satisfy a homogeneous linear relation with constant coefficients. The condition $\Delta = 0$ is necessary and sufficient.

If the linear complex, to which the generators of the ruled surface belong, is special, additional relations must be fulfilled. For any two linear complexes which are not special can be transformed into each other by a projective transformation, while a non-special complex can never be so transformed into a special complex.

A special complex consists of all lines which intersect a given straight line, called the axis of the complex. If

$$\omega_{12}a_{34} + \omega_{23}a_{14} + \omega_{31}a_{24} + \omega_{14}a_{23} + \omega_{24}a_{31} + \omega_{34}a_{12} = 0$$

is its equation, then a_{ik} are the line-coördinates of its axis, and

$$a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24} = 0.$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 18.

Every ruled surface contained in such a complex, therefore has upon it a straight line which is not a generator of the surface, namely the axis of the complex. Take this as our fundamental curve C_y, and also as one of the edges of the tetrahedron of reference.

Then we can put

$$y_3 = y_4 = 0 \tag{28}$$

while $y_1, y_2; z_1, z_2, z_3, z_4$ may be arbitrary functions of x. Since $y_3 = y_4 = 0$ we can take as the first differential equation of our system the linear differential equation of the second order of which y_1, y_2 form a fundamental system, so that

$$p_{12} = q_{12} = 0,$$

whence follows

$$u_{12} = v_{12} = w_{12} = 0, \tag{29}$$

which makes

$$\theta_{10} = \theta_{15} = \theta_{18} = 0, \tag{30}$$

while θ_4 and θ_6 do not vanish, unless there is a further specialization. Since, moreover, in this case

$$\theta_4 = (u_{11} - u_{22})^2,$$

we may assume

$$u_{11} - u_{22} \neq 0. \tag{29a}$$

The conditions (29) and (29a) are sufficient to insure that the integrating ruled surface shall be generated by lines belonging to a special linear complex, for from them follows $p_{12} = q_{12} = 0$. But they are not necessary. The necessary and sufficient conditions must form an invariant system of equations, which the conditions here found do not do. The conditions (30) on the other hand are necessary, as they form an invariant system, but they are not sufficient.

To find conditions for this case, which are both necessary and sufficient we proceed as follows. If the ruled surface belongs to a special linear complex, it must be possible to transform the given system, by a transformation of the form

$$y = a_{11}\eta + a_{12}\zeta,$$

$$z = a_{21}\eta + a_{22}\zeta,$$

into another, for which $\pi_{12} = \rho_{12} = 0$, if the coefficients of the transformed system be denoted by Greek letters.

But the conditions $\pi_{12} = \rho_{12} = 0$, give

$$2(a'_{12}a_{22} - a'_{22}a_{12}) + p_{11}a_{12}a_{22} + p_{12}a_{22}^2 - p_{21}a_{12}^2 - p_{22}a_{22}a_{12} = 0, \tag{31}$$

and

$$a''_{12}a_{22} - a''_{22}a_{12} + p_{11}a'_{12}a_{22} + p_{12}a'_{22}a_{22} - p_{21}a'_{12}a_{12} - p_{22}a'_{22}a_{12} + q_{11}a_{12}a_{22} + q_{12}a_{22}^2 - q_{21}a_{12}^2 - q_{22}a_{22}a_{12} = 0. \tag{32}$$

Differentiating the left member of (31) and subtracting from the resulting expression twice the left member of (32), we find

$$(33)\quad \begin{aligned}(a_{12}a'_{22} - a'_{12}a_{22})(p_{11}+p_{22}) + a_{12}a_{22}[p'_{11} - 2q_{11} - (p'_{22} - 2q_{22})] \\ + a_{22}^2(p'_{12} - 2q_{12}) - a_{12}^2(p'_{21} - 2q_{21}) = 0\,.\end{aligned}$$

Multiplying (31) by $-(p_{11}+p_{22})$ and (33) by 2, and adding we find

$$(34)\quad a_{12}a_{22}(u_{11} - u_{22}) + a_{22}^2u_{12} - a_{12}^2u_{21} = 0\,,$$

where the quantities u_{ik} are the same as those so denoted in our previous paper.

Dividing (34) by a_{22}^2, we have

$$\frac{a_{12}}{a_{22}}(u_{11} - u_{22}) + u_{12} - \left(\frac{a_{12}}{a_{22}}\right)^2 u_{21} = 0\,,$$

whence by differentiation

$$(35)\quad \left(u_{11} - u_{22} - 2\,\frac{a_{12}}{a_{22}}u_{21}\right)\frac{d}{dx}\frac{a_{12}}{a_{22}} = -u'_{12} - \frac{a_{12}}{a_{22}}(u'_{11} - u'_{22}) + \left(\frac{a_{12}}{a_{22}}\right)^2 u'_{21}.$$

But from (31) we have

$$2\,\frac{d}{dx}\frac{a_{12}}{a_{22}} = -\frac{a_{12}}{a_{22}}(p_{11} - p_{22}) - p_{12} + \left(\frac{a_{12}}{a_{22}}\right)^2 p_{21},$$

which gives on substitution in (35), after simplifying and clearing of fractions,

$$(36)\quad a_{12}a_{22}(v_{11} - v_{22}) + a_{22}^2v_{12} - a_{12}^2v_{21} = 0\,.$$

Differentiating (36), we get by a repetition of this process

$$(37)\quad a_{12}a_{22}(w_{11} - w_{22}) + a_{22}^2w_{12} - a_{12}^2w_{21} = 0\,.$$

We have then three equations (34), (36) and (37) for

$$a = \frac{a_{12}}{a_{22}}.$$

It is at once seen that Δ must vanish, and if we put

$$(38)\quad \begin{aligned} &2(v_{12}w_{21} - v_{21}w_{12}) = U, \quad 2(w_{12}u_{21} - w_{21}u_{12}) = V, \quad 2(u_{12}v_{21} - u_{21}v_{12}) = W, \\ &vw_{12} - wv_{12} = U_{12}, \qquad wu_{12} - uw_{12} = V_{12}, \qquad uv_{12} - vu_{12} = W_{12}, \\ &vw_{21} - wv_{21} = U_{21}, \qquad wu_{21} - uw_{21} = V_{21}, \qquad uv_{21} - vu_{21} = W_{21}, \\ &u_{11} - u_{22} = u\,, \quad v_{11} - v_{22} = v\,, \quad w_{11} - w_{22} = w\,, \end{aligned}$$

we obtain

$$(39)\quad \begin{aligned} a &= -\frac{U}{2U_{21}} = -\frac{V}{2V_{21}} = -\frac{W}{2W_{21}}, \\ a^2 &= \frac{U_{12}}{U_{21}} = \frac{V_{12}}{V_{21}} = \frac{W_{12}}{W_{21}}, \end{aligned}$$

whence

$$
\begin{aligned}
&VW_{21} - WV_{21} \quad = WU_{21} - UW_{21} \quad = UV_{21} - VU_{21} = 0\,, \\
&V_{12}W_{21} - W_{12}V_{21} = W_{12}U_{21} - U_{12}W_{21} = U_{12}V_{21} - V_{12}U_{21} = 0\,, \\
&U^2 - 4U_{12}U_{21} \quad = V^2 - 4V_{12}V_{21} \quad = W^2 - 4W_{12}W_{21} = 0\,.
\end{aligned} \tag{40}
$$

It will be noted that U, V, W, etc., are the nine minors of the second order of the determinant Δ, and that *all of the minors of the second order of their determinant*

$$
\begin{vmatrix} U & U_{12} & U_{21} \\ V & V_{12} & V_{21} \\ W & W_{12} & W_{21} \end{vmatrix} \tag{41}
$$

vanish in consequence of the relations (40). Moreover the quantities U, U_{12}, U_{21}; V, V_{12}, V_{21}; W, W_{12}, W_{21}; are all cogredient with each other and u, u_{12}, u_{21}, etc., for transformations of the dependent variables, and we can write

$$
\theta_{10} = \tfrac{1}{4}W^2 - W_{12}W_{21}\,. \tag{42}
$$

The conditions (40) which are necessary for the case under consideration are also sufficient. For, let us suppose these conditions fulfilled for a system of differential equations. Then let us transform this system as before, putting now

$$
a_{12} = -\tfrac{1}{2}W, \quad a_{22} = W_{21}\,. \tag{43}
$$

We find by actual computation that the left member of (31), which is equal to π_{12}, vanishes. Denoting this left member, in general, by π_{12}, and the left member of (32) by ρ_{12}, the left member of (34) will be

$$
2\pi'_{12} - 4\rho_{12} - (p_{11} + p_{22})\pi_{12}\,.
$$

But this vanishes under the given conditions, and since $\pi_{12} = 0$, ρ_{12} also will vanish. Therefore the first equation of the transformed system will actually have the form

$$
\frac{d^2\eta}{dx^2} + \pi_{11}\frac{d\eta}{dx} + \rho_{11}\eta = 0\,, \tag{44}
$$

which proves that the ruled surface in this case belongs to a special linear complex. This proof is insufficient only if $W = W_{21} = 0$.

If $W = W_{21} = 0$, we have either $u_{21} = v_{21} = 0$, or else also $W_{12} = 0$. In the first case, if $u_{21} = v_{21} = 0$, and $u_{11} - u_{22} \neq 0$ we find from the equations defining these quantities that p_{21} and q_{21} must vanish, which means that in this case no transformation is necessary, one of the original equations already being of form (44). If however, in addition to $u_{21} = v_{21} = 0$, we have $u = 0$, we find from (40) that either $W_{12} = 0$, which comes under our second case, or else U, V, U_{21} and

V_{21} must all vanish. But this again gives either $u = 0$, $u_{12} = 0$, $u_{21} = 0$, which we shall see later makes the ruled surface of the second order, or else $w_{21} = 0$. This again gives rise to two sub-cases. Either $p_{21} = q_{21} = 0$ as in a former case, or $v = 0$ and therefore $u_{12} = 0$, which is again the case of a surface of the second order. In all of these cases the ruled surface belongs to a special linear complex.

In the second case mentioned above, we have $W = W_{12} = W_{21} = 0$, or

$$u_{12}v_{21} - u_{21}v_{12} = 0, \quad uv_{12} - vu_{12} = 0, \quad uv_{21} - vu_{21} = 0,$$

whence, denoting by ρ a proportionality factor,

$$v = \rho u, \quad v_{12} = \rho u_{12}, \quad v_{21} = \rho u_{21}, \tag{45}$$

unless $u = u_{12} = u_{21} = 0$, which is the case of a surface of the second degree.

Using the equations (32) of our former paper,* which defined v_{ik} we find from (45)

$$\begin{aligned} \rho u &= 2u' + 2(p_{12}u_{21} - p_{21}u_{12}), \\ \rho u_{12} &= 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}u, \\ \rho u_{21} &= 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}u, \end{aligned}$$

and similarly, from the equations (39) in our former paper,

$$w = (\rho^2 + 2\rho')u, \quad w_{12} = (\rho^2 + 2\rho')u_{12}, \quad w_{21} = (\rho^2 + 2\rho')u_{21}, \tag{46}$$

but these expressions of v_{ik}, w_{ik} show that, in this case, not only

$$W = W_{12} = W_{21} = 0,$$

but also

$$U = U_{12} = U_{21} = V = V_{12} = V_{21} = 0,$$

i. e., all of the minors of the second order of Δ vanish. In this case also, the ruled surface belongs to a special linear complex although our method of reduction fails. That this is so will, however, be apparent from what follows.

We have seen, then, that the equations (40) *are the necessary and sufficient conditions for an integrating ruled surface belonging to a special linear complex.* Moreover we have seen how the axis of this complex, i. e., the straight line on the surface, which is not one of the generators, may be found, for this is the geometrical meaning of the transformation which we have just completed. The cases in which this transformation fails are those in which more than one such straight line exists on the surface.

If the generators of the ruled surface belong to two different linear complexes,

$$\begin{aligned} \Omega_1 &= \omega_{12}a_{34} + \omega_{23}a_{14} + \omega_{31}a_{24} + \omega_{14}a_{23} + \omega_{24}a_{31} + \omega_{34}a_{12} = 0, \\ \Omega_2 &= \omega_{12}b_{34} + \omega_{23}b_{14} + \omega_{31}b_{24} + \omega_{14}b_{23} + \omega_{24}b_{31} + \omega_{34}b_{12} = 0, \end{aligned} \tag{47}$$

*Transactions of the American Mathematical Society, vol. 2, no. 1, p. 8.

or if they are lines of what PLÜCKER calls a linear congruence, they belong to every one of the ∞^1 complexes

$$\Omega_1 + \lambda\Omega_2 = 0. \tag{48}$$

Now, among these there are in general two, which are special. For (48) represents a special linear complex, if and only if,

$$B\lambda^2 + C\lambda + A = 0, \tag{49}$$

where we have put

$$\begin{aligned} A &= a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24}, \\ B &= b_{12}b_{34} + b_{23}b_{14} + b_{31}b_{24}, \\ C &= a_{12}b_{34} + a_{23}b_{14} + a_{31}b_{24} + a_{34}b_{12} + a_{14}b_{23} + a_{24}b_{31}. \end{aligned} \tag{50}$$

But (49) has two distinct and determinate roots if

$$C^2 - 4AB \neq 0.$$

In this case we can assume that the complexes (47) are special, so that $A = B = 0$, and consequently $C \neq 0$. The congruence consists of all of the lines intersecting two distinct, non-intersecting straight lines, the directrices of the congruence.

If $C^2 - 4AB = 0$, we have a congruence with coincident directrices.

A sub-case of this,

$$A = B = C = 0,$$

must be excluded from our investigation. For in this case the directrices of the congruence intersect, so that the congruence consists of all straight lines intersecting a pair of intersecting straight lines. In other words, all of the lines of the congruence are situated either in the plane of the directrices, or else pass through their point of intersection. Any ruled surface contained in such a congruence is either a plane or a cone, and therefore a developable surface. But a developable surface cannot be the integrating ruled surface of our system of differential equations.

It will be easily seen that the necessary and sufficient conditions, which must be fulfilled in order that the generators of the integrating ruled surface may belong to a linear congruence, are that all of the minors of the second order of Δ *must vanish.*

In this case then

$$U = U_{12} = U_{21} = V = V_{12} = V_{21} = W = W_{12} = W_{21} = 0, \tag{51}$$

and as a consequence,

$$\theta_{10} = \theta_{15} = \theta_{18} = 0.$$

In the case of a congruence with coincident directrices additional conditions must be fulfilled. Such a congruence may be described in a number of ways. It consists of all of the lines of a complex which intersect a given line of the complex. Or, we may consider the line with which the two coincident directrices of the congruence coincide, as a generator of a surface of the second order. All of the tangents to this surface, passing through the given generator, constitute the lines of the congruence. Finally, for actual use, we can define such a congruence in the following manner. Take rectangular axes x, y, z in space. Through a point P on the x axis, and the origin O pass a plane, whose projection on the yz plane makes an angle λ with the z axis, where

$$\tan \lambda = \frac{k}{x},$$

k being a constant, and x the distance of P from the origin. Then the lines of this plane, which pass through P, are the lines of the congruence.*

Let us take the x axis, which must be a line on our surface, intersected by all of the generators, as the fundamental curve C_y, and the abscissa $OP = x$ of the point P as the independent variable of our system of differential equations. Moreover, suppose that we have chosen our homogeneous coördinates so that the ratios

$$\frac{y_1}{y_4}, \quad \frac{y_2}{y_4}, \quad \frac{y_3}{y_4}; \qquad \frac{z_1}{z_4}, \quad \frac{z_2}{z_4}, \quad \frac{z_3}{z_4};$$

are the cartesian coördinates of the points (y) and (z) respectively. Then we can put

$$y_1 = x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1. \tag{52}$$

Moreover the equation of the plane mentioned above, will be

$$y = z \tan \lambda,$$

whence follows the single relation between z_2 and z_3,

$$z_2 = z_3 \tan \lambda = z_3 \frac{k}{x},$$

or

$$xz_2 - kz_3 = 0. \tag{53}$$

There exists then, in this case, a system of differential equations, equivalent to the given system, for which

$$\begin{aligned} &y_1 = x, \quad y_2 = 0, \quad y_3 = 0, \quad y_4 = 1, \\ &z_1 = z_1, \quad z_2 = z_2, \quad z_3 = \frac{x}{k} z_2, \quad z_4 = 1, \end{aligned} \tag{54}$$

* PLÜCKER, *Neue Geometrie des Raumes*, pp. 57 and 73.

where z_1 and z_2 are arbitrary functions of x.

Substituting these values in equations (9), we find

$$
\begin{aligned}
&p_{11} = p_{12} = q_{11} = q_{12} = 0, \\
&p_{21} = -\frac{1}{z_2^2}\left[z_1''z_2^2 - 2z_2z_1'z_2' - (x - z_1)\{2(z_2')^2 - z_2z_2''\}\right], \\
&p_{22} = -2\frac{z_2'}{z_2}, \\
&q_{21} = -\frac{1}{z_2^2}\left[2(z_2')^2 - z_2z_2''\right], \quad q_{22} = +\frac{1}{z_2^2}\left[2(z_2')^2 - z_2z_2''\right],
\end{aligned}
\tag{55}
$$

whence we derive the relations

$$
\begin{aligned}
&2p_{22}' - 4q_{22} + p_{22}^2 = 0, \\
&q_{21} + q_{22} = 0, \\
&p_{11} = p_{12} = q_{11} = q_{12} = 0.
\end{aligned}
\tag{56}
$$

Therefore

$$
u_{11} = u_{12} = u_{22} = 0, \quad v_{11} = v_{12} = v_{22} = 0, \quad w_{11} = w_{12} = w_{22} = 0. \tag{57}
$$

Substituting these values, it will be seen that *all of the invariants vanish.*

The conditions (57) are sufficient to make the ruled surface belong to a congruence with coincident directrices. For, if we assume in the first place the additional relation $u_{21} = 0$, the surface will be of the second order, as we shall show very soon, and therefore it will be a surface of the kind here considered. If $u_{21} \neq 0$, we find from the conditions $u_{11} = u_{12} = v_{11} = 0$, that p_{12} must vanish. From $u_{12} = 0$ then follows $q_{12} = 0$, and from $u_{11} = u_{22} = 0$,

$$
2p_{11}' - 4q_{11} + p_{11}^2 = 2p_{22}' - 4q_{22} + p_{22}^2 = 0. \tag{58}
$$

Our system therefore has the form

$$
\begin{aligned}
&y'' + p_{11}y' + q_{11}y = 0, \\
&z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0,
\end{aligned}
$$

with the relations (58). If we now transform, by putting

$$
y = e^{-\frac{1}{2}\int p_{11}dx}\eta, \qquad z = e^{-\frac{1}{2}\int p_{22}dx}\zeta,
$$

the system will become, owing to the relations (58),

$$
\begin{aligned}
&\eta'' = 0, \\
&\zeta'' + \pi\eta' + \kappa\eta = 0,
\end{aligned}
$$

where π and κ are functions of x subject to no condition.

But this system has the fundamental system of solutions

$$\eta_1 = x, \qquad \eta_2 = 0, \quad \eta_3 = 0, \quad \eta_4 = 1,$$
$$\zeta_1 = -\int\int (\pi + \kappa x)\, dx^2, \quad \zeta_2 = k, \quad \zeta_3 = x, \quad \zeta_4 = -\int\int \kappa dx^2,$$

where k is a constant, which must be different from zero, for we find that

$$D(\eta_k', \zeta_k', \eta_k, \zeta_k) = k.$$

Making another transformation, by putting

$$\bar{\zeta} = \frac{\zeta}{-\int\int \kappa dx^2},$$

it will be seen that η_1, η_2, η_3, η_4 and $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\zeta}_3$, $\bar{\zeta}_4$ form a fundamental system of solutions of a system equivalent to the given system, and of the form (54). This proves that any system of differential equations, for which the relations (57) are satisfied, has a ruled surface which belongs to a linear congruence with coincident directrices.

But the sufficient conditions (57) are not necessary. The following conditions

$$I^2 - 4J = 0, \quad K - I'^2 = 0, \quad L - 4I''^2 = 0, \tag{59}$$

are both necessary and sufficient. For, in the first place it may be verified that they form an invariant system of equations.

In the second place, any system of differential equations of the kind here considered can be so transformed that the relations (57) shall be satisfied. But then the relations (59) will also be satisfied, and moreover, since these relations form an invariant system of equations, they must have been satisfied for the untransformed system of differential equations. The equations (59) are then necessary conditions. But they are also sufficient. For, suppose that they are satisfied. We can then assume

$$u_{11} - u_{22} = 0, \quad u_{12}u_{21} = 0.$$

For, we can, by a transformation of the form $y = \lambda\eta$, $z = \mu\zeta$, make u_{11} and u_{22} equal to any functions of x whatever, and therefore it may be so chosen as to make $u_{11} = 0$ and $u_{22} = 0$.

Moreover, since then $u_{12}u_{21} = 0$, either u_{12} or u_{21} must vanish. Suppose $u_{21} \neq 0$; then $u_{12} = 0$, and $v_{12} = 0$, while

$$v_{11} - v_{22} = 2p_{12}u_{21}$$

must vanish, on account of (59). But since $u_{21} \neq 0$, we have $p_{12} = 0$, whence $v_{11} = v_{22} = 0$, and also $w_{11} = w_{22} = w_{12} = 0$, or, taken together, the sufficient conditions (57). If $u_{21} = 0$, together with $u_{11} = u_{22} = u_{12} = 0$, we have the case of a ruled surface of the second order, which also belongs to a linear congruence with

coincident directrices. Had we assumed $u_{12} \neq 0$, instead of $u_{21} \neq 0$, we should have found the conditions

$$u_{11} = u_{21} = u_{22} = 0, \quad v_{11} = v_{21} = v_{22} = 0, \quad w_{11} = w_{21} = w_{22} = 0,$$

which are the same as (57) except that the order of the two differential equations is interchanged, i. e., except for the notation.

The necessary and sufficient conditions for a ruled surface belonging to a linear congruence with coincident directrices, are therefore

$$\begin{aligned} &(u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0, \\ &(v_{11} - v_{22})^2 + 4v_{12}v_{21} = 0, \\ &(w_{11} - w_{22})^2 + 4w_{12}w_{21} = 0, \end{aligned} \tag{60}$$

for these are equivalent to (59).

If the ruled surface is made up of the lines common to three distinct linear complexes, it is a surface of the second order (not a cone), and all of the elements of the determinant Δ must vanish.

Therefore the necessary and sufficient conditions for an integrating ruled surface of the second order are

$$u_{11} - u_{22} = u_{12} = u_{21} = 0. \tag{61}$$

This is the case treated in our former paper in § 4. It is possible to so transform the system that the new dependent variables η and ζ satisfy the same linear differential equation of the second order

$$\eta'' + p\eta' + q\eta = 0;$$

the geometrical sense of this reduction is that any two generators of the second kind are taken as the fundamental curves C_y and C_z. The differential equations for y and z are then the same because the point-rows cut out of any two generators of the second kind by a moving generator of the surface are projective.

This reduction can be made analytically by reducing to the semi-canonical form.

§ 4. *Geometrical interpretation of the semi-canonical form of the system of differential equations.*

We have shown in our former paper, that every system of differential equations of the form here considered, can be reduced to the form

$$\begin{aligned} &y'' + \rho_{11}y + \rho_{12}z = 0, \\ &z'' + \rho_{21}y + \rho_{22}z = 0, \end{aligned} \tag{62}$$

which we have called the semi-canonical form. We have also determined the

most general subgroup of the group (2), which leaves this semi-canonical form invariant. Its finite equations are,

$$(63) \qquad \xi = \xi(x), \quad \eta = \sqrt{\frac{d\xi}{dx}}\,(C_{11}y + C_{12}z), \quad \zeta = \sqrt{\frac{d\xi}{dx}}\,(C_{21}y + C_{22}z),$$

where C_{ik} are constants.

Now, of course, equations (62) define a pair of curves on the integrating ruled surface. This is transformed by equations (63) into another pair, whose defining system of differential equations also has the semi-canonical form.

Since (63) is the most general transformation which leaves the semi-canonical form invariant, *we see that there exists upon the integrating ruled surface a single infinity of curves, whose defining systems of differential equations have the semi-canonical form. It is also clear from equations* (63) *that a moving generator of the ruled surface intersects any four of these curves in a row of four points whose anharmonic ratio is constant.*

On any ruled surface the generating straight lines constitute one set of asymptotic lines. *We shall now prove that the curves just mentioned constitute the other set of asymptotic lines.*

Let the homogeneous coördinates be so chosen that

$$\frac{y_1}{y_4}, \frac{y_2}{y_4}, \frac{y_3}{y_4}; \quad \frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4};$$

are the cartesian coördinates of the points describing the curves C_y and C_z respectively. Then if X, Y, Z denote the coördinates of any point in the osculating plane of the curve C_y at the point (y_1, y_2, y_3, y_4), the equation of this plane is

$$\begin{vmatrix} X - \frac{y_1}{y_4}, & Y - \frac{y_2}{y_4}, & Z - \frac{y_3}{y_4} \\ \frac{d}{dx}\frac{y_1}{y_4}, & \frac{d}{dx}\frac{y_2}{y_4}, & \frac{d}{dx}\frac{y_3}{y_4} \\ \frac{d^2}{dx^2}\frac{y_1}{y_4}, & \frac{d^2}{dx^2}\frac{y_2}{y_4}, & \frac{d^2}{dx^2}\frac{y_3}{y_4} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} Xy_4 - y_1, & Yy_4 - y_2, & Zy_4 - y_3 \\ y_1y_4' - y_4y_1', & y_2y_4' - y_4y_2', & y_3y_4' - y_4y_3' \\ y_1y_4'' - y_4y_1'', & y_2y_4'' - y_4y_2'', & y_3y_4'' - y_4y_3'' \end{vmatrix} = 0.$$

But since y_i, z_i are solutions of (62), we have

$$\begin{aligned} y_i'' &= -\rho_{11}y_i - \rho_{12}z_i, \\ z_i'' &= -\rho_{21}y_i - \rho_{22}z_i \end{aligned} \qquad (i = 1, 2, 3, 4),$$

and, therefore,

$$y_k y_i'' - y_i y_k'' = -\rho_{12}(z_i y_k - y_i z_k),$$

which shows that the osculating plane of C_y at the point (y_1, y_2, y_3, y_4) also passes through (z_1, z_2, z_3, z_4), i. e., it coincides with the plane tangent to the surface at the point (y_1, y_2, y_3, y_4). This proves that C_y is an asymptotic curve of the surface.

Our remarks furnish a new proof of PAUL SERRET'S theorem: *the double ratio of the four points, in which any generator intersects four fixed asymptotic curves of the second kind, is constant.*

We shall leave the geometrical interpretation of the canonical form for a future occasion, when we shall also treat some other important problems not touched upon in this paper. In particular, it will be interesting to introduce the idea of duality into this theory.

UNIVERSITY OF CALIFORNIA,
BERKELEY, *March* 30, 1901.

THEORY OF LINEAR GROUPS IN AN ARBITRARY FIELD*

BY

LEONARD EUGENE DICKSON

§ 1. *Introduction.*

Various branches of group theory may be correlated by a treatment of groups of transformations in a given field or domain of rationality. In view of the simplicity of their treatment and of their importance as well in applications as in the general theory,† groups of linear transformations offer a natural starting place in the construction of a theory of groups in a given domain of rationality.

The chief result of the present paper is the exhibition of four infinite systems of groups of transformations which are simple groups in every domain of rationality. For the case of the field of all complex numbers these groups are the simple continuous groups of LIE. By the well known investigations of KILLING and CARTAN, the latter groups give the only systems of simple continuous groups of a finite number of parameters.

As in the theories of algebraic and differential equations, so also in the theory of groups of transformations, it is of first importance that the definitions, conceptions and developments shall have reference to a given field or domain of rationality. For example, it is important to have a theory of continuous groups in the field of complex numbers and a theory in the field of real numbers. Two real continuous groups may not be isomorphic, although the corresponding complex groups are isomorphic.‡ If we allow complex transformations to canonical types, there results a complete list of real groups; allowing only reductions by real transformations, the list is often more extensive.

---

* Presented to the Society at the Ithaca Meeting, August 20, 1901. Received for publication June 7, 1901.

† For example, a continuous group G with a finite number of parameters is simply or multiply isomorphic with a linear homogeneous group called the adjoint group of G. If G be simple, it may be exhibited as a linear fractional (projective) group. From the fundamental rôle played by the adjoint group and by the simple groups, the theory of linear groups is of capital importance in LIE's theory of continuous groups.

‡ DICKSON: Bulletin of the American Mathematical Society, (2) vol. 7 (1901), p. 340; SLOCUM: Proceedings of the American Academy of Arts and Sciences, vol. 36 (1900), p. 105.

The chief results in the theory of linear groups in a finite field are presented in the author's treatise on Linear Groups, * to which reference will here be made by the initials L. G.

In certain questions cencerning continuous groups, as that of the structure of a mixed group, the methods here presented often give additional knowledge, not obtained by following LIE'S method.

In §§ 9–10 is investigated a group in an arbitrary field which corresponds to the simple continuous group of 14 parameters, an isolated group not in the four systems of LIE. For the case of a finite field of order p^n, we are led to a new simple group of order $p^{6n}(p^{6n}-1)(p^{2n}-1)$.

§ 2. *Definition of fields and groups.*

A set of operators forms a *group* if the following properties hold:

(a) The product (compound) of any two operators of the set is itself an operator of the set.

(b) The composition of operators is associative: if A, B, C are any operators of the set, then $(AB)C = A(BC)$.

(c) To every operator A of the set corresponds an operator A_1 of the set such that $AA_1 = A_1A = I$, where I is the operator identity, which leaves unaltered all possible operands. This A_1 is called the *inverse* of A and is designated A^{-1}.

A set of elements forms a *field* † if they can be combined by addition, subtraction, multiplication and division, the divisor not being the element zero (necessarily in the set), these operations being subject to laws of elementary algebra, and if the resulting sum, difference, product or quotient be uniquely determined as an element of the set.

A field may therefore be characterized by the property that *the rational operations of algebra can be performed within the field.*

As examples of fields may be noted the finite fields, ‡ the field R of all rational numbers, the field $R(i)$ of numbers $a + bi$, where a and b are rational, the field of all real numbers, the field C of all complex numbers, the field $R(\theta)$ of all rational functions of the algebraic number θ, a root of an equation belonging to and irreducible in the field R.

§ 3. *General linear homogeneous and linear fractional groups.*

Let $\xi_1, \xi_2, \cdots, \xi_m$ be arbitrary variables. Consider the linear homogeneous transformation

$$A: \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, 2, \cdots, m)$$

* B. G. Teubner, Leipsic, 1901.

† Domain of rationality or Körper. See WEBER'S *Algebra*.

‡ Each is necessarily a Galois field of order a power of a prime (MOORE).

with coefficients a_{ij} in a given field F such that the determinant

$$|A| \equiv |a_{ij}| \neq 0 \qquad (i, j = 1, 2, \cdots, m).$$

Such a transformation A will be said to belong to the field F. Consider a second transformation belonging to F,

$$A': \qquad \xi'_i = \sum_{j=1}^{m} a'_{ij}\xi_j \qquad (i = 1, 2, \cdots, m).$$

By the compound, or product, of A and A' we mean the transformation

$$A'': \qquad \xi'_i = \sum_{j=1}^{m} a''_{ij}\xi_j \qquad (i = 1, 2, \cdots, m),$$

where

$$a''_{ij} \equiv \sum_{k=1}^{m} a'_{ik}a_{kj} \qquad (i, j = 1, 2, \cdots, m).$$

We have here compounded the transformations in the order A, A'; the relation between A, A', A'' is written

$$AA' = A''.$$

By the theorem for the multiplication of determinants,

$$|A''| \equiv |a''_{ij}| = |a'_{ik}| \cdot |a_{kj}| = |A'| \cdot |A| \neq 0.$$

Since a_{ij} and a'_{ij} belong to the field F by hypothesis, the coefficients a''_{ij} also belong to F. Hence the product A'' is a transformation belonging to the field F. The transformation

$$A^{-1}: \qquad \xi'_i = \sum_{j=1}^{m} \frac{A_{ji}}{|A|}\xi_j \qquad (i = 1, 2, \cdots, m),$$

where A_{ji} is the adjoint (first minor with proper sign) of a_{ji} in $|a_{ji}|$, has its coefficients in the field F and has the determinant $|A|^{-1}$. The product AA^{-1} is the identity I; indeed, it replaces ξ_i by the function *

$$\sum_{j,k}^{1,\cdots,m} \frac{A_{ji}a_{jk}}{|A|}\xi_k = \sum_{k=1}^{m} \frac{\delta_{ik}|A|}{|A|}\xi_k = \xi_i.$$

Hence the inverse A^{-1} of A is a transformation in the field F. Moreover, the transformations of the form A are seen to obey the associative law [§ 2, property (b)]. It follows that the totality of transformations A constitutes a group. It will be called *the general linear homogeneous group on m variables with coefficients in the field F* and denoted by the symbol † $GLH(m, F)$.

* In KRONECKER's notation, $\delta_{ii} = 1$, $\delta_{ik} = 0\,(k \neq i)$.

† A finite field is uniquely defined by its order, necessarily a power of a prime number, p^n (MOORE). The corresponding group is $GLH(m, p^n)$. [See L. G., §§ 97–98.]

The group $GLH(m, F)$ is generated by the transformations

$$D_{1,\lambda}: \quad \xi_1' = \lambda\xi_1, \quad \xi_i' = \xi_i \qquad (i = 2, 3, \cdots, m),$$

$$B_{r,s,\lambda}: \quad \xi_r' = \xi_r + \lambda\xi_s, \quad \xi_i' = \xi_i \qquad (i = 1, \cdots, m;\ i \neq r, r \neq s),$$

where λ is an arbitrary quantity $\neq 0$ in the field F.

The proof is identical with that in L. G., § 100. The proof shows that any transformation A of the group can be expressed uniquely as a product $A_1 D_{1,\lambda}$, where $\lambda = |A|$ and where A_1 is derived from the transformations $B_{r,s,k}$, all of which have determinant unity.

The totality of transformations A of determinant unity forms a group Γ called *the special linear homogeneous group* $SLH(m, F)$. It is generated by the transformations $B_{r,s,\lambda}$.

The product $B^{-1}AB$ is called *the transform of A by B*. Since

$$|B^{-1}AB| = |B^{-1}| \cdot |A| \cdot |B| = |B|^{-1} \cdot |A| \cdot |B| = |A|,$$

the transform of A has the same determinant as A.

A subgroup Γ of G is called *invariant* (*self-conjugate*) *under* G if the transform of each transformation of Γ by an arbitrary transformation of G belongs to Γ, i. e., symbolically, if $g^{-1}\gamma g = \gamma'$.

The group $SLH(m, F)$ is an invariant subgroup of $GLH(m, F)$.

By making A_1 correspond to $A \equiv A_1 D_{1,|A|}$, we establish an isomorphism of $SLH(m, F)$ with $GLH(m, F)$. The identity corresponds to the (commutative) group of the transformations $D_{1,\lambda}$. The latter is called the *quotient-group* of the general by the special linear group.

The special linear homogeneous group Γ contains an invariant subgroup H composed of the transformations

$$M_\mu: \quad \xi_i' = \mu\xi_i \qquad [\mu^m = 1] \qquad (i = 1, \cdots, m).$$

Let J be an invariant subgroup of Γ which contains all the transformations M_μ and still other transformations. By the proof in L. G., § 104, interpreted for an infinite field F, it follows that $J \equiv \Gamma$ if $m > 2$; while, by § 105, it follows, for $m = 2$, that J contains a transformation of the form $B_{2,1,\lambda\rho^2}$, with $\lambda \neq 0$, and ρ an arbitrary quantity in the field F. Having $B_{2,1,\lambda\sigma^2}$, the *group* J contains its inverse $B_{2,1,-\lambda\sigma^2}$. Hence J contains the product, in which ρ and σ are arbitrary in F,

$$B_{2,1,\lambda\rho^2} B_{2,1,-\lambda\sigma^2} \equiv B_{2,1,\lambda(\rho^2-\sigma^2)}.$$

To make $\rho^2 - \sigma^2 = \tau$, an arbitrary quantity in F, we set

$$\rho = \tfrac{1}{2}(\tau + 1), \quad \sigma = \tfrac{1}{2}(\tau - 1),$$

thereby excluding the case in which F has a modulus $p = 2$.

With $B_{2,1,\lambda\tau}$, J contains every $B_{2,1,\kappa}$, κ arbitrary in F. But $B_{2,1,\kappa}$ is transformed into $B_{1,2,-\kappa}$ by $(\xi_1' = \xi_2,\ \xi_2' = -\xi_1)$, which belongs to Γ. Since $B_{1,2,\kappa}$ and $B_{2,1,\kappa}$ generate Γ, we have $J = \Gamma$. Hence, for any m, H is a maximal invariant subgroup of Γ. The quotient-group Γ/H is therefore *simple.*

The group $SLH(m, F)$ has $(f, 1)$ isomorphism with a simple group, where f is the number of solutions in F of $x^m = 1$, and F is any infinite field or any finite field of order p^n, provided $p^n > 3$ if $m = 2$.*

Introducing the linear fractional transformations

$$x_i' = \frac{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im-1}x_{m-1} + a_{im}}{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm-1}x_{m-1} + a_{mm}} \qquad (i = 1, \cdots, m-1),$$

we may derive, as in L. G., § 108, the following theorem:

The group $LF(m, F)$ of all linear fractional transformations on $m-1$ variables with coefficients in an infinite field F and of determinant unity is a simple group.

§ 4. *The Abelian linear group.*

A linear homogeneous transformation on $2m$ variables with coefficients in a field F is called *Abelian* if, when operating simultaneously upon two sets of variables $\xi_i, \eta_i;\ \bar{\xi}_i, \bar{\eta}_i (i = 1, \cdots, m)$, it leaves formally invariant (up to a factor belonging to F) the function

$$\phi \equiv \sum_{i=1}^{m} (\xi_i \bar{\eta}_i - \eta_i \bar{\xi}_i).$$

The totality of such transformations constitutes the *general Abelian linear group* $GA(2m, F)$. Those transformations which leave ϕ absolutely invariant form a subgroup called the *special Abelian linear group* $SA(2m, F)$.

If F be a continuous field (real or complex), the group $SA(2m, F)$ is simple† (in LIE'S sense). If we take for ϕ the function

$$\sum_{i=1}^{m} (\xi_i d\eta_i - \eta_i d\xi_i),$$

we recognize $SA(2m, F)$ to be the homogeneous form of the largest projective group on $2m-1$ variables which leaves invariant a linear complex.‡

---

* In speaking of the index of an invariant subgroup H of a group Γ of infinite order, we mean the number of right-hand multipliers M_i such that the products hM_i, when h runs through the set of operators of H, give once and but once every operator of Γ. If H is of order f, we say that Γ has $(f, 1)$ isomorphism with Γ/H.

† Bulletin of the American Mathematical Society (2), vol. 3 (1897), pp. 267–270. With LIE a simple group is one containing no invariant continuous subgroup.

‡ Ibid., pp. 270, 271.

THEOREM. *In any field F, the group $SA(2m, F)$ is generated by the transformations, all of determinant unity,*

$$M_i: \quad \xi_i' = \eta_i, \quad \eta_i' = -\xi_i;$$
$$L_{i,\lambda}: \quad \xi_i' = \xi_i + \lambda\eta_i;$$
$$N_{i,j,\lambda}: \quad \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j + \lambda\eta_i;$$

where λ is an arbitrary quantity in F. The group has a maximal invariant subgroup formed of the identity I and the transformation T which changes the signs of the $2m$ variables.‡ The case $m = 1$ is exceptional if F be of order 2 or 3.

The proof proceeds as in L. G., §§ 110, 111, 114, 116, the statements on p. 97, lines 1–3, being replaced by the following argument. Since J contains $L_{2,\lambda\tau^2}$, in which $\lambda \neq 0$ and τ is arbitrary in F, it contains the inverse $L_{2,-\lambda\tau^2}$ and therefore the product

$$L_{2,\lambda\tau_1^2} L_{2,-\lambda\tau_2^2} \equiv L_{2,\lambda(\tau_1^2-\tau_2^2)}.$$

Taking $\tau_1 = \frac{1}{2}(\kappa+1)$, $\tau_2 = \frac{1}{2}(\kappa-1)$, we reach $L_{2,\lambda\kappa}$, where κ is arbitrary in F. Hence J contains $L_{2,\mu}$, μ being arbitrary in F. An analogous change is to be made on p. 97, lines 25–28.

The group obtained as the quotient-group of $SA(2m, F)$ by $\{I, T\}$ will be designated by $A(2m, F)$. It is simple except in the cases $m = 1$, F of order 2 or 3.

THEOREM. *A transformation of period 2 of $SA(2m, F)$ is conjugate within that group with one of the m non-conjugate transformations*

$$T_{1,-1}, \quad T_{1,-1}T_{2,-1}, \quad T_{1,-1}T_{2,-1}T_{3,-1}, \quad \cdots, \quad T \equiv T_{1,-1}T_{2,-1}\cdots T_{m,-1},$$

where $T_{i,-1}$ alters only ξ_i and η_i whose signs it changes.

The proof proceeds as in L. G., §§ 120–121.

The study of the conjugacy of the operators of period 2 in the quotient-group $A(2m, F)$ is not so simple for infinite fields as for finite fields (L. G., §§ 122–123). For the simplest case $m = 1$, a transformation of period 4 in $SA(2, F)$ is conjugate within that group with one of the transformations

$$S_\gamma \equiv \begin{pmatrix} 0 & \gamma \\ -\gamma^{-1} & 0 \end{pmatrix}.$$

The most general transformation of determinant unity which transforms S_γ into S_δ has the form

$$\begin{pmatrix} d\delta\gamma^{-1}, & b\delta \\ -b\gamma^{-1}, & d \end{pmatrix} \qquad (\gamma/\delta = b^2 + d^2).$$

‡ For the second part of the theorem and for the remainder of this section, it is assumed that, *if* there be a modulus p for an infinite field F, $p \neq 2$.

Hence S_γ and S_δ are conjugate within $SA(2, F)$ if, and only if, the ratio γ/δ is expressible as the sum of two squares (including zero) in the field F.

For the field of rational numbers, S_1 is conjugate with S_γ for $\gamma = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \cdots, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \cdots$, a series containing every prime number of the form $4t + 1$, but no prime number of the form $4t + 3$.* It follows readily that there is an infinite number of non-conjugate transformations of period 4, viz., $S_1, S_3, S_7, S_{21}, S_{11}, S_{33}, S_{77}, S_{231}, S_{19}, \cdots$, including every S_τ for which τ is a prime number of the form $4t + 3$, of which there are an infinite number by DIRICHLET'S theorem.

In the field C of all complex numbers, every S_γ is conjugate with $S_1 \equiv M_1$ within the group. As in L. G., §§ 122–123, we obtain the theorem:

If $s \equiv m/2$ or $(m-1)/2$ according as m is even or odd, the group $A(2m, C)$ contains exactly $s + 1$ sets of conjugate operators of period 2. As representatives we may take

$$M \equiv M_1 M_2 \cdots M_m, \quad T_{1,-1}, \quad T_{1,-1}T_{2,-1}, \cdots, T_{1,-1}T_{2,-1}\cdots T_{s,-1}.$$

In the field of all real numbers every S_γ is conjugate within the group $SA(2, F)$ either with $S_1 \equiv M_1$ or else with $S_{-1} \equiv M_1 T_{1,-1}$. By a simple modification of the developments in L. G., § 123, we find that, within $SA(2m, F)$, every transformation S, such that $S^2 = T_{1,-1}T_{2,-1}\cdots T_{m,-1}$, is conjugate with one of the transformations M, $MT_{1,-1}$, $MT_{1,-1}T_{2,-1}$, $\cdots$, $MT_{1,-1}\cdots T_{m,-1}$. For $m = 1$ these transformations have been shown to be not conjugate. That they are not conjugate when $m = 2$ may be shown as follows. A transformation which transforms M_1M_2 into either $M_1M_2T_{1,-1}$ or $M_1M_2T_{1,-1}T_{2,-1}$ must have the form

$$\begin{Bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \gamma_{11} & -\alpha_{11} & \gamma_{12} & -\alpha_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix},$$

so that, by one of the Abelian conditions,

$$-\alpha_{11}^2 - \gamma_{11}^2 - \alpha_{12}^2 - \gamma_{12}^2 = 1.$$

The commutative transformation $T_{1,-1}T_{2,-1}$ being introduced, it follows that

$$M_1M_2T_{1,-1}T_{2,-1}$$

is not conjugate with $M_1M_2T_{2,-1}$ or with M_1M_2.

For general m, it follows by a similar proof that neither M nor MT is conjugate with any one of the series $MT_{1,-1}, MT_{1,-1}T_{2,-1}, \cdots, MT_{1,-1}\cdots T_{m-1,-1}$ nor M with MT.

---

*Cf. WEBER, *Algebra*, vol. 1, 1st ed. (1895), p. 585.

For the field F of all real numbers, every operator of period two of the group $A(2m, F)$ is conjugate within the group with one of the following operators

$$T_{1,-1},\ T_{1,-1}T_{2,-1},\ \cdots,\ T_{1,-1}T_{2,-1}\cdots T_{s,-1},\ M \equiv M_1M_2\cdots M_m,$$

$$MT_{1,-1},\ MT_{1,-1}T_{2,-1},\ \cdots,\ MT_{1,-1}T_{s,-1}\cdots T_{3,-1},$$

where $s = m/2$ or $(m-1)/2$ according as m is even or odd.

At least in the cases $m = 1$ and $m = 2$, no two of these $2s+1$ operators are conjugate within the group.

§ 5. *A generalization of the Abelian linear group.*

Those linear homogeneous transformations on mq variables with coefficients in any field F which, if operating simultaneously upon q sets each of mq variables, the jth set being exhibited by the notation

$$x_{i1}^{(j)},\ x_{i2}^{(j)},\ \cdots,\ x_{iq}^{(j)} \qquad (i=1, 2, \cdots, m),$$

leave formally and absolutely invariant the function

$$\phi = \sum_{i=1}^{m} \begin{vmatrix} x_{i1}^{(1)}, & x_{i2}^{(1)}, & \cdots, & x_{iq}^{(1)} \\ \cdot & \cdot & \cdots & \cdot \\ x_{i1}^{(q)}, & x_{i2}^{(q)}, & \cdots, & x_{iq}^{(q)} \end{vmatrix},$$

form a group $G(m, q, F)$. For $q = 2$, it is the group $SA(m, F)$ of § 4.

Proceeding as in L. G., §§ 124–128, we obtain the theorems:*

For $q > 2$, the group $G(m, q, F)$ is generated by the substitutions

$$P_{ij} \equiv (x_{i1}x_{j1})(x_{i2}x_{j2})\cdots(x_{iq}x_{jq}) \qquad (i, j=1, 2, \cdots, m),$$

and the totality of transformations in F of determinant unity,

$$x'_{1l} = \sum_{k=1}^{q} a_{lk}x_{1k} \qquad (l=1, 2, \cdots, q).$$

For $q > 2$, $G(m, q, F)$ has an invariant subgroup which is the direct product of m commutative groups each the special linear homogeneous group

* For the case in which F is a continuous field, these theorems were established by the author (using the LIE theory) in the Bulletin of the American Mathematical Society (2), vol. 3, pp. 271–273, May, 1897.

in F on q variables, the quotient-group being generated by the substitutions P_{ij} and thus holoedrically isomorphic with the symmetric group on m letters.*

§ 6. *The compounds of a linear homogeneous group.*

To the transformation (a_{ij}) of § 3 we make correspond the transformation

$$[a]_q: \qquad Y'_{i_1, i_2, \cdots, i_q} = \sum_{l_1, \cdots, l_q} \begin{vmatrix} a_{i_1 l_1} & a_{i_1 l_2} \cdots a_{i_1 l_q} \\ a_{i_2 l_1} & a_{i_2 l_2} \cdots a_{i_2 l_q} \\ \cdots & \cdots \\ a_{i_q l_1} & a_{i_q l_2} \cdots a_{i_q l_q} \end{vmatrix} Y_{l_1, l_2, \cdots, l_q},$$

where the sets $i_1, i_2, \cdots, i_q$ and $l_1, l_2, \cdots, l_q$ take independently all the distinct combinations q at a time of the integers $1, 2, \cdots, m$. Likewise, to the transformation (a'_{ij}) we make correspond the transformation $[a']_q$. Then (L. G., § 153) to the product $(a''_{ij}) \equiv (a_{ij})(a'_{ij})$ corresponds $[a'']_q \equiv [a]_q [a']_q$. Hence, if the transformations (a_{ij}) form a group, the transformations $[a]_q$ form an isomorphic group called the qth *compound* of the former.

By simple modifications of §§ 154–164 of L. G., we obtain the results:

The general linear homogeneous group $GLH(m, F)$ has $(d, 1)$ isomorphism with its qth compound if the equation $x^q = 1$ has exactly d roots in F.

The special linear homogeneous group $SLH(m, F)$ has $(g, 1)$ isomorphism with its qth compound if the equations $x^q = 1$, $x^m = 1$ have in the field F exactly g common solutions.

The second compound of $GLH(m, F)$ leaves invariant the Pfaffian† $[1, 2, \cdots, m]$. For m odd, the transformation $[a]_2$ gives rise to the transformation

$$F'_i = \sum_{j=1}^{m} A_{ij} F_j \qquad (i = 1, 2, \cdots, m)$$

upon the Pfaffians $F_j \equiv [1, 2, \cdots, j-1, j+1, \cdots, m]$, where A_{ij} denotes the first minor of a_{ij} in the determinant $|a_{ij}|$. The transformation $[a]_2$ effects upon the $\frac{1}{2}m(m-1)$ Pfaffians $[i_1, i_2, \cdots, i_{m-2}]$, where each $i = 1, 2, \cdots, m$,

---

* The transformations of the ith group are given by the formula

$$x'_{ij} = \sum_{k=1}^{q} \beta_{ik}^{ij} x_{ik}, \; x'_{sj} = x_{sj} \qquad (s = 1, \cdots, m;\; s \neq i;\; j = 1, \cdots, q),$$

where, for each $i \leqq m$, the determinant $|\beta_{ik}^{ij}| = 1 (j, k = 1, \cdots, q)$.

† Inversely, in a continuous field F, the largest linear group on $\frac{1}{2}m(m-1)$ variables which leaves the Pfaffian invariant is the second compound of $GLH(m, F)$, Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 338–342.

such that $i_1 < i_2 < \cdots < i_{m-2}$, *a linear homogeneous transformation identical with the transformation* $[a]_{m-2}$ *of the* $(m-2)$*th compound.* *

The q*th and* $(m-q)$*th compounds of* $SLH(m, F)$ *are holoedrically isomorphic.* Indeed, the number of solutions in F of $x^m = 1$, $x^q = 1$ equals the number of solutions in F of $x^m = 1$, $x^{m-q} = 1$.

The general Abelian group $GA(2m, F)$ *is the largest linear homogeneous group in the field* F *on* $2m$ *variables whose second compound has the relative invariant*

$$Z \equiv Y_{12} + Y_{34} + \cdots + Y_{2m-1\,2m}.$$

The second compound of $SA(2m, F)$ *is a simple*† *group with the absolute invariants* Z *and the Pfaffian* $[1, 2 \cdots, 2m]$.

The simple group $A(4, F)$ *is holoedrically isomophic with a subgroup of the quinary linear group in* F *which leaves absolutely invariant*

$$Y^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

According as the field F *does not or does contain a primitive fourth root of unity, the second compound of* $SLH(4, F)$ *is holoedrically isomorphic with the simple group* $LF(4, F)$ *or has a maximal invariant subgroup* $\{I, T\}$, *where* T *changes the signs of the six variables, the quotient-group being holoedrically isomorphic with* $LF(4, F)$.

The second compound of $SLH(4, F)$ *contains the transformation*

$$Y'_{12} = \nu Y_{12}, \quad Y'_{13} = Y_{13}, \quad Y'_{14} = Y_{14}, \quad Y'_{23} = Y_{23}, \quad Y'_{24} = Y_{24}, \quad Y'_{34} = \nu^{-1} Y_{34}$$

if, and only if, ν *be a square in the field* F.

By § 3, the group $SLH(4, F)$ is generated by the transformations $B_{r,s,\lambda}$ $(r, s = 1, 2, 3, 4; r \neq s)$, and hence by $B_{1,2,\lambda}$ and

$$A_{12}: (\xi'_1 = \xi_2, \xi'_2 = -\xi_1), \quad A_{13}: (\xi'_1 = \xi_3, \xi'_3 = -\xi_1), \quad A_{34}: (\xi'_3 = \xi_4, \xi'_4 = -\xi_3).$$

The second compounds of these transformations are respectively

$$\begin{aligned}
B'_{1,2,\lambda}: &\quad Y'_{13} = Y_{13} + \lambda Y_{23}, \quad Y'_{14} = Y_{14} + \lambda Y_{24};\\
A'_{12}: &\quad Y'_{13} = Y_{23}, \quad Y'_{14} = Y_{24}, \quad Y'_{23} = -Y_{13}, \quad Y'_{24} = -Y_{14};\\
A'_{13}: &\quad Y'_{12} = -Y'_{23}, \quad Y'_{14} = Y_{34}, \quad Y'_{23} = Y_{12}, \quad Y'_{34} = -Y_{14};\\
A'_{34}: &\quad Y'_{13} = Y_{14}, \quad Y'_{14} = -Y_{13}, \quad Y'_{23} = Y_{24}, \quad Y'_{24} = -Y_{23}.
\end{aligned}$$

The second compound possesses the absolute invariant

$$\theta_6 \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

* For the case of a continuous field F, these theorems were established by the author in the Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 120–135.

† If there be a modulus p, we assume that $p \neq 2$; as also in the rest of the section.

The second compound of $SLH(4, F)$ *may be exhibited as that senary linear group* G_6, *leaving* $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$ *invariant, which is generated by the transformations* $S_{ij} \equiv (\xi_i\eta_i)(\xi_j\eta_j)$ *and*

$$W_{i,j,\lambda}: \qquad \xi_i' = \xi_i + \lambda\eta_j, \ \xi_j' = \xi_j - \lambda\eta_i.$$

The group G_6 will therefore contain the transformations

$$S_{ij}^{-1} W_{i,j,\lambda} S_{ij} \equiv V_{i,j,\lambda}: \qquad \eta_i' = \eta_i + \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\xi_i;$$
$$S_{jk}^{-1} W_{i,j,\lambda} S_{jk} = Q_{i,j,\lambda}: \qquad \xi_i' = \xi_i + \lambda\xi_j, \ \eta_j' = \eta_j - \lambda\eta_i;$$
$$Q_{j,i,-1} Q_{i,j,1} Q_{j,i,-1} \equiv P_{ij} T_{j,-1};$$

where $P_{ij} \equiv (\xi_i\xi_j)(\eta_i\eta_j)$ and $T_{i,\lambda}$ denotes $\xi_i' = \lambda\xi_i$, $\eta_i' = \lambda^{-1}\eta_i$. Set

$$(1) \quad Y_{12} \equiv \xi_3, \ Y_{13} \equiv \xi_1, \ Y_{14} \equiv \xi_2, \ Y_{23} \equiv -\eta_2, \ Y_{24} \equiv \eta_1, \ Y_{34} \equiv -\eta_3.$$

Then $B_{1,2,\lambda}'$, A_{12}', A_{13}', A_{34}' become $W_{1,2,-\lambda}$, $S_{12}T_{2,-1}P_{12}$, $S_{23}P_{23}T_{2,-1}$, $P_{12}T_{2,-1}$, respectively. Hence G_6 contains the transformations which correspond to the generators of the second compound. Inversely, from them we derive $W_{1,2,\lambda}$, $P_{12}T_{2,-1}$, S_{12} and therefore the transform S_{13} of S_{12} by $S_{23}P_{23}T_{2,-1}$. We then derive $S_{12}S_{13} \equiv S_{23}$ and $P_{23}T_{2,-1}$. The latter transforms $W_{1,2,\lambda}$ into $W_{1,3,\lambda}$; and $P_{12}T_{2,-1}$ transforms $W_{1,3,-\lambda}$ into $W_{2,3,\lambda}$. We have therefore derived the generators S_{ij}, $W_{i,j,\lambda}$ of G_6.

By § 4 the special Abelian group $SA(4, F)$ is generated by M_1, M_2, $L_{1,\lambda}$, $L_{2,\lambda}$ and $N_{1,2,\lambda}$. Their second compounds are respectively A_{12}', A_{34}', $B_{1,2,\lambda}'$, $B_{3,4,\lambda}'$ and $N_{1,2,\lambda}'$, the first three being exhibited above,* while

$$B_{3,4,\lambda}': \qquad Y_{13}' = Y_{13} + \lambda Y_{14}, \quad Y_{23}' = Y_{23} + \lambda Y_{24};$$
$$N_{1,2,\lambda}': \qquad \begin{cases} Y_{12}' = Y_{12} - \lambda Y_{24}, & Y_{34}' = Y_{34} + \lambda Y_{24}, \\ Y_{13}' = Y_{13} + \lambda Y_{12} - \lambda Y_{34} - \lambda^2 Y_{24}. \end{cases}$$

The last five transformations leave invariant θ_6 and $Z \equiv Y_{12} + Y_{34}$, by the theorem stated above. We introduce the new variables †

$$(2) \quad \xi_0 = \tfrac{1}{2}(Y_{12} - Y_{34}), \quad \xi_1 = Y_{13}, \quad \eta_1 = Y_{24}, \quad \xi_2 = Y_{14}, \quad \eta_2 = -Y_{23}.$$

Then A_{12}', A_{34}', $B_{1,2,\lambda}'$ and $B_{3,4,\lambda}'$ become $S_{12}T_{2,-1}P_{12}$, $P_{12}T_{2,-1}$, $W_{1,2,-\lambda}$ and $Q_{1,2,\lambda}$ respectively. Finally, $N_{1,2,\lambda}'$ becomes $X_{0,1,\lambda}$, if we use the notation

$$X_{0,j,\lambda}: \qquad \xi_0' = \xi_0 - \lambda\eta_j, \quad \xi_j' = \xi_j + 2\lambda\xi_0 - \lambda^2\eta_j.$$

For later use, we introduce, for the transform of $X_{0,j,\lambda}$ by $S_{jk}(k \neq j)$,

$$Y_{0,j,\lambda}: \qquad \xi_0' = \xi_0 - \lambda\xi_j, \quad \eta_j' = \eta_j + 2\lambda\xi_0 - \lambda^2\xi_j.$$

---

* The present variables ξ_1, η_1, ξ_2, η_2 correspond to the former ξ_1, ξ_2, ξ_3, ξ_4 respectively.

† The last four have the same definition as in the case of the group G_6.

The second compound of $SA(4, F)$ is a simple group which may be exhibited as that quinary linear group G_5 leaving $\xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2$ invariant which is generated by $P_{12}T_{2,-1}$, S_{12}, $W_{1,2,\lambda}$, $Q_{1,2,\lambda}$ and $X_{0,1,\lambda}$.

We obtain an important subgroup of G_5 as follows. The second compound of the special Abelian transformation

$$\begin{Bmatrix} a & 0 & 0 & \beta \\ 0 & \delta & \gamma & 0 \\ 0 & \beta & a & 0 \\ \gamma & 0 & 0 & \delta \end{Bmatrix} \qquad (a\delta - \beta\gamma = 1)$$

affects only four of the six variables Y_{ij} and has the form

	Y_{12}	Y_{13}	Y_{24}	Y_{34}
$Y'_{12} =$	$a\delta$	$a\gamma$	$-\beta\delta$	$-\beta\gamma$
$Y'_{13} =$	$a\beta$	a^2	$-\beta^2$	$-a\beta$
$Y'_{24} =$	$-\gamma\delta$	$-\gamma^2$	δ^2	$\gamma\delta$
$Y'_{34} =$	$-\beta\gamma$	$-a\gamma$	$\beta\delta$	$a\delta$

The latter leaves $Y_{12} + Y_{34}$ invariant. Expressed in terms of the variables ξ_0, ξ_1, η_1 defined by (2), it takes the form

(3)

	ξ_0	ξ_1	η_1
$\xi'_0 =$	$a\delta + \beta\gamma$	$a\gamma$	$-\beta\delta$
$\xi'_1 =$	$2a\beta$	a^2	$-\beta^2$
$\eta'_1 =$	$-2\gamma\delta$	$-\gamma^2$	δ^2

$(a\delta - \beta\gamma = 1)$,

a transformation of determinant unity leaving $\xi_0^2 + \xi_1\eta_1$ absolutely invariant.* *The transformations* (3) *form a simple group isomorphic with* $LF(2, F)$.

To exhibit a subgroup of both G_5 and G_6 which leaves $\xi_1\eta_1 + \xi_2\eta_2$ invariant, we form the second compound of

$$\begin{Bmatrix} a & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & D \end{Bmatrix} \qquad \begin{pmatrix} a\delta - \beta\gamma = 1 \\ AD - BC = 1 \end{pmatrix},$$

* Among them occur $X_{0,1,\lambda}$ and $Y_{0,1,\lambda}$, the latter for $a = \delta = 1$, $\beta = 0$, $\gamma = -\lambda$.

and obtain a transformation affecting only Y_{13}, Y_{14}, Y_{23}, Y_{24}. Introducing $\xi_1, \xi_2, -\eta_2, \eta_1$, respectively, for the former [as in (1) or (2)], we obtain the transformation

$$(4)\qquad \begin{array}{c|cccc} & \xi_1 & \eta_1 & \xi_2 & \eta_2 \\ \hline \xi_1' = & \alpha A & \beta B & \alpha B & -\beta A \\ \eta_1' = & \gamma C & \delta D & \gamma D & -\delta C \\ \xi_2' = & \alpha C & \beta D & \alpha D & -\beta C \\ \eta_2' = & -\gamma A & -\delta B & -\gamma B & \delta A \end{array}$$

The group of these transformations has, in view of its origin, the factor groups $LF(2, F), LF(2, F)$, and $\{I, T\}$, where T changes the sign of each variable.*

§ 7. *Concerning linear groups with quadratic invariants.*

Consider the group $G(m, F)$ of linear homogeneous transformations

$$S: \quad \begin{cases} \xi_i' = a_{i0}\xi_0 + \sum_{j=1}^{m} (a_{ij}\xi_j + \gamma_{ij}\eta_j) & (i = 0, 1, \cdots, m), \\ \eta_i' = \beta_{i0}\xi_0 + \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) & (i = 1, \cdots, m), \end{cases}$$

with coefficients in a field† F, which have the absolute invariant

$$g_m \equiv \xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

The conditions for the formal invariance of g_m are

$$(5)\qquad a_{00}^2 + \sum_{i=1}^{m} a_{i0}\beta_{i0} = 1,$$

$$(6)\qquad 2a_{0j}a_{0k} + \sum_{i=1}^{m} (a_{ij}\beta_{ik} + a_{ik}\beta_{ij}) = 0 \qquad (j = 0, 1, \cdots, m;\ k = 1, \cdots, m),$$

$$(7)\qquad 2\gamma_{0j}\gamma_{0k} + \sum_{i=1}^{m} (\gamma_{ij}\delta_{ik} + \gamma_{ik}\delta_{ij}) = 0 \qquad (j, k = 1, \cdots, m),$$

$$(8)\qquad 2a_{0j}\gamma_{0k} + \sum_{i=1}^{m} (a_{ij}\delta_{ik} + \gamma_{ik}\beta_{ij}) = \begin{cases} 1 & (k = j), \\ 0 & (k \neq j), \end{cases} \qquad (j = 0, 1, \cdots, m;\ k = 1, \cdots, m).$$

* To compare with the earlier proof for finite fields, American Journal of Mathematics, vol. 21 (1899), p. 248, we have only to replace β by $-\gamma$, γ by $-\beta$, B by C, C by B.

† If there be a modulus p, we assume that $p \neq 2$. For a finite field of order 2^n, the structure was given by the writer in the American Journal of Mathematics, vol. 21 (1899), p. 243.

It follows from these relations that the inverse of S is

$$S^{-1}: \quad \begin{cases} \xi_0' = a_{00}\xi_0 + \frac{1}{2}\sum_{j=1}^{m}(\beta_{j0}\xi_j + a_{j0}\eta_j) \\ \xi_i' = 2\gamma_{0i}\xi_0 + \sum_{j=1}^{m}(\delta_{ji}\xi_j + \gamma_{ji}\eta_j) \qquad (i=1,\cdots,m). \\ \eta_i' = 2a_{0i}\xi_0 + \sum_{j=1}^{m}(\beta_{ji}\xi_j + a_{ji}\eta_j) \end{cases}$$

The conditions for the invariance of g_m under S^{-1} are seen to be

$$a_{00}^2 + 4\sum_{i=1}^{m} a_{0i}\gamma_{0i} = 1, \tag{9}$$

$$\tfrac{1}{2}a_{j0}a_{k0} + \sum_{i=1}^{m}(a_{ji}\gamma_{ki} + a_{ki}\gamma_{ji}) = 0 \tag{10}$$
$$(j=0,1,\cdots,m\,;\,k=1,\cdots,m),$$

$$\tfrac{1}{2}\beta_{j0}\beta_{k0} + \sum_{i=1}^{m}(\beta_{ji}\delta_{ki} + \beta_{ki}\delta_{ji}) = 0 \qquad (j,k=1,\cdots,m), \tag{11}$$

$$\tfrac{1}{2}a_{j0}\beta_{k0} + \sum_{i=1}^{m}(a_{ji}\delta_{ki} + \beta_{ki}\gamma_{ji}) = \begin{cases} 1 & (k=j) \\ 0 & (k\neq j) \end{cases} \tag{12}$$
$$(j=0,1,\cdots,m\,;\,k=1,\cdots,m).$$

S and S^{-1} have equal determinants, so that $|S| = \pm 1$. In fact, we take the variables in the order $\xi_0, \xi_1, \eta_1, \cdots, \xi_m, \eta_m$, and reflect on its main diagonal the determinant of S^{-1}, then interchange the second and third rows, the fourth and fifth rows, $\cdots$, and likewise the corresponding columns. The resulting determinant, aside from the factors 2 and $\frac{1}{2}$ which may be dropped, is identical with the determinant of S.

If in the above formulæ we drop the variable ξ_0 and the coefficients $a_{i0}, a_{0i}, \gamma_{i0}, \cdots$, we obtain results valid for the group $Q(m, F)$ of all linear transformations in F which leave invariant *

$$q_m \equiv \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_m\eta_m.$$

With such modifications, the following investigation of the structure of the group $G(m, F)$ will hold for the group $Q(m, F)$.

Let S be an arbitrary transformation of $G(m, F)$. We proceed to determine a transformation Σ, derived from

$$W_{i,j,\lambda},\ V_{i,j,\lambda},\ Q_{i,j,\lambda},\ X_{0,j,\lambda},\ Y_{0,j,\lambda},\ T_{i,\lambda},\ S_{ij},\ P_{ij}, \tag{13}$$

* For the case in which F is the Galois field of order p^n, this group was studied by the writer in the Proceedings of the London Mathematical Society, vol. 30 (1898), pp. 70–98. The calculations of pp. 77–80 are here avoided by the use of the simplicity of the group G_6.

which replaces ξ_1 by the same function that S does, viz.,

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j),$$

where by (10), for $j = k = 1$,

$$\tfrac{1}{4}a_{10}^2 + \sum_{i=1}^{m} a_{1i}\gamma_{1i} = 0. \tag{14}$$

The a_{1i}, $\gamma_{1i}(i = 1, \cdots, m)$ are not all zero, since otherwise $a_{10} = 0$ and $f_1 \equiv 0$.

(a) If $a_{11} \neq 0$, we may take for Σ the product

$$T_{1,\,a_{11}} X_{0,\,1,\,\frac{1}{2}a_{10}} Q_{1,\,2,\,a_{12}} W_{1,\,2,\,\gamma_{12}} \cdots Q_{1,\,m,\,a_{1m}} W_{1,\,m,\,\gamma_{1m}},$$

which replaces ξ_1 by

$$a_{10}\xi_0 + a_{11}\xi_1 - a_{11}^{-1}(\tfrac{1}{4}a_{10}^2 + a_{12}\gamma_{12} + \cdots + a_{1m}\gamma_{1m})\eta_1 + \sum_{j=2}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \equiv f_1.$$

(b) If $\gamma_{11} \neq 0$, we may choose for Σ the product

$$S_{12} T_{1,\,\gamma_{11}} X_{0,\,1,\,\frac{1}{2}a_{10}} Q_{1,\,2,\,\gamma_{12}} W_{1,\,2,\,a_{12}} \cdot Q_{1,\,3,\,a_{13}} W_{1,\,3,\,\gamma_{13}} \cdots Q_{1,\,m,\,a_{1m}} W_{1,\,m,\,\gamma_{1m}}.$$

(c) Let $a_{1j} = \gamma_{1j} = 0\,(j = 1, \cdots, s-1)$, while a_{1s}, γ_{1s} are not both zero. By case (a) or (b), we obtain a transformation Σ' which replaces ξ_s by f_1. Then will $\Sigma \equiv \Sigma' P_{1s}$ replace ξ_1 by f_1.

We may therefore set $S = S\Sigma'$, where S' is a transformation of $G(m, F)$ which leaves ξ_1 fixed. Let S' replace η_1 by

$$f_1' \equiv \beta_{10}\xi_0 + \sum_{j=1}^{m}(\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

For S' we have $a_{10} = 0$, $a_{11} = 1$, $a_{1j} = 0\,(j = 2, \cdots, m)$, $\gamma_{1j} = 0\,(j = 1, \cdots, m)$. Then by (12) for $j = k = 1$, we have $\delta_{11} = 1$. By (11) for $j = k = 1$, we have

$$\tfrac{1}{4}\beta_{10}^2 + \sum_{i=1}^{m}\beta_{1i}\delta_{1i} = 0. \tag{15}$$

The transformation

$$\Sigma_1 \equiv Y_{0,\,1,\,\frac{1}{2}\beta_{10}} V_{2,\,1,\,-\beta_{12}} Q_{2,\,1,\,-\delta_{12}} \cdots V_{m,\,1,\,-\beta_{1m}} Q_{m,\,1,\,-\delta_{1m}}$$

leaves ξ_1 fixed and replaces η_1 by f_1'. We may therefore set

$$S' = \Sigma_1 S_1, \quad S \equiv \Sigma\,\Sigma_1 S_1,$$

where S_1 is a transformation of $G(m, F)$ which leaves ξ_1 and η_1 fixed. Hence S_1 is of the form S, written above, with

$$a_{10} = \beta_{10} = 0, \quad a_{11} = \delta_{11} = 1, \quad a_{1j} = \delta_{1j} = 0 \qquad (j = 2, \cdots, m),$$

$$\gamma_{1j} = \beta_{1j} = 0 \qquad (j = 1, \cdots, m).$$

Hence, by (11) and (12), for $j=1$, we get

$$\beta_{k1}=0, \quad \delta_{k1}=0 \qquad (k=2, \cdots, m).$$

By (10) and (12), for $k=1$, we get

$$\gamma_{j1}=0, \quad \alpha_{j1}=0 \qquad (j=0, 2, \cdots, m).$$

Hence S_1 is a transformation of $G(m, F)$ involving only the variables

$$\xi_0, \quad \xi_i, \quad \eta_i \qquad (i=2, \cdots, m).$$

We proceed with S_1 as we did with S. After $m-1$ such steps, we reach a transformation S_{m-1} affecting only ξ_0, ξ_m, η_m. Let it replace ξ_m by

$$f_m \equiv \alpha_{m0}\xi_0 + \alpha_{mm}\xi_m + \gamma_{mm}\eta_m,$$

where, by (10),

$$\tfrac{1}{4}\alpha_{m0}^2 + \alpha_{mm}\gamma_{mm}=0.$$

If $\alpha_{mm}\neq 0$, the transformation $T_{m,\,\alpha_{mm}}X_{0,\,m,\,\frac{1}{2}\alpha_{m0}}$ replaces ξ_m by f_m. If $\alpha_{mm}=0$, then $\alpha_{m0}=0$ and $\gamma_{mm}\neq 0$, so that we may set

$$S_{m-1} \equiv (\xi_m \eta_m) T_{m,\,\gamma_{mm}} K,$$

where K leaves also ξ_m fixed.* Let K replace η_m by

$$f'_m \equiv \beta_{m0}\xi_0 + \beta_{mm}\xi_m + \delta_{mm}\eta_m.$$

By (11) and (12) for $i=j=k=m$, we get

$$\delta_{mm}=1, \quad \tfrac{1}{4}\beta_{m0}^2 + \beta_{mm}\delta_{mm}=0.$$

Hence $K=Y_{0,\,m,\,\frac{1}{2}\beta_{m0}}K'$, where K' leaves ξ_m and η_m fixed, and is therefore the identity or C_0, where C_0 alters only ξ_0 whose sign it changes. But

$$C_0 \equiv X_{0i1}Y_{0i1}X_{0i1}T_{i,\,-1}(\xi_i\eta_i). \tag{16}$$

It follows that an arbitrary transformation of $G(m, F)$ or of $Q(m, F)$ may be given one of the two forms A or $A(\xi_m\eta_m)$, where A is derived from the transformations (13) of determinant unity. Hence these groups contain subgroups of index 2, designated by $G'(m, F)$ and $Q'(m, F)$, generated by the transformations (13).

Consider, for $m\geqq 3$, the following subgroups of $G'(m, F)$ and $Q'(m, F)$:

$$G_1(m, F) \equiv \{S_{ij}, \quad W_{i,j,\lambda}, \quad X_{0,j,\lambda} \ (i, j=1, \cdots, m; i\neq j)\},$$

$$Q_1(m, F) \equiv \{S_{ij}, \quad W_{i,j,\lambda} \qquad (i, j=1, \cdots, m; i\neq j)\},\dagger$$

* For the group $Q(m, F)$, K is necessarily the identity.

† For $m=3$, $Q_1(m, F)$ is the group G_6 of § 6.

where λ is arbitrary in the field F. By the formulæ of § 6, they contain the transformations

$$V_{i,j,\lambda}, \quad Q_{i,j,\lambda}, \quad P_{ij}T_{j,-1} \qquad (i, j=1, \cdots, m;\ i \neq j),$$

while $G_1(m, F)$ contains also $Y_{0,j,\lambda}$. Hence they contain

$$T_{1,\mu}T_{2,u} \equiv S_{12}P_{12}T_{1,-1}V_{1,2,\mu-1}W_{1,2,\mu}V_{1,2,\mu-1}, \tag{17}$$

$$T_{1,\mu^2} \equiv T_{1,\mu}T_{2,\mu} \cdot T_{2,\mu-1}T_{3,\mu-1} \cdot T_{3,\mu}T_{1,\mu}. \tag{18}$$

But $T_{k,\lambda}$ transforms $W_{i,j,\lambda}$, $V_{i,j,\lambda}$, $Q_{i,j,\lambda}$, $X_{0,j,\lambda}$, $Y_{0,j,\lambda}$, $T_{i,\lambda}$ into transformations of the same respective forms. Also

$$T_{i,\lambda}S_{ij} = S_{ij}T_{i,\lambda-1}, \quad T_{i,\lambda}P_{ij} = P_{ij}T_{j,\lambda}. \tag{19}$$

Hence every transformation of the group $G'(m, F)$ or $Q'(m, F)$ may be given one of the forms Σ, $\Sigma T_{m,\nu}$, where Σ belongs to $G_1(m, F)$ or $Q_1(m, F)$, respectively, while ν runs through the series of elements of F which are not-squares and whose ratios are all not-squares.

These results hold true for the group $G_1(2, F) \equiv G_5$ of § 6, viz.,

$$G_1(2, F) \equiv \{P_{12}T_{2,-1},\ S_{12},\ W_{1,2,\lambda},\ Q_{1,2,\lambda},\ X_{0,1,\lambda}\}.$$

Indeed, the latter group contains $Y_{0,1,\lambda}$, $X_{0,2,\lambda}$, $Y_{0,2,\lambda}$, $V_{1,2,\lambda}$, $Q_{2,1,\lambda}$, $T_{1,\mu}T_{2,\mu}$ and, by (16), $C_0(\xi_1\eta_1)T_{1,-1}$. The latter transforms $T_{1,\mu}T_{2,\mu}$ into $T_{1,\mu-1}T_{2,\mu}$, so that the product T_{2,μ^2} belongs to the group.

The group $G'(m, F)$, for $m \geqq 3$, contains the invariant subgroup $G_1(m, F)$; the group $Q'(m, F)$, for $m \geqq 2$, contains the invariant subgroup $Q_1(m, F)$. The invariant subgroup is extended to the main group by the right-hand extenders $T_{m,\nu}$, where ν runs through the series of those not-squares of F, the ratio of no two of which is a square.

§ 8. *Structure of the groups $G_1(m, F)$ and $Q_1(m, F)$.*

By § 6 we have the results: *

The group $G_1(2, F)$ on five variables is simple. The senary group $Q_1(3, F)$ is simple or has the maximal invariant subgroup composed of the identity I and the transformation $T_{1,-1}T_{2,-1}T_{3,-1}$ according as -1 is a not-square or a square in the field F.

We employ these theorems in dealing with the case of general m. Let J be an invariant subgroup of $G_1(m, F)$ containing a transformation S not the identity I. To treat simultaneously the group $Q_1(m, F)$, let J be an invariant subgroup containing a transformation S neither the identity nor

$$T \equiv T_{1,-1}T_{2,-1}\cdots T_{m,-1},$$

* If there be a modulus p, we assume, in this section, that $p \neq 2$.

in case the latter belongs to $Q_1(m, F)$. The groups $G_1(m, F)$ and $Q_1(m, F)$ will be considered together under the notation G. We assume that $m \geqq 3$.

LEMMA I. *The group J contains a transformation which multiplies ξ_1 by a constant and does not reduce to I or T.*

By hypothesis, J contains a transformation S neither I nor T. Let S replace ξ_1 by

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{m}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \neq a_{11}\xi_1,$$

the coefficients being subject to the condition (14).

(*a*) If $\gamma_{11} \neq 0$, the group G contains the product

$$P \equiv T_{1,\gamma_{11}^{-1}} T_{2,\gamma_{11}^{-1}} V_{0,1,\frac{1}{2}a_{10}} V_{2,1,-\gamma_{11}a_{12}} Q_{2,1,-\gamma_{11}^{-1}\gamma_{12}} K,$$

where K denotes the transformation

$$K \equiv V_{3,1,-a_{13}} Q_{3,1,-\gamma_{13}} \cdots V_{m,1,-a_{1m}} Q_{m,1,-\gamma_{1m}}.$$

Employing (14), we find that P replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by f_1. Hence J contains $S_1 \equiv P^{-1}SP$, which replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$.

If S_1 multiplies ξ_2 by a constant, J contains its transform S_1' by $P_{12}T_{2,-1}$. This S_1' multiplies ξ_1 by a constant and is neither I nor T.

If S_1 does not multiply ξ_2 by a constant, there exists in G a transformation B leaving ξ_1 and η_1 unaltered and not commutative with S_1, so that J contains $S_1^{-1}B^{-1}S_1B$, which leaves ξ_1 fixed and is neither I nor T. In fact, if S_1 be commutative with $V_{2,3,\lambda}$, we find, on equating the expressions by which $S_1V_{2,3,\lambda}$ and $V_{2,3,\lambda}S_1$ replace η_3, that

$$\eta_3' - \lambda\xi_2' = \eta_3' + (\ )\xi_2 + (\ )\xi_3.$$

Similarly, if S_1 be commutative with $Q_{3,2,\lambda}$, we get

$$\xi_3' + \lambda\xi_2' = \xi_3' + (\ )\xi_2 + (\ )\eta_3.$$

Hence would $\xi_2' = (\ )\xi_2$, contrary to hypothesis.

(*b*) If $\gamma_{11} = 0$, we may take* $a_{12} \neq 0$. Then G contains

$$R \equiv T_{2,a_{12}} T_{3,a_{12}} X_{0,2,\frac{1}{2}a_{10}} Q_{2,1,a_{11}} Q_{2,3,a_{12}^{-1}a_{13}} W_{2,3,a_{12}\gamma_{13}} K_1,$$

$$K_1 \equiv Q_{2,4,a_{14}} W_{2,4,\gamma_{14}} \cdots Q_{2,m,a_{1m}} W_{2,m,\gamma_{1m}}.$$

Employing (14), we find that R replaces ξ_2 by f_1 without altering ξ_1. Then J contains $S_2 \equiv R^{-1}SR$ which replaces ξ_1 by ξ_2.

If S_2 multiplies ξ_3 by a constant, J contains its transform S_2' by $P_{13}T_{1,-1}$. But S_2' multiplies ξ_1 by a constant and is neither I nor T.

---

*If $a_{12} = a_{13} = \cdots = a_{1m} = 0$, then $a_{10} = 0$ by (14). Not every γ_{1j} is zero by hypothesis. If $\gamma_{12} \neq 0$, for example, we take in place of S its transform by S_{23}, for which $a_{12}' \neq 0$.

If S_2 does not multiply ξ_3 by a constant, S_2 is not commutative with both $V_{3,1,\lambda}$ and $V_{3,2,\lambda}$, since $S_2 V_{3,k,\lambda}$ and $V_{3,k,\lambda} S_2$ replace η_k by $\eta_k' - \lambda\xi_3'$ and $\eta_k' + (\ )\xi_3 + (\ )\xi_k$, respectively. Hence J contains

$$S_2^{-1} V_{3,k,\lambda}^{-1} S_2 V_{3,k,\lambda} \qquad (k=1, 2)$$

which leave ξ_1 fixed (so that neither is T), and are not both the identity.

LEMMA II. *The group J contains a transformation which leaves ξ_1 and η_1 unaltered and is different from the identity.*

In view of Lemma I, we may suppose that J contains a transformation S, different from I and T, which replaces ξ_1 by $a\xi_1$ and η_1 by

$$\beta_{10}\xi_0 + \sum_{j=1}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j),$$

where by (12), for $j = k = 1$, we have $\delta_{11} = a^{-1}$, and by (11), for $j = k = 1$,

$$\tfrac{1}{4}\beta_{10}^2 + \sum_{i=1}^{m} \beta_{1i}\delta_{1i} = 0. \tag{15}$$

(*A*) Let $\beta_{11} = 0$, $\beta_{1j} = \delta_{1j} = 0\,(j = 2, \cdots, m)$. Then $\beta_{10} = 0$ by (15). Hence $S = T_{1,a} S_1$, where S_1 leaves ξ_1 and η_1 unaltered. Hence,* S_1 involves only the variables $\xi_0, \xi_i, \eta_i\,(i = 2, \cdots, m)$. If $a = 1$, the Lemma is proved. Let next $a \neq 1$.

If $S_1 = I$, or if, when $G \equiv Q_1(m, F)$, $S = T_{2,-1} T_{3,-1} \cdots T_{m,-1} \equiv \tau$, the group J contains $T_{1,a}$ or $S^2 = T_{1,a^2}$ respectively. In the second case, $a \neq -1$, since $S \neq T$. Transforming by $P_{12} T_{2,-1}$, we obtain in either case a transformation leaving ξ_1 and η_1 fixed and not the identity.

If S_1 be neither I nor τ, there exists in G a transformation Σ_1 affecting the same variables as S_1 and not commutative with S_1.† Hence J contains

$$S^{-1}\Sigma_1^{-1} S \Sigma_1 \equiv S_1^{-1}\Sigma_1^{-1} S_1 \Sigma_1 \neq I,$$

which leaves ξ_1 and η_1 unaltered.

(*B*) Let $\beta_{11} = 0$, and $\beta_{1j}, \delta_{1j}\,(j = 2, \cdots, m)$ be not all zero. Then, by § 7, G contains a transformation L which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$\beta_{10}\xi_0 + \sum_{j=2}^{m} (\beta_{1j}\xi_j + \delta_{1j}\eta_j) \qquad \left[\tfrac{1}{4}\beta_{10}^2 + \sum_{j=2}^{m} \beta_{1j}\delta_{1j} = 0\right].$$

Hence J contains $S_1 = L^{-1} S L$, which replaces ξ_1 by $a\xi_1$ and η_1 by $\xi_2 + a^{-1}\eta_1$. The latter function is invariant under the transformations $Q_{3,2,\lambda}$, $V_{2,3,\lambda}$, T_{3,μ^2} and $T_{1,\lambda} T_{2,\lambda^{-1}}$ belonging to G. If any one of these, say Σ, is not commuta-

* See § 7, case (*c*).

† For $G \equiv G_1(m, F)$, the transformation changing the signs of $\xi_0, \xi_2, \eta_2, \cdots, \xi_m, \eta_m$ is of determinant -1 and hence does not belong to G.

tive with S_1, then J contains $S_1^{-1}\Sigma^{-1}S_1\Sigma \neq I$, which leaves ξ_1 and η_1 fixed. Suppose next that they are all commutative with S_1. Equating the two expressions by which $S_1 V_{2,3,\lambda}$ and $V_{2,3,\lambda}S_1$ replace η_2, and the two by which they replace η_3, we get

$$\xi_3' = \delta_{22}\xi_3 - \delta_{23}\xi_2, \quad \xi_2' = -\delta_{32}\xi_3 + \delta_{33}\xi_2.$$

Equating the expressions by which $S_1 Q_{3,2,\lambda}$ and $Q_{3,2,\lambda}S_1$ replace ξ_3, and those by which they replace η_2, we find that

$$\xi_2' = a_{33}\xi_2 - \gamma_{32}\eta_3, \quad \eta_3' = -\beta_{23}\xi_2 + \delta_{22}\eta_3.$$

The field F contains an element λ different from 0 and 1. If we equate the expressions by which $T_{1,\lambda}T_{2,\lambda-1}S_1$ and $S_1 T_{1,\lambda}T_{2,\lambda-1}$ replace η_2, we find that $\beta_{23} = \delta_{23} = 0$. Hence S_1 merely multiplies ξ_3 and η_3 by the same constant δ_{22}. Transforming S_1 by $P_{13}T_{1,-1}$, we obtain a transformation of the kind treated in case (A).

(C) Let $\beta_{11} \neq 0$, β_{1j}, δ_{1j} $(j = 2, \cdots, m)$ be not all zero. By a simple transformation, we may take $\delta_{12} \neq 0$. Transforming S by $T_{2,\delta_{12}}T_{3,\delta_{12}}$, we reach a transformation S' with $\beta_{11} \neq 0$, $\delta_{12} = 1$. Then

$$\omega \equiv V_{0,2,\frac{1}{2}\beta_{10}} Q_{3,2,-\delta_{13}} V_{2,3,\beta_{13}} \cdots Q_{m,2,-\delta_{1m}} V_{2,m,\beta_{1m}}$$

leaves ξ_1, η_1, and ξ_2 unaltered and replaces η_2 by

$$\beta_{10}\xi_0 + (\beta_{11}\delta_{11} + \beta_{12})\xi_2 + \eta_2 + \sum_{j=3}^{m}(\beta_{1j}\xi_j + \delta_{1j}\eta_j).$$

Then J contains $S_1 \equiv \omega^{-1}S'\omega$, which replaces ξ_1 by $a\xi_1$, and η_1 by

$$\beta_{11}\xi_1 + a^{-1}\eta_1 - a^{-1}\beta_{11}\xi_2 + \eta_2.$$

Let $\mu \equiv -\beta_{11}a^{-1} \neq 0$. If among the transformations $Q_{3,2,\mu}W_{2,3,1} \equiv \sigma$, $T_{3,\mu}T_{2,\mu}S_{23}$, etc., of G, which leave ξ_1, η_1, and $\mu\xi_2 + \eta_2$ invariant, there exists one, say R, which is not commutative with S_1, then J contains $S_1^{-1}R^{-1}S_1R$ which leaves ξ_1 and η_1 fixed and differs from the identity. In the contrary case, we find, on equating the functions by which $S_1\sigma$ and σS_1 replace ξ_2, that

$$\eta_3' = (\ )\eta_3 - a_{23}\eta_2 + \mu a_{23}\xi_2.$$

Then, by (11) for $j = k = 3$, we have $a_{23} = 0$. Finally, if S_1 be commutative with $T_{3,\mu}T_{2,\mu}S_{23}$, it must multiply ξ_3 and η_3 by the same constant.

(D) Let $\beta_{11} \neq 0$, $\beta_{1j} = \delta_{1j} = 0$ $(j = 2, \cdots, m)$. Then, by (15),

$$\tfrac{1}{4}\beta_{10}^2 + \beta_{11}\delta_{11} = 0, \quad \delta_{11} = a^{-1}.$$

Hence S replaces ξ_1 by $a\xi_1$ and replaces η_1 by

$$\beta_{10}\xi_0 + \beta_{11}\xi_1 + a^{-1}\eta_1 = a^{-1}(\eta_1 + 2\lambda\xi_0 - \lambda^2\xi_1),$$

if we set $2\lambda \equiv a\beta_{10}$. Hence $S = Y_{0,1,\lambda} T_{1,a} S_1$, where S_1 leaves ξ_1 and η_1 unaltered and so involves only $\xi_0, \xi_i, \eta_i (i = 2, \cdots, m)$. Then

$$T_{1,\rho}^{-1} S T_{1,\rho} \cdot S^{-1} \equiv Y_{0,1,\lambda\rho^{-1}-\lambda}$$

belongs to J and is not the identity if $\rho \neq 1$. Transforming it by $P_{12} T_{2,-1}$, we obtain in J a transformation $\neq I$, which leaves ξ_1 and η_1 unaltered.

In the proofs of lemmas I and II, we assumed the existence of the variables $\xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3$ only. If $m \geqq 4$, we may therefore conclude that J contains a transformation different from I and T which leaves $\xi_1, \eta_1, \xi_2, \eta_2$ unaltered. After $m - 2$ applications of the lemmas, we reach in J a transformation, neither I nor T, which affects only $\xi_0, \xi_{m-1}, \eta_{m-1}, \xi_m, \eta_m$. Transforming it by $P_{1\,m-1} P_{2m}$, which belongs to G, we obtain a transformation, neither I nor T, which affects only $\xi_0, \xi_1, \eta_1, \xi_2, \eta_2$. From the simplicity of $G_1(2, F)$, it follows, when $G \equiv G_1(m, F)$, that J contains all the transformations of $G_1(2, F)$. Transforming them by the $P_{ij} T_{j,-1}$, we reach in J all the generators of $G_1(m, F)$. Since $Q_1(3, F)$ is simple or has the maximal invariant subgroup $\{I, T\}$, it follows, when $G \equiv Q_1(m, F)$, that J contains the generators of $Q_1(3, F)$ and, therefore, by transformation by the $P_{ij} T_{j,-1}$, the generators of $Q_1(m, F)$.

If $m \geqq 2$, the group $G_1(m, F)$ is simple. If $m \geqq 3$, the group $Q_1(m, F)$ is simple or has the maximal invariant subgroup $\{I, T\}$, according as -1 is a not-square or a square in the field F.

§ 9. *Definition and generators of a subgroup of* $G_1(3, F)$.

We next define and investigate the septenary group in an arbitrary* field F which becomes, for the case of a continuous field, the continuous group of fourteen parameters studied by KILLING, ENGEL,† and CARTAN.‡

The totality of linear homogeneous transformations S on seven variables with coefficients in F which leave absolutely invariant

$$g_3 \equiv \xi_0^2 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

form a group $G(3, F)$. Taking $m = 3$, and giving S the notation of § 7, we may take as the conditions upon S the relations (9), (10), (11), (12). We study the group H of transformations S, belonging to $G(3, F)$, which, when operating cogrediently upon the two sets of variables

$$(20) \qquad \xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3; \quad \bar{\xi}_0, \bar{\xi}_1, \bar{\eta}_1, \bar{\xi}_2, \bar{\eta}_2, \bar{\xi}_3, \bar{\eta}_3,$$

* As in §§ 7–8, we assume that $p \neq 2$, if F has a modulus p.

† LIE-ENGEL, *Transformationsgruppen*, vol. 3, pp. 763–765, 777.

‡ CARTAN, *Thèses*, Paris, 1894, pp. 146, 149–151.

leave invariant the system of equations

$$(21)\qquad \begin{cases} X_1 + Y_{23} = 0, & X_2 + Y_{31} = 0, & X_3 + Y_{12} = 0, \\ Y_1 + X_{23} = 0, & Y_2 + X_{31} = 0, & Y_3 + X_{12} = 0. \end{cases}$$

We have here employed the notations

$$X_i \equiv \begin{vmatrix} \xi_0 & \xi_i \\ \bar{\xi}_0 & \bar{\xi}_i \end{vmatrix}, \quad Y_i \equiv \begin{vmatrix} \xi_0 & \eta_i \\ \bar{\xi}_0 & \bar{\eta}_i \end{vmatrix}, \quad X_{ij} \equiv \begin{vmatrix} \xi_i & \xi_j \\ \bar{\xi}_i & \bar{\xi}_j \end{vmatrix}, \quad Y_{ij} \equiv \begin{vmatrix} \eta_i & \eta_j \\ \bar{\eta}_i & \bar{\eta}_j \end{vmatrix}, \quad Z_{ij} \equiv \begin{vmatrix} \xi_i & \eta_j \\ \bar{\xi}_i & \bar{\eta}_j \end{vmatrix}.$$

Multiplying the equations in the first row of (21) by η_1, η_2, η_3, respectively, and adding the results, we get $X_1\eta_1 + X_2\eta_2 + X_3\eta_3 = 0$, since

$$Y_{23}\eta_1 + Y_{31}\eta_2 + Y_{13}\eta_3 \equiv \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \bar{\eta}_1 & \bar{\eta}_2 & \bar{\eta}_3 \end{vmatrix} \equiv 0.$$

Similarly from the equations in the second row of (21), we get

$$Y_1\xi_1 + Y_2\xi_2 + Y_3\xi_3 = 0.$$

In view of the identity

$$Y_i\xi_i - X_i\eta_i \equiv \xi_0(\xi_i\bar{\eta}_i - \bar{\xi}_i\eta_i) \equiv \xi_0 Z_{ii},$$

we derive the equation* [a consequence of equations (21)]

$$(22)\qquad Z_{11} + Z_{22} + Z_{33} = 0.$$

An inspection of equations (21) and (22) leads to a proof of the following

LEMMA: *A linear equation involving X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij} will be a consequence of equations* (21) *if, and only if, the coefficient of X_i equals that of Y_{jk}, the coefficient of Y_i equals that of X_{jk}, the coefficients of Z_{11}, Z_{22}, Z_{33} are all equal, and the coefficient of each $Z_{r,s}$ $(r, s = 1, 2, 3;\ r \neq s)$ is zero; where i, j, k is any cyclic permutation of $1, 2, 3$.*

When operating cogrediently upon the variables (20), the transformation

$$S: \begin{cases} \xi_i' = a_{i0}\xi_0 + a_{i1}\xi_1 + a_{i2}\xi_2 + a_{i3}\xi_3 + \gamma_{i1}\eta_1 + \gamma_{i2}\eta_2 + \gamma_{i3}\eta_3 & (i = 0, 1, 2, 3) \\ \eta_i' = \beta_{i0}\xi_0 + \beta_{i1}\xi_1 + \beta_{i2}\xi_2 + \beta_{i3}\xi_3 + \delta_{i1}\eta_1 + \delta_{i2}\eta_2 + \delta_{i3}\eta_3 & (i = 1, 2, 3) \end{cases}$$

* Other equations may be derived from (21); for example,

$$Y_{12}X_{12} + Y_{23}X_{23} + Y_{31}X_{31} = 0, \quad Y_{12}^2 + Z_{32}X_{32} + Z_{31}X_{31} = 0.$$

Since they are quadratic in each set of variables $\xi, \eta; \bar{\xi}, \bar{\eta}$, they do not enter into the discussion of the invariance of equations (21).

replaces the function X_l by the expression

$$\sum_{i=1}^{3} \begin{vmatrix} a_{00} & a_{0i} \\ a_{l0} & a_{li} \end{vmatrix} X_i + \sum_{i=1}^{3} \begin{vmatrix} a_{00} & \gamma_{0i} \\ a_{l0} & \gamma_{li} \end{vmatrix} Y_i + \sum_{i,j}^{1,2,3} \begin{vmatrix} a_{0i} & \gamma_{0j} \\ a_{li} & \gamma_{lj} \end{vmatrix} Z_{ij}$$
$$+ \sum_{\substack{i,j=1,2,3 \\ i<j}} \left\{ \begin{vmatrix} a_{0i} & a_{0j} \\ a_{li} & a_{lj} \end{vmatrix} X_{ij} + \begin{vmatrix} \gamma_{0i} & \gamma_{0j} \\ \gamma_{li} & \gamma_{lj} \end{vmatrix} Y_{ij} \right\}.$$

Let l, m, n be any cyclic permutation of $1, 2, 3$. Then S replaces Y_{mn} by

$$\sum_{i=1}^{3} \begin{vmatrix} \beta_{m0} & \beta_{mi} \\ \beta_{n0} & \beta_{ni} \end{vmatrix} X_i + \sum_{i=1}^{3} \begin{vmatrix} \beta_{m0} & \delta_{mi} \\ \beta_{n0} & \delta_{ni} \end{vmatrix} Y_i + \sum_{i,j} \begin{vmatrix} \beta_{mi} & \delta_{mj} \\ \beta_{ni} & \delta_{nj} \end{vmatrix} Z_{ij}$$
$$+ \sum_{i,j} \left\{ \begin{vmatrix} \beta_{mi} & \beta_{mj} \\ \beta_{ni} & \beta_{nj} \end{vmatrix} X_{ij} + \begin{vmatrix} \delta_{mi} & \delta_{mj} \\ \delta_{ni} & \delta_{nj} \end{vmatrix} Y_{ij} \right\}.$$

Hence S replaces the equation $X_l + Y_{mn} = 0$ by an equation which is linear in X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij}. Applying the above lemma, we see that the resulting equation will be a consequence of equations (21) if, and only if, the conditions (23), (24), (25) below are satisfied. Similarly, S replaces the equation $Y_l + X_{mn} = 0$ by an equation which is linear in X_i, Y_i, X_{ij}, Y_{ij}, Z_{ij} with coefficients obtained from the corresponding coefficients in the earlier equation by interchanging a_{r0} with β_{r0}, a_{ri} with β_{ri}, γ_{ri} with δ_{ri}, for $r = 1, 2, 3$; $i = 1, 2, 3$. In view of the lemma, the resulting equation will be a consequence of equations (21), if, and only if, the conditions (26), (27), (28) are satisfied.

(23) $$\begin{vmatrix} a_{00} & a_{0i} \\ a_{l0} & a_{li} \end{vmatrix} + \begin{vmatrix} \beta_{m0} & \beta_{mi} \\ \beta_{n0} & \beta_{ni} \end{vmatrix} = \begin{vmatrix} \gamma_{0j} & \gamma_{0k} \\ \gamma_{lj} & \gamma_{lk} \end{vmatrix} + \begin{vmatrix} \delta_{mj} & \delta_{mk} \\ \delta_{nj} & \delta_{nk} \end{vmatrix}.$$

(24) $$\begin{vmatrix} a_{00} & \gamma_{0i} \\ a_{l0} & \gamma_{li} \end{vmatrix} + \begin{vmatrix} \beta_{m0} & \delta_{mi} \\ \beta_{n0} & \delta_{ni} \end{vmatrix} = \begin{vmatrix} a_{0j} & a_{0k} \\ a_{lj} & a_{lk} \end{vmatrix} + \begin{vmatrix} \beta_{mj} & \beta_{mk} \\ \beta_{nj} & \beta_{nk} \end{vmatrix}.$$

(25) $$\begin{vmatrix} a_{0i} & \gamma_{0j} \\ a_{li} & \gamma_{lj} \end{vmatrix} + \begin{vmatrix} \beta_{mi} & \delta_{mj} \\ \beta_{ni} & \delta_{nj} \end{vmatrix} \equiv C_{ij}, \quad \begin{cases} C_{11} = C_{22} = C_{33}. \\ C_{ij} = 0 \end{cases} \qquad (i, j = 1, 2, 3;\ i \neq j).$$

(26) $$\begin{vmatrix} a_{00} & a_{0i} \\ \beta_{l0} & \beta_{li} \end{vmatrix} + \begin{vmatrix} a_{m0} & a_{mi} \\ a_{n0} & a_{ni} \end{vmatrix} = \begin{vmatrix} \gamma_{0j} & \gamma_{0k} \\ \delta_{lj} & \delta_{lk} \end{vmatrix} + \begin{vmatrix} \gamma_{mj} & \gamma_{mk} \\ \gamma_{nj} & \gamma_{nk} \end{vmatrix}.$$

(27) $$\begin{vmatrix} a_{00} & \gamma_{0i} \\ \beta_{l0} & \delta_{li} \end{vmatrix} + \begin{vmatrix} a_{m0} & \gamma_{mi} \\ a_{n0} & \gamma_{ni} \end{vmatrix} = \begin{vmatrix} a_{0j} & a_{0k} \\ \beta_{lj} & \beta_{lk} \end{vmatrix} + \begin{vmatrix} a_{mj} & a_{mk} \\ a_{nj} & a_{nk} \end{vmatrix}.$$

(28) $$\begin{vmatrix} a_{0i} & \gamma_{0j} \\ \beta_{li} & \delta_{lj} \end{vmatrix} + \begin{vmatrix} a_{mi} & \gamma_{mj} \\ a_{ni} & \gamma_{nj} \end{vmatrix} = d_{ij}, \quad \begin{cases} d_{11} = d_{22} = d_{33}, \\ d_{ij} = 0 \end{cases} \qquad (i, j = 1, 2, 3;\ i \neq j).$$

The formulæ hold for any cyclic permutations i, j, k; l, m, n of $1, 2, 3$.

The transformations S of the group H may be defined by the conditions (9), (10), (11), (12) *and* (23)–(28).

Among the transformations of H occur the simple types: *

$$Q_{i,j,\tau},\quad T_{i,\tau-2}T_{j,\tau}T_{k,\tau},\quad X_{0,i,\tau}V_{j,k,\tau},\quad Y_{0,i,\tau}W_{j,k,\tau}, \tag{29}$$

where τ is arbitrary in the field F, and i, j, k form any cyclic permutation of $1, 2, 3$. From them we obtain, as in § 6,

$$T_{i,-1}P_{ij} \equiv P_{ij}T_{j,-1} = Q_{j,i,-1}Q_{i,j,1}Q_{j,i,-1};$$

$$\begin{aligned} &X_{0,1,1}V_{2,3,1}\cdot Y_{0,1,1}W_{2,3,1}\cdot X_{0,1,1}V_{2,3,1} \\ &\quad = X_{0,1,1}Y_{0,1,1}X_{0,1,1}\cdot V_{2,3,1}W_{2,3,1}V_{2,3,1} \\ &\quad = C_0(\xi_1\eta_1)T_{1,-1}\cdot T_{2,-1}P_{23}S_{23} = (\xi_1\eta_1)(\xi_2\eta_2)(\xi_3\eta_3)C_0T_{1,-1}T_{2,-1}P_{23}, \end{aligned}$$

upon applying formulæ (16), (17), the latter with $\mu = 1$, and with $2, 3$ in place of $1, 2$. Since $T_{2,-1}P_{23}$ belongs to H, it follows that H contains

$$\Sigma \equiv (\xi_1\eta_1)(\xi_2\eta_2)(\xi_3\eta_3)C_0T_{1,-1}. \tag{30}$$

The transformations $T_{i,\tau}T_{j,\tau-1}$ leave invariant g_3 and the equations (21), and therefore belong to the group H. From the transformations (29) we readily derive $T_{i,\tau}T_{j,\tau-1}$, for the case in which τ is a cube in the field.

THEOREM: *The transformations of H of determinant $+1$ form a subgroup H' of index 2 which is generated by the transformations*

$$Q_{i,j,\tau},\quad T_{i,\tau}T_{j,\tau-1},\quad X_{0,i,\tau}V_{j,k,\tau},\quad Y_{0,i,\tau}W_{j,k,\tau}. \tag{31}$$

Let S be an arbitrary transformation of H and let it be exhibited in the above form. We are to prove that there exists a transformation K derived from the types (31) such that $KS = I$, the identity, or $C_0T_{3,-1}$, the latter of determinant -1.

We may assume that $a_{11} \neq 0$ in S. For, if $a_{1i} \neq 0$, the product $T_{i,-1}P_{i1}S$ has $a_{11} \neq 0$; if $\gamma_{1i} \neq 0$, the product ΣS has $a_{1i} \neq 0$; while the case $a_{1i} = \gamma_{1i} = 0$ $(i = 1, 2, 3)$ is excluded, since then $a_{10} = 0$ by (14). The product $S_1 \equiv T_{1,a_{11}^{-1}}T_{2,a_{11}}S$ replaces ξ_1 by a function of the form

$$f \equiv a_{10}\xi_0 + \xi_1 + a_{12}\xi_2 + a_{13}\xi_3 + \gamma_{11}\eta_1 + \gamma_{12}\eta_2 + \gamma_{13}\eta_3.$$

Then $Y_{0,2,\gamma_{13}}W_{3,1,\gamma_{13}}S_1 \equiv S_2$ replaces ξ_1 by a function of the form f with $\gamma_{13} = 0$. Then $Q_{1,3,-a_{13}}S_2 \equiv S_3$ replaces ξ_1 by a function f with $\gamma_{13} = a_{13} = 0$. Next, $Q_{1,2,-a_{12}}S_3 \equiv S_4$ replaces ξ_1 by a function

$$f' \equiv a_{10}\xi_0 + \xi_1 + \gamma_{11}\eta_1 + \gamma_{12}\eta_2.$$

Then, by (14), we have $\frac{1}{4}a_{10}^2 + \gamma_{11} = 0$. If $\gamma_{11} \neq 0$, $Q_{2,1,\kappa}S_4 \equiv S_5$ replaces ξ_1

* They correspond respectively to the generators X_{ij}, X_{ii}, X_{i0}, $-X_{0i}$ of CARTAN, p. 146.

by a function of the form f' with $\gamma_{12}-\kappa\gamma_{11}$ in place of γ_{12}. By a proper choice of κ, we make the new γ_{12} equal to zero. Hence S_5 replaces ξ_1 by

$$\xi_1 + a_{10}\xi_0 - \tfrac{1}{4}a_{10}^2\eta_1,$$

so that $(X_{0,1,\frac{1}{2}a_{10}}V_{2,3,\frac{1}{2}a_{10}})^{-1}S_5$ leaves ξ_1 unaltered. But, if $\gamma_{11}=0$, then $f' \equiv \xi_1 + \gamma_{12}\eta_2$, so that $Y_{0,3,-\gamma_{12}}W_{1,2,-\gamma_{12}}S_4$ leaves ξ_1 unaltered.

Consider a transformation S' which replaces ξ_1 by ξ_1, and η_1 by

$$f_1 \equiv \beta_{10}\xi_0 + \beta_{11}\xi_1 + \eta_1 + \beta_{12}\xi_2 + \beta_{13}\xi_3 + \delta_{12}\eta_2 + \delta_{13}\eta_3.$$

Then, by applying in succession as left-hand multipliers $Q_{3,1,\delta_{13}}$, $X_{0,2,\beta_{13}}V_{3,1,\beta_{13}}$, $Q_{2,1,\delta_{12}}$, we obtain a transformation S'' which replaces ξ_1 by ξ_1, and η_1 by $\beta_{10}\xi_0 + \beta_{11}\xi_1 + \eta_1 + \beta_{12}\xi_2$. By (15), $\beta_{11} = -\tfrac{1}{4}\beta_{10}^2$. If $\beta_{11}=0$, S'' replaces η_1 by $\eta_1 + \beta_{12}\xi_2$, so that

$$X_{0,3,-\beta_{12}}V_{1,2,-\beta_{12}}S''$$

leaves ξ_1 and η_1 unaltered. If $\beta_{11} \neq 0$,

$$(Y_{0,1,\frac{1}{2}\beta_{10}}W_{2,3,\frac{1}{2}\beta_{10}})^{-1}S'' \equiv S'''$$

replaces ξ_1 by ξ_1, and η_1 by $\eta_1 + \beta_{12}\xi_2 - \tfrac{1}{2}\beta_{10}\eta_3$. Then $Q_{3,1,-\frac{1}{2}\beta_{10}}S'''$ replaces ξ_1 by ξ_1, and η_1 by $\eta_1 + \beta_{12}\xi_2$, a case just considered.

It remains to discuss those substitutions S of H which do not alter ξ_1 and η_1 and therefore have (compare § 7)

$$a_{11} = \delta_{11} = 1, \quad a_{12} = a_{13} = a_{21} = a_{31} = 0, \quad \delta_{12} = \delta_{13} = \delta_{21} = \delta_{31} = 0,$$

$$a_{01} = a_{10} = \beta_{10} = \gamma_{01} = 0, \quad \beta_{1i} = \beta_{i1} = \gamma_{1i} = \gamma_{i1} = 0 \quad (i = 1, 2, 3).$$

By (25), for $(l, i, j) = (1, 1, 2), (1, 1, 3), (2, 2, 1), (2, 3, 1), (3, 2, 1), (3, 3, 1)$, we get respectively

$$\gamma_{02} = 0, \quad \gamma_{03} = 0, \quad \beta_{32} = 0, \quad \beta_{33} = 0, \quad \beta_{22} = 0, \quad \beta_{23} = 0.$$

By (28), for $(l, i, j) = (1, 2, 1), (1, 3, 1), (2, 1, 2), (2, 1, 3), (3, 1, 2), (3, 1, 3)$, we get respectively

$$a_{02} = 0, \quad a_{03} = 0, \quad \gamma_{32} = 0, \quad \gamma_{33} = 0, \quad \gamma_{22} = 0, \quad \gamma_{23} = 0.$$

By (24), for $(l, i) = (3, 1)$, $\beta_{20} = 0$; for $(l, i) = (2, 1)$, $\beta_{30} = 0$. By (26), for $(l, i) = (3, 1)$, $a_{20} = 0$; for $(l, i) = (2, 1)$, $a_{30} = 0$. By (27), for $(l, i) = (3, 2), (2, 3), (3, 3), (2, 2)$, we get respectively

$$-a_{23} = a_{00}\delta_{32}, \quad -a_{32} = a_{00}\delta_{23}, \quad a_{22} = a_{00}\delta_{33}, \quad a_{33} = a_{00}\delta_{22}.$$

Finally, by (23), for $(l, i) = (1, 1)$, and by (9), we get respectively

$$(32) \qquad \begin{vmatrix} \delta_{22} & \delta_{23} \\ \delta_{32} & \delta_{33} \end{vmatrix} = a_{00}, \quad a_{00}^2 = 1.$$

Hence S has the following form:

$$\xi_0' = a_{00}\xi_0, \quad \eta_2' = \delta_{22}\eta_2 + \delta_{23}\eta_3, \quad \xi_2' = a_{00}\delta_{33}\xi_2 - a_{00}\delta_{32}\xi_3,$$
$$\eta_3' = \delta_{32}\eta_2 + \delta_{33}\eta_3, \quad \xi_3' = -a_{00}\delta_{23}\xi_2 + a_{00}\delta_{22}\xi_3,$$

subject to relations (32). Its determinant equals

$$a_{00}^3 \begin{vmatrix} \delta_{22} & \delta_{23} \\ \delta_{32} & \delta_{33} \end{vmatrix}^2 = a_{00} = \pm 1.$$

Denote by S_1 the preceding substitution S for the case $a_{00} = +1$. For $a_{00} = -1$, $S = S' C_0 T_{3,-1}$, where S' is of the form S_1, of determinant $+1$.

The general substitution S_1 has the form

$$\eta_2' = \kappa\eta_2 + \lambda\eta_3, \quad \eta_3' = \mu\eta_2 + \nu\eta_3, \quad \xi_2' = \nu\xi_2 - \mu\xi_3, \quad \xi_3' = -\lambda\xi_2 + \kappa\xi_3, \tag{33}$$
$$\kappa\nu - \lambda\mu = 1.$$

If $\kappa = \nu = 0$, then $S_1 = T_{2,\mu-1} T_{3,\mu} T_{3,-1} P_{23}$. If κ, ν are not both zero, we may take $\nu \neq 0$; for, if $\nu = 0$, $\kappa \neq 0$, the transform of S_1 by $T_{3,-1}P_{23}$ has $\nu \neq 0$. Then $Q_{2,3,\mu\nu-1} S_1$ may be expressed as the product

$$Q_{3,2,-\lambda\nu-1} T_{2,\nu} T_{3,\nu-1}.$$

THEOREM.—*The group H' in the $GF[p^n]$ has the order* *

$$\Omega \equiv p^{6n}(p^{6n}-1)(p^{2n}-1). \tag{34}$$

It has been shown that H' contains a transformation which replaces ξ_1 by

$$\sum_{j=0}^{3} a_{1j}\xi_j + \sum_{j=1}^{3} \gamma_{1j}\eta_j,$$

in which $a_{10}, \cdots, \gamma_{13}$ are arbitrary marks, not all zero, such that

$$\tfrac{1}{4}a_{10}^2 + a_{11}\gamma_{11} + a_{12}\gamma_{12} + a_{13}\gamma_{13} = 0. \tag{35}$$

By various methods, this equation is seen to have p^{6n} sets of solutions in the $GF[p^n]$. Hence $\Omega = p^{6n}\Omega_1$, where Ω_1 denotes the number of transformations of H' which leave ξ_1 fixed. As shown above, H' contains a transformation which leaves ξ_1 fixed and replaces η_1 by f_1', in which $\beta_{10}, \cdots, \delta_{13}$ are any marks which satisfy (15), viz.;

$$\tfrac{1}{4}\beta_{10}^2 + \beta_{11} + \beta_{12}\delta_{12} + \beta_{13}\delta_{13} = 0. \tag{36}$$

Hence $\beta_{10}, \beta_{12}, \delta_{12}, \beta_{13}, \delta_{13}$ may be chosen arbitrarily, the value of β_{11} being then determined uniquely. Hence $\Omega_1 = p^{5n}\Omega_2$, where Ω_2 denotes the number of transformations S_1 of H' which leaves ξ_1 and η_1 unaltered. As shown above,

* As a check, we note that p^n, the order of the field, enters to the power 14, which is the number of parameters in the continuous group.

S_1 must be of the form (33). Inversely, all the $p^n(p^{2n}-1)$ transformations (33) belong to H'. Hence the order of H' is given by (34).

We observe the special values:

$$p^n = 3, \quad \Omega = 2^6 \cdot 3^6 \cdot 7 \cdot 13 \equiv 4{,}245{,}696;$$

$$p^n = 5, \quad \Omega = 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31 \equiv 5{,}859{,}000{,}000.$$

These simple groups are not in the writer's list of known simple groups. *

§ 10. *Simplicity of the group H'.*

Suppose that H' contains an invariant subgroup J which possesses a transformation S not the identity I. Let S replace ξ_1 by

$$f_1 \equiv a_{10}\xi_0 + \sum_{j=1}^{3}(a_{1j}\xi_j + \gamma_{1j}\eta_j) \qquad \text{[subject to (35)]}.$$

LEMMA: *J contains a transformation $\neq I$ which multiplies ξ_1 by a constant.*

(*a*) Let first $\gamma_{11} \neq 0$. It was shown in § 9 that H' contains a transformation R which replaces ξ_1 by ξ_1 and η_1 by

$$\beta_{10}\xi_0 + \beta_{11}\xi_1 + \eta_1 + \beta_{12}\xi_2 + \delta_{12}\eta_2 + \beta_{13}\xi_3 + \delta_{13}\eta_3,$$

in which β_{1i}, δ_{1i} are any quantities of the field F satisfying (36). By suitable choice of these quantities, the product

$$P \equiv T_{1,\,\gamma_{11}^{-1}} T_{2,\,\gamma_{11}} R$$

replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by f_1. Hence J contains

$$S_1 \equiv P^{-1}SP$$

which replaces ξ_1 by $\gamma_{11}^{-1}{}_1\eta_1$. If H' contains a transformation B leaving ξ_1 and η_1 unaltered and not commutative with S_1, then J contains $S_1^{-1}B^{-1}S_1B$, which leaves ξ_1 fixed and is not the identity. In the contrary case S_1 is commutative with $Q_{3,\,2,\,\lambda}$, $T_{2,\,-1}P_{23}$. Equating the functions by which $S_1Q_{3,\,2,\,\lambda}$ and $Q_{3,\,2,\,\lambda}S_1$ replace η_2 and the functions by which they replace ξ_3, we see that S_1 must replace ξ_2 and η_3 by respectively

$$\xi_2' = a\xi_2 + b\eta_3, \quad \eta_3' = c\eta_3 - d\xi_2.$$

Since S_1 is to be commutative with $T_{2,\,-1}P_{23}$, it replaces η_2 and ξ_3 by

$$\eta_2' = c\eta_2 + d\xi_3, \quad \xi_3' = a\xi_3 - b\eta_2.$$

Then (23), for $l = 1$, $i = 1$, gives $c = 0$. Then J contains

$$S_1^{-1}(T_{1,\,\mu}T_{2,\,\mu-1})^{-1}S_1(T_{1,\,\mu}T_{2,\,\mu-1})$$

* Bulletin of the American Mathematical Society, ser. 2, vol. 5 (1899), p. 470–475.

which replaces η_2 by $\mu\eta_2$. Its transform by $T_{1,-1}P_{12}\Sigma$ replaces ξ_1 by $\mu\xi_1$ and is not the identity if $\mu \neq 0, 1$, where Σ is defined by (30).

(*b*) Let $\gamma_{11} = 0$, but a_{12} and a_{13} not both zero. By an evident transformation within H', we may make $a_{12} = 1$. Transforming S by

$$Y_{0,1,\gamma_{13}} W_{2,3,\gamma_{13}} \cdot Q_{2,3,a_{13}},$$

we reach in J a transformation S' which replaces ξ_1 by

$$a_{10}\xi_0 + a_{11}\xi_1 + \xi_2 + \gamma_{12}\eta_2 \qquad (\tfrac{1}{4}a_{10}^2 + \gamma_{12} = 0).$$

Then the transform of S' by $Q_{2,1,a_{11}}$ replaces ξ_1 by $\xi_2 + a_{10}\xi_0 - \frac{1}{4}a_{10}^2\eta_2$. Transforming by $X_{0,2,\frac{1}{2}a_{10}} V_{3,1,\frac{1}{2}a_{10}}$, we reach in J a transformation S_2 which replaces ξ_1 by ξ_2. Then J contains

$$S_2^{-1} \cdot T_{2,-1}T_{3,-1}S_2T_{2,-1}T_{3,-1}, \tag{37}$$

which replaces ξ_1 by $-\xi_1$.

(*c*) Let $\gamma_{11} = a_{12} = a_{13} = 0$. Then $a_{10} = 0$ by (35). If $\gamma_{12} = \gamma_{13} = 0$, S replaces ξ_1 by $a_{11}\xi_1$. In the contrary case, S is conjugate with a transformation S' with $\gamma_{12} = 1$; it replaces ξ_1 by $a_{11}\xi_1 + \eta_2 + \gamma_{13}\eta_3$. The transform of S' by $Q_{3,2,-\gamma_{13}}$ replaces ξ_1 by $a_{11}\xi_1 + \eta_2$. Transforming it by $X_{0,3,-a_{11}} V_{1,2,-a_{11}}$, we obtain a transformation S_2 which replaces ξ_1 by η_2. Then J contains (37), which replaces ξ_1 by $-\xi_1$.

LEMMA. *The group J contains a transformation, not the identity, which does not alter ξ_1 or η_1.*

In view of the preceding lemma, J contains a transformation $S \neq I$ which replaces ξ_1 by $a\xi_1$ and η_1 by

$$\beta_{10}\xi_0 + \sum_{j=1}^{3} (\beta_{1j}\xi_j + \delta_{1j}\eta_j) \qquad [\text{subject to (15)}; \delta_{11} = a^{-1}].$$

According as δ_{12}, δ_{13} are both zero or not both zero, β_{12}, β_{13} both zero or not both zero, we distinguish four cases. Transforming by one or more of the transformations $T_{2,-1}P_{23}$, $Q_{2,3,\lambda}$, $Q_{3,2,\lambda}$, $T_{2,\lambda}T_{3,\lambda^{-1}}$, we obtain a transformation in J which replaces ξ_1 by $a\xi_1$ and η_1 by one of the four functions

$$f \equiv \beta_{10}\xi_0 + \beta_{11}\xi_1 + a^{-1}\eta_1, \quad f + \xi_2, \quad f + \beta_{12}\xi_2 + \eta_2, \quad f + \beta_{13}\xi_3 + \eta_2.$$

Let first S replace ξ_1 by f. By (15), $a^{-1}\beta_{11} + \frac{1}{4}\beta_{10}^2 = 0$. Hence

$$S = T_{1,a}T_{2,a^{-1}}Y_{0,1,\beta}W_{2,3,\beta}S_1 \qquad (\beta \equiv \tfrac{1}{2}\beta_{10}),$$

where S_1 leaves ξ_1 and η_1 fixed, belongs to H', and hence has the form (33). If $\beta = 0$, the lemma is proved when $a = 1$. For $\beta = 0$, $a \neq 1$, S_1 does not reduce to the identity I or to $T_{2,-1}T_{3,-1}$ and hence is not commutative with every (33). If S_1 be not commutative with Σ_1 of the form (33), then J contains

$$S^{-1}\Sigma_1^{-1}S\Sigma_1 \equiv S_1^{-1}\Sigma_1^{-1}S_1\Sigma_1 \neq I$$

which leaves ξ_1 and η_1 unaltered. Let next $\beta \neq 0$. Then S is transformed by $T_{2,-1}P_{23}$ into $S_2 \equiv T_{1,\alpha}T_{3,\alpha^{-1}}Y_{0,1,\beta}W_{2,3,\beta}S'$, where

$$S' \equiv P_{23}T_{2,-1}S_1T_{2,-1}P_{23}.$$

Hence J contains the transformation

$$S_2' \equiv T_{2,\alpha}T_{3,\alpha^{-1}}S^{-1}T_{2,\alpha^{-1}}T_{3,\alpha}S_2 \equiv T_{2,\alpha}T_{3,\alpha^{-1}}S_1^{-1}S',$$

which leaves ξ_1 and η_1 unaltered. If S_2' be the identity, $\nu = a\kappa$, $\mu = -a\lambda$ in (33). Again, $T_{1,\alpha^{-1}}T_{2,\alpha}$ transforms S into $S_3 \equiv Y_{0,1,\beta}W_{2,3,\beta}S_1T_{1,\alpha^{-1}}T_{2,\alpha}$. Then J contains the product

$$S_4 \equiv S_3^{-1}Q_{2,3,1}^{-1}S_3Q_{2,3,1} \equiv T_{2,\alpha^{-1}}S_1^{-1}Q_{2,3,1}^{-1}S_1T_{2,\alpha}Q_{2,3,1},$$

since $Q_{2,3,\tau}$ and $W_{2,3,\beta}$ are commutative. Now S_4 leaves ξ_1 and η_1 fixed. If S_4 reduce to the identity, we find that $\lambda = 0$, $\kappa = a\nu$ in (33). Hence if S_2' and S_4 are both the identity, (33) becomes $T_{2,\kappa^{-1}}T_{3,\kappa}$. In this case J contains

$$T_{1,-1}T_{2,-1}S^{-1}T_{1,-1}T_{2,-1}S \equiv T_{2,\kappa}T_{3,\kappa^{-1}}Y_{0,1,2\beta}W_{2,3,2\beta}T_{2,\kappa^{-1}}T_{3,\kappa},$$

and therefore its transform by $T_{2,\kappa}T_{3,\kappa^{-1}}$, giving $Y_{0,1,2\beta}W_{2,3,2\beta}$. Hence J contains every $Y_{0,1,\tau}W_{2,3,\tau}$ and $X_{0,1,\tau}V_{2,3,\tau}$. As in § 9, we derive $\Sigma' = \Sigma T_{2,-1}P_{23}$, where Σ is given by (30). Transforming Σ' by $P_{23}T_{2,-1}$, we reach $T_{2,-1}P_{23}\Sigma$. The product of the two gives $T_{2,-1}T_{3,-1}$, which leaves ξ_1 and η_1 unaltered and is not the identity.

The remaining three cases may be treated in a similar manner.

In view of the two lemmas, the group J contains a transformation, not the identity, which does not alter ξ_1 or η_1 and hence (§ 9) has the form (33). If it be $T_{2,-1}T_{3,-1}$, it is transformed into $Q_{1,2,2\lambda}T_{2,-1}T_{3,-1}$ by $Q_{1,2,-\lambda}$. Hence would J contain $Q_{1,2,2\lambda}$ and therefore $Q_{2,3,2\lambda}$. It follows that J contains a transformation (33) neither I nor $T_{2,-1}T_{3,-1}$. But the transformations (33) form a group holoedrically isomorphic with the binary group $SLH(2, F)$. It follows that J contains all the transformations (33) and hence every $Q_{i,j,\tau}$, $T_{i,-1}P_{i,j}$, $T_{i,\mu}T_{j,\mu-1}$. But $X_{0,3,\tau}V_{1,2,\tau}$ transforms $T_{2,-1}T_{3,-1}$ into $X_{0,3,-2\tau}V_{1,2,-2\tau}T_{2,-1}T_{3,-1}$. Hence J contains every $X_{0,3,\lambda}V_{1,2,\lambda}$. Transforming it by suitable $T_{i,-1}P_{i,j}$ and by Σ, we reach every $X_{0,i,\tau}V_{j,k,\tau}$, $Y_{0,i,\tau}W_{j,k,\tau}$. Hence $J \equiv H'$, so that H' is simple in any field F not having the modulus $p = 2$.

§ 11. *Linear groups with invariants of degree $d > 2$.*

Consider the group $G(q, r, F)$ of linear transformations S on rq variables with coefficients in an arbitrary field F which leave formally and absolutely invariant the function

$$\sum_{i=1}^{r} \xi_{i1}\xi_{i2}\cdots\xi_{iq}.$$

For $q > 2$, we may express every S as a product, $S = AB$, where A merely multiplies each variable ξ_{ij} by a constant a_{ij}, while B is a substitution on the letters ξ_{ij} having the imprimitive systems*

$$\xi_{11}, \xi_{12}, \cdots, \xi_{1q}; \quad \xi_{21}, \xi_{22}, \cdots, \xi_{2q}; \quad \cdots; \quad \xi_{r1}, \xi_{r2}, \cdots, \xi_{rq}.$$

The transformations A form a commutative group invariant under $G(q, r, F)$. The quotient-group $\{B\}$ has an invariant subgroup R, the direct product of r symmetric groups, the general one being the symmetric group on the q letters $\xi_{i1}, \xi_{i2}, \cdots, \xi_{iq}$. The quotient-group $\{B\}/R$ is the symmetric group on r letters, the above imprimitive systems.

Consider the group $H(m, r, F)$ of linear transformations S on m variables with coefficients in a field F, not possessing a modulus, which leave formally and absolutely invariant the function

$$\lambda_1 \xi_1^r + \lambda_2 \xi_2^r + \cdots + \lambda_m \xi_m^r \qquad (\text{each } \lambda_i \neq 0 \text{ in } F).$$

If $r > 2$, we may set $S = AL$, where A is a transformation of the form

$$\xi_i' = a_i \xi_i \quad [a_i^r = 1] \qquad (i = 1 \cdots, m),$$

and L is a literal substitution on the letters $\xi_1, \xi_2, \cdots, \xi_m$.†

On the other hand, there exist linear groups in an arbitrary field F which possess invariants of degree $d > 2$ and which lead to simple groups. Examples of such groups are furnished by the second compounds of the groups $GLH(m, F)$ and $GA(2m, F)$, each possessing an invariant Pfaffian (see § 6).

§ 12. *Canonical forms of linear homogeneous transformations.*

Consider a transformation with coefficients in a field F,

$$S: \qquad \xi_i' = a_{i1}\xi_1 + a_{i2}\xi_2 + \cdots + a_{im}\xi_m \qquad (i = 1, 2, \cdots, m).$$

The determination of a linear function which S multiplies by a constant K depends upon the *characteristic equation*

$$\Delta(K) \equiv \begin{vmatrix} a_{11} - K & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} - K & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} - K \end{vmatrix} = 0.$$

If we introduce new variables defined by the transformation

$$T: \qquad \eta_i \equiv \beta_{i1}\xi_1 + \beta_{i2}\xi_2 + \cdots + \beta_{im}\xi_m \qquad (i = 1, 2, \cdots, m),$$

---

*Proceedings of the London Mathematical Society, vol. 30 (1899), pp. 200–208. The factor C should be $t_1! \, t_2! \cdots t_\tau!$ For an elementary treatment of the case $q = 3$, see L. G., §§ 211–212.

†Mathematische Annalen, vol. 52, p. 563; L. G., §§ 139–141.

the transformation S becomes the transformation $S_1 \equiv T^{-1}ST$ on the variables η_i and the characteristic determinant $\Delta(K)$ of S_1 is equal to the characteristic determinant of S (L. G., §§ 101, 102). *The characteristic equation is unaltered under linear transformation.*

Let $\Delta(K)$ be decomposed in the field F into irreducible factors,

$$\Delta(K) \equiv [F_k(K)]^\alpha [F_l(K)]^\beta \cdots \qquad (k\alpha + l\beta + \cdots = m).$$

Let $F_k(K) = 0$ have the roots $K_1, K_2, \cdots, K_k$; $F_l(K) = 0$ the roots $L_1, L_2, \cdots, L_l$; etc. To exhibit compactly the general type of canonical form of transformations S, let $\alpha, \beta, \cdots$ be partitioned into positive integers,

$$\alpha = a_1 + a_2 + \cdots + a_{r+1}, \quad \beta = b_1 + b_2 + \cdots + b_{s+1}, \quad \cdots$$

Let $a, b, \cdots$, denote an arbitrary one of the respective sets of integers

$$(a) \qquad 1, \quad a_1 + 1, \quad a_1 + a_2 + 1, \quad \cdots, \quad a_1 + a_2 + \cdots + a_r + 1;$$

$$(b) \qquad 1, \quad b_1 + 1, \quad b_1 + b_2 + 1, \quad \cdots, \quad b_1 + b_2 + \cdots + b_s + 1;$$

etc. Let A denote an arbitrary positive integer $\geqq a$ and not an a; let B denote an arbitrary positive integer $\geqq \beta$ and not a b; etc.

Proceeding as in L. G., §§ 214–218, we obtain the theorems:

By a suitable linear homogeneous transformation of variables (not belonging to F in general), S can be reduced to a canonical form

$$\eta'_{ia} = K_i \eta_{ia}, \quad \eta'_{iA} = K_i \eta_{iA} + K_i \eta_{iA-1} \qquad (i = 1, 2, \cdots, k)$$

$$\zeta'_{ib} = L_i \zeta_{ib}, \quad \zeta'_{iB} = L_i \zeta_{iB} + L_i \zeta_{iB-1} \qquad (i = 1, 2, \cdots, l)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

in which the new variables $\eta_{ij}, \zeta_{ij}, \cdots$ have the properties:

(1) *The variables $\eta_{1j}(j = 1, 2, \cdots, a)$ are linear homogeneous functions of the ξ_i whose coefficients are polynomials in K_1 with coefficients in F;*

(2) *The variables η_{ij} are obtained from the η_{1j} by replacing K_1 by K_i;*

(3) *The variables $\zeta_{1j}(j = 1, 2, \cdots, \beta)$ are linear functions of the ξ_i whose coefficients are polynomials in L_1 with coefficients in F;*

(4) *The variables ζ_{ij} are obtained from the ζ_{1j} by replacing L_1 by L_i;*

(5) *The ka variables $\eta_{ij}(i = 1, \cdots, k; j = 1, \cdots, a)$ may be replaced by ka linear homogeneous functions y_{ij} of the ξ_i with coefficients in F, such that S replaces each y_{ij} by a linear function of the y_{ij} with coefficients in F; similarly for the $l\beta$ variables ζ_{ij}; etc.*

Two linear homogeneous transformations S_1 and S_2 belonging to a field F have the same canonical form if, and only if, S_1 is the transform of S_2 by a linear homogeneous transformation T in the field F and on the same variables.

To determine all linear homogeneous transformations T commutative with a given one S, each in the field F and affecting m variables ξ_i, we apply the transformation of indices which reduces S to its canonical form S',

$$S' \equiv Y_1 Y_2 \cdots Y_k Z_1 Z_2 \cdots Z_l \cdots,$$

where each transformation Y_i, $Z_i, \cdots$, is defined thus:

$$Y_i: \quad \eta'_{ia} = K_i \eta_{ia}, \quad \eta'_{iA} = K_i \eta_{iA} + K_i \eta_{iA-1} \qquad \text{(for every } a, A);$$

$$Z_i: \quad \zeta'_{ib} = L_i \zeta_{ib}, \quad \zeta'_{iB} = L_i \zeta_{iB} + L_i \zeta_{iB-1} \qquad \text{(for every } b, B).$$

If T' be commutative with S', then

$$T' \equiv Y'_1 Y'_2 \cdots Y'_k Z'_1 Z'_2 \cdots Z'_l \cdots,$$

where Y'_i, Z'_i, $\cdots$, are of the form

$$Y'_i: \quad \eta'_{ij} = \sum_{u=1}^{\alpha} \rho_{ju}(K_i)\, \eta_{iu} \qquad (j = 1, \cdots, \alpha);$$

$$Z'_i: \quad \zeta'_{ij} = \sum_{v=1}^{\beta} \sigma_{jv}(L_i)\, \zeta_{iv} \qquad (j = 1, \cdots, \beta),$$

the coefficients of the polynomials $\rho_{ju}(K_i), \cdots$, belonging to F.

Inversely, if T' have the above form and if Y'_1 be commutative with Y_1, Z'_1 commutative with $Z_1, \cdots$, then the transformation T (T' expressed in the in itial variables ξ_i) will be commutative with S and will have its coefficients in the field F.

Y'_1 and Y_1 are commutative if, and only if, for every a, A, A',

$$\rho_{aA} = 0, \quad \rho_{A-1\,a-1} = 0, \quad \rho_{A-1\,a} = 0, \quad \rho_{A-1\,A'-1} = \rho_{AA'}.$$

THE UNIVERSITY OF CHICAGO,
June 1, 1901.

ON CERTAIN AGGREGATES OF DETERMINANT MINORS*

BY

W. H. METZLER

Introduction and General Notations.

1. Since the announcement † by KRONECKER, in 1882, of his now well-known theorem regarding linear relations between the minors of an axisymmetric determinant various papers ‡ have appeared treating of the subject. Dr. MUIR in his paper of 1888 showed that a similar relation exists between the minors of a centrosymmetric determinant and in his paper of 1900 he gives the following two theorems:

THEOREM A: *If μ and ν be any integers, μ being the less, taken from the series $n, n+1, n+2, \cdots, 2n$ and $a, \beta, \gamma, \cdots, \omega$ be what the series becomes when μ is removed, and $a, \beta, \gamma, \cdots, \psi$ what it becomes when both are removed; then in connection with any even-ordered determinant $\left|\begin{smallmatrix}1&2&\cdots&2n\\1&2&\cdots&2n\end{smallmatrix}\right|$ we have*

$$\sum_{\mu}(-1)^{n-\mu}\begin{vmatrix}1\,2\,3\cdots\overline{n-1}\;\mu\\ a\,\beta\,\gamma\cdots\cdots\;\omega\end{vmatrix}=\sum_{\mu,\nu}(-1)^{3n-(\mu+\nu)}\begin{vmatrix}1\,2\,3\cdots\overline{n-1}\\ a\,\beta\,\gamma\cdots\;\psi\end{vmatrix}\binom{\mu}{\nu}\binom{\nu}{\mu}.$$

THEOREM B: *If the symbol*

$$\sum_{\mu=1}^{n}\begin{vmatrix}\overline{n+1}\;\overline{n+2}\cdots\overline{n-\mu+1}\cdots 2n\\ \overline{n+1}\;\overline{n+2}\cdots\;\overline{n+\mu}\;\cdots 2n\end{vmatrix}$$

stand for the sum of the n determinants whose column-indices are in every case

---

* Presented to the Society April 27, 1901. Received for publication August 31, 1901.

† KRONECKER, L., *Die Subdeterminanten symmetricher Systeme*, Berliner Berichte, 1882.

‡ RUNGE, C., *Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetricher Systeme*, Crelle's Journal, vol. 93.

MEHMKE, R., *Bemerkung über die Subdeterminanten symmetricher Systeme*, Mathematische Annalen, vol. 26.

SCHENDEL, L., *Die Kronecker'sche Subdeterminantensatz*, Zeitschrift für Mathematik und Physik, vol. 32.

MUIR, T., *On Vanishing Aggregates of Determinants*, Proceedings of the Royal Society of Edinburgh, 1888. *On Certain Aggregates of Determinant Minors*, ibid., 1900.

WHITE, H. S., *Kronecker's Linear Relation among Minors of a Symmetric Determinant*, Bulletin of the American Mathematical Society, 1896.

$n+1, n+2, \cdots, 2n$ *and whose row-indices are the same except that for one of them there has been substituted its defect from* $2n+1$; *and if*

$$\left|\begin{matrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1}' \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{matrix}\right|$$

be taken to indicate that in the determinant

$$\left|\begin{matrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{matrix}\right|$$

each element $\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}$ *of the* μ*th row is to be diminished by the element* $\begin{smallmatrix}2n+1-\alpha\\ 2n+1-\beta\end{smallmatrix}$; *then,*

$$\sum_{\mu=1}^{n} \left\{ \left|\begin{matrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{matrix}\right| - \left|\begin{matrix} \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1} \cdots 2n \end{matrix}\right| \right\}$$

$$= \sum_{\mu=1}^{n} \left|\begin{matrix} \overline{n+1}\ \overline{n+2} \cdots \overline{n-\mu+1}' \cdots 2n \\ \overline{n+1}\ \overline{n+2} \cdots \quad \overline{n+\mu} \quad \cdots 2n \end{matrix}\right|.$$

If the determinant $\left|\begin{smallmatrix}1\,2\cdots 2n\\ 1\,2\cdots 2n\end{smallmatrix}\right|$ is axisymmetric then $\begin{smallmatrix}\mu\\ \nu\end{smallmatrix} = \begin{smallmatrix}\nu\\ \mu\end{smallmatrix}$ and the right-hand side of theorem A vanishes, giving KRONECKER'S relation. If the determinant is centrosymmetric then the right-hand side of theorem B vanishes, giving the relation pointed out by MUIR in 1888. In this paper three theorems [(1), (2), (3), below] and two formulas [I, II, below] are developed. Theorems A and B, which may be characterized by the notations $A(n, n-1)$ and $B(n, 1)$ respectively, are extended, giving theorems (1) and (3), characterized by the notations $A(n, k)$ and $B(n, k)$, $(1 \leqq k \leqq n-1)$, respectively. If we call the left-hand side of a Kronecker relation a Kronecker expression, then theorem (2) expresses a sum of Kronecker expressions as a sum of products of sums of Kronecker expressions of lower order. Formula I gives for a centrosymmetric determinant the number of true vanishing aggregates of each type, and formula II gives the number of those which are linearly independent. Part I (arts. 3–7) contains the work relating to theorem A and Part II (arts. 8–11) contains the work relating to theorem B.

2. If we are given any combination (selection) of $2n$ numbers k at a time, the combination of the remaining $2n-k$ numbers is said to be the complementary with respect to $2n$ of the given combination. Let us use here, as elsewhere,*

* Cf. American Journal of Mathematics, vol. 22, no. 1.

$^{(2n\ k)}_{\alpha}$* to denote the αth combination k at a time of the $2n$ numbers $1, 2, \cdots, 2n$, $^{(2n|k|\ l)}_{\alpha\ \beta}$ to denote the βth combination l at a time of the numbers in the αth combination k at a time of the $2n$ numbers, and let $^{(2\bar{n}|k)}_{\alpha}$ denote the complementary with respect to $2n$ of the combination $^{(2n|k)}_{\alpha}$; and similarly $^{(2n|\bar{k}|\ l)}_{\alpha\ \beta}$ will denote the complementary with respect to k of the combination $^{(2n|k|\ l)}_{\alpha\ \beta}$, i. e., the combination formed by the numbers remaining after the numbers in the combination $^{(2n|k|\ l)}_{\alpha\ \beta}$ are taken from the combination $^{(2n|k)}_{\alpha}$. The symbol $^{(2\bar{n}|\bar{k}|\ l)}_{\alpha\ \beta}$ will denote the complementary of $^{(2n|k|\ l)}_{\alpha\ \beta}$ with respect to k, i. e., the combination formed by the $2n - k - l$ numbers remaining after the l numbers in the combination $^{(2\bar{n}|k|\ l)}_{\alpha\ \beta}$ are taken from the combination $^{(2\bar{n}|k)}_{\alpha}$. Similarly $(2\bar{n}\,|\bar{k}|\,\bar{l}\,|h\cdots)$ is the complementary of $(2n\,|\bar{k}|\,l\,|h\cdots)$ with respect to l.

If (a), (b) denote combinations of distinct numbers the notations $(a\,\backslash\!\backslash\,b)$ and $(a)\,(b)$ denote the combination of all the numbers involved in (a) and (b) taken respectively *in loco* or in natural order according as the brackets are or are not interlocked.

The notation s_t will be used for $s!/t!(s-t)!$

Part I.

3. In this part of the paper expressions occur containing minors and products of minors of the following form: $\left|\begin{smallmatrix}(x_1)\\(x_2)\end{smallmatrix}\right|$, $\left|\begin{smallmatrix}(y_1)\\(y_2)\end{smallmatrix}\right| \cdot \left|\begin{smallmatrix}(z_1)\\(z_2)\end{smallmatrix}\right|$, where $(x_1\,\backslash\!\backslash\,x_2)$ and $(y_1\,\backslash\!\backslash\,z_1\,\backslash\!\backslash\,y_2\,\backslash\!\backslash\,z_2)$ are permutations of the $2n$ numbers $1, 2, \cdots, 2n$. In general the sign factor for any term $\left|\begin{smallmatrix}(x_1)\\(x_2)\end{smallmatrix}\right| \cdot \left|\begin{smallmatrix}(y_1)\\(y_2)\end{smallmatrix}\right| \cdot \left|\begin{smallmatrix}(z_1)\\(z_2)\end{smallmatrix}\right| \cdots$ is $(-1)^{\nu}$ where ν is the number of inversions of order in $(x_1\,\backslash\!\backslash\,y_1\,\backslash\!\backslash\,z_1)\cdots(x_2\,\backslash\!\backslash\,y_2\,\backslash\!\backslash\,z_2)\cdots$.

With respect to a parent determinant of order $2n$ the sum of minors of order n

$$\sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n\,|k\,\backslash\!\backslash\,2\bar{n}\,|\bar{k}|\,n-k) \\ (2\bar{n}\,|k|\,n-k) \end{vmatrix},$$

which has k rows the same throughout may obviously be written in the form

$$\frac{1}{n_k} \sum_{i_1=1}^{(2n-k)_k} \sum_{i_2=1}^{(2n-2k)_{n-k}} (-1)^{\mu} \begin{vmatrix} (2n\,|k\,\backslash\!\backslash\,2n\,|\bar{k}|\,k|\,n-k) \\ (2\bar{n}\,|k|k\,\backslash\!\backslash\,2\bar{n}\,|\bar{k}|\,k|\,n-k) \end{vmatrix}$$

which, on expanding each term by Laplace's theorem in terms of minors formed from the αth selection of k rows with their complementaries, and finally collecting all the terms containing

$$\begin{vmatrix} (2n\,|\,k) \\ (2\bar{n}\,|\,k\,|\,k) \end{vmatrix}$$

* Unless otherwise stated, it is understood that the numbers in a combination are arranged in their natural order (order of magnitude).

as a factor, takes the form

$$\sum_{i_1=1}^{(2n-k)_k}(-1)^{\lambda}\begin{vmatrix}(2n\,|\,k)\\ \alpha\\ (2\overline{n}\,|\,\overline{k}\,|\,k)\\ \alpha\ i_1\end{vmatrix}\sum_{i_2=1}^{(2n-2k)_{n-k}}\begin{vmatrix}(2\overline{n}\,|\,\overline{k}\,|\,k\,|\,n-k)\\ \alpha\ i_1\ i_2\\ (2\overline{n}\,|\,\overline{k}\,|\,k\,|\,n-k)\\ \alpha\ i_1\ i_2\end{vmatrix}.$$

On examining the inner sum of this form one will easily see that any given selection of h, $1 \leqq h \leqq n-k$, of the $2n-2k$ numbers in the combination ${}^{(2n|\overline{k}|k)}_{\alpha\ i_1}$ is constant as rows running through $(2n-2k-h)_{n-k}$ of the terms and constant as columns through as many more terms. We therefore arrive at the following:

THEOREM (1) or $A(n, k)$:

(1)

$$\sum_{i_1=1}^{(2n-k)_n}(-1)^{\nu_1}\begin{vmatrix}(2n\,|\,k\}2\overline{n}\,|\,\overline{k}\,|\,n-k)\\ \alpha\qquad\alpha\qquad i_1\\ (2\overline{n}\,|\,\overline{k}\,|\,n-k)\\ \alpha\qquad i_1\end{vmatrix}=\sum_{i_1=1}^{(2n-k)_k}(-1)^{\nu_2}\begin{vmatrix}(2n\,|\,k)\\ \alpha\\ (2\overline{n}\,|\,\overline{k}\,|\,k)\\ \alpha\ i_1\end{vmatrix}\frac{1}{2(n-k)_h}\sum_{\beta=1}^{(2n-2k)_h}S_\beta,$$

where S_β denotes the sum of the two aggregates:

$$\sum_{i_2=1}^{(2n-2k-h)_{n-k}}(-1)^{\nu_3}\begin{vmatrix}(2\overline{n}\,|\,\overline{k}\,|\,k\,|\,\overline{h}\}2\overline{n}\,|\,\overline{k}\,|\,\overline{k}\,|\,\overline{h}\,|\,n-k-h)\\ \alpha\ i_1\ \beta\qquad\alpha\ i_1\ \beta\ i_2\\ (2\overline{n}\,|\,\overline{k}\,|\,\overline{k}\,|\,\overline{h}\,|\,n-k-h)\\ \alpha\ i_1\ \beta\ i_2\end{vmatrix},$$

$$\sum_{i_2=1}^{(2n-2k-h)_{n-k}}(-1)^{\nu_4}\begin{vmatrix}(2\overline{n}\,|\,\overline{k}\,|\,k\,|\,\overline{h}\,|\,n-k-h)\\ \alpha\ i_1\ \beta\ i_2\\ (2\overline{n}\,|\,\overline{k}\,|\,k\,|\,\overline{h}\}2\overline{n}\,|\,\overline{k}\,|\,\overline{k}\,|\,\overline{h}\,|\,n-k-h)\\ \alpha\ i_1\ \beta\qquad\alpha\ i_1\ \beta\ i_2\end{vmatrix},$$

in which, if

$$(2n\,|\,k),\quad(2\overline{n}\,|\,\overline{k}\,|\,k\,|\,h),\quad(2\overline{n}\,|\,\overline{k}\,|\,\overline{k}\,|\,\overline{h}\,|\,n-k-h),\quad(2\overline{n}\,|\,k\,|\,k),\quad(2\overline{n}\,|\,\overline{k}\,|\,\overline{k}\,|\,\overline{h}\,|\,n-k-h)$$

are denoted by

$$(a),\ (b),\ (c),\ (d),\ (e)$$

respectively, the integers ν_2, ν_3, ν_4 denote the numbers of inversions in

$$(a\}b\}c\}d\}e),\quad(b\}c\}e),\quad(e\}b\}c)$$

respectively.

The identity of this theorem (1) or $A(n, k)$ depends upon any combination ${}^{(2n|k)}_{\alpha}$ and any integer h $(1 \leqq h \leqq n-k)$.

For $h = 1$ and $k = n-1$ theorem (1) reduces to MUIR'S theorem A.

As an illustration of (1) we have for $(n, k, h) = (4, 2, 1)$:

$$\begin{vmatrix}1234\\5678\end{vmatrix} - \begin{vmatrix}1235\\4678\end{vmatrix} + \begin{vmatrix}1236\\4578\end{vmatrix} - \begin{vmatrix}1237\\4568\end{vmatrix} + \begin{vmatrix}1238\\4567\end{vmatrix} + \begin{vmatrix}1245\\3678\end{vmatrix} - \begin{vmatrix}1246\\3578\end{vmatrix} + \begin{vmatrix}1247\\3568\end{vmatrix}$$

$$- \begin{vmatrix}1248\\3567\end{vmatrix} + \begin{vmatrix}1256\\3478\end{vmatrix} - \begin{vmatrix}1257\\3468\end{vmatrix} + \begin{vmatrix}1258\\3467\end{vmatrix} + \begin{vmatrix}1267\\3458\end{vmatrix} - \begin{vmatrix}1268\\3457\end{vmatrix} + \begin{vmatrix}1278\\3456\end{vmatrix}$$

$$= \begin{vmatrix}12\\34\end{vmatrix} \left\{ \begin{vmatrix}56\\78\end{vmatrix} - \begin{vmatrix}57\\68\end{vmatrix} + \begin{vmatrix}58\\67\end{vmatrix} + \begin{vmatrix}67\\58\end{vmatrix} - \begin{vmatrix}68\\57\end{vmatrix} + \begin{vmatrix}78\\56\end{vmatrix} \right\}$$

$$- \begin{vmatrix}12\\35\end{vmatrix} \left\{ \begin{vmatrix}46\\78\end{vmatrix} - \begin{vmatrix}47\\68\end{vmatrix} + \begin{vmatrix}48\\67\end{vmatrix} + \begin{vmatrix}67\\48\end{vmatrix} - \begin{vmatrix}68\\47\end{vmatrix} + \begin{vmatrix}78\\46\end{vmatrix} \right\}$$

$$+ \text{ etc.}$$

$$+ \begin{vmatrix}12\\78\end{vmatrix} \left\{ \begin{vmatrix}34\\56\end{vmatrix} - \begin{vmatrix}35\\46\end{vmatrix} + \begin{vmatrix}36\\45\end{vmatrix} + \begin{vmatrix}45\\36\end{vmatrix} - \begin{vmatrix}46\\35\end{vmatrix} + \begin{vmatrix}56\\34\end{vmatrix} \right\}.$$

4. The conjugate of every minor in the first aggregate of S_β in (1) is found in the second with the same or opposite sign according as $n - k$ is even or odd. This is readily seen on observing that in bringing $(b \lozenge c \lozenge e)$ to the form $(e \lozenge b \lozenge c)$ the $n - k$ numbers in (e) are carried over the $n - k$ numbers in (c) and (b) and therefore the exponents of the sign factors for the two differ, mod. 2, by $(n - k)^2$ or $n - k$.*

5. Since each aggregate of S_β on the right of (1) is an expression involving minors of order $n - k$ the same in form as those on the left-hand side of the equation we may, if we choose, apply the theorem to them. If $n - k$ is not odd we may select h so that, after applying the theorem a second time, $n - k - h$ is odd and shall have within each of the resulting S_β's conjugate minors with opposite signs.

6. If now we take the sum

$$(n-k)_g \sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n \underset{\alpha}{|} k \lozenge 2\bar{n} \underset{\alpha}{|} k | \, n-k)_{i_1} \\ (2\bar{n} \underset{\alpha}{|} k | \, n-k)_{i_1} \end{vmatrix}$$

and, if instead of expanding each minor in terms of minors formed from the αth selection of k rows as in art. 3, we expand in terms of minor formed from the βth selection of h of the αth selection of k rows and g others and finally collect like terms we get the following:

THEOREM (2):

$$(2) \qquad (n-k)_g \sum_{i_1=1}^{(2n-k)_n} (-1)^{\nu_1} \begin{vmatrix} (2n \underset{\alpha}{|} k \lozenge 2\bar{n} \underset{\alpha}{|} k | \, n-k)_{i_1} \\ (2\bar{n} \underset{\alpha}{|} \bar{k} | \, n-k)_{i_1} \end{vmatrix} = \sum_{i_1=1}^{(2n-k)_{h+2g}} (-1)^{\nu_2} P_{i_1},$$

* Cf. American Journal of Mathematics, vol. 22, no. 1, art. 2, p. 56.

where P_{i_1} denotes the product of the two aggregates:

$$\sum_{i_2=1}^{(h+2g)_g}(-1)^{\nu_3}\begin{vmatrix}(2n\,|\underset{\alpha}{k}|\,\underset{\beta}{h}\,\}\,\overline{2n}\,|\underset{\alpha}{k}|\,\overline{h+2g}\,|\underset{i_1}{g})\\ (\overline{2n}\,|\underset{\alpha}{k}|\,\overline{h+2g}\,|\underset{i_1}{g})\end{vmatrix},$$

$$\sum_{i_3=1}^{(2n-2g-h-k)_{n-h-g}}(-1)^{\nu_4}\begin{vmatrix}(2n\,|\underset{\alpha}{k}|\,\underset{\beta}{h}\,\}\,\overline{2n}\,|\underset{\alpha}{k}|\,\overline{h+2g}\,|n-k-g)\\ (\overline{2n}\,|\underset{\alpha}{k}|\,\overline{h+2g}\,|n-k-g)\end{vmatrix}.$$

If $h=k$ and $g=1$, then (2) reduces to (1).

As an example of (2) for $(n, k, g, h)=(3, 1, 1, 1)$ we have

$$2\left\{\begin{vmatrix}123\\456\end{vmatrix}-\begin{vmatrix}124\\356\end{vmatrix}+\begin{vmatrix}125\\346\end{vmatrix}-\begin{vmatrix}126\\345\end{vmatrix}+\begin{vmatrix}134\\256\end{vmatrix}-\begin{vmatrix}135\\246\end{vmatrix}+\begin{vmatrix}136\\245\end{vmatrix}+\begin{vmatrix}145\\236\end{vmatrix}-\begin{vmatrix}146\\235\end{vmatrix}+\begin{vmatrix}156\\234\end{vmatrix}\right\}$$

$$=\binom{2}{3}\binom{3}{2}\left\{\begin{vmatrix}14\\56\end{vmatrix}-\begin{vmatrix}15\\46\end{vmatrix}+\begin{vmatrix}16\\45\end{vmatrix}\right\}-\binom{2}{4}\binom{4}{2}\left\{\begin{vmatrix}13\\56\end{vmatrix}-\begin{vmatrix}15\\36\end{vmatrix}+\begin{vmatrix}16\\35\end{vmatrix}\right\}$$

$$+\binom{2}{5}\binom{5}{2}\left\{\begin{vmatrix}13\\46\end{vmatrix}-\begin{vmatrix}14\\36\end{vmatrix}+\begin{vmatrix}16\\34\end{vmatrix}\right\}-\binom{2}{6}\binom{6}{2}\left\{\begin{vmatrix}13\\45\end{vmatrix}-\begin{vmatrix}14\\35\end{vmatrix}+\begin{vmatrix}15\\34\end{vmatrix}\right\}$$

$$+\binom{3}{4}\binom{4}{3}\left\{\begin{vmatrix}12\\56\end{vmatrix}-\begin{vmatrix}15\\26\end{vmatrix}+\begin{vmatrix}16\\25\end{vmatrix}\right\}-\binom{3}{5}\binom{5}{3}\left\{\begin{vmatrix}12\\46\end{vmatrix}-\begin{vmatrix}14\\26\end{vmatrix}+\begin{vmatrix}16\\24\end{vmatrix}\right\}$$

$$+\binom{3}{6}\binom{6}{3}\left\{\begin{vmatrix}12\\45\end{vmatrix}-\begin{vmatrix}14\\25\end{vmatrix}+\begin{vmatrix}15\\24\end{vmatrix}\right\}+\binom{4}{5}\binom{5}{4}\left\{\begin{vmatrix}12\\36\end{vmatrix}-\begin{vmatrix}13\\26\end{vmatrix}+\begin{vmatrix}16\\23\end{vmatrix}\right\}$$

$$-\binom{4}{6}\binom{6}{4}\left\{\begin{vmatrix}12\\35\end{vmatrix}-\begin{vmatrix}13\\25\end{vmatrix}+\begin{vmatrix}15\\23\end{vmatrix}\right\}+\binom{5}{6}\binom{6}{3}\left\{\begin{vmatrix}12\\34\end{vmatrix}-\begin{vmatrix}13\\24\end{vmatrix}+\begin{vmatrix}14\\23\end{vmatrix}\right\}.$$

7. If now we impose the conditions for axisymmetry the right-hand side of (1) becomes zero and for $k=n-1$ is the general Kronecker relation. Under the same conditions of axisymmetry the left-hand side of (2), which in general we shall denote by $(n-k)_g K(n, k)$, is a sum of Kronecker expressions and the theorem (2) may be written schematically

$$(n-k)_g K(n, k)=\sum K(h+g, h)K(n-g-h, k-h).$$

If $g=1$, then $K(h+g, h)$ is a Kronecker expression and if in addition $k=n-2$, then $K(n-g-h, k-h)$ is also a Kronecker expression, so that under these conditions (2) expresses *a sum of Kronecker expressions for minors of order n as a sum of products of Kronecker expressions for minors of orders $h+g$ and $n-h-g$ respectively.* It is apparent that, by repeated applications of the theorem and by a proper choice of g, we may reduce both factors of each term on the right of (2) to Kronecker expressions.

If the original determinant is skew then the two aggregates of S_β in (1) are equal.

Part II.

8. If (m) denotes any combination of the $2n$ numbers $1, 2, \cdots, 2n$, then $(\overline{m})$ shall denote the combination formed by replacing every number in (m) by its defect from $2n+1$.

The extension of theorem B is now easily stated as follows:

THEOREM (3) or $B(n, k)$:

$$(3)\quad \sum_{i_1=1}^{n_k} \begin{vmatrix} (\overline{2n\,|n|\,k})(2n\,|n|\,k) \\ {\scriptstyle a\ \ i_1}\qquad\quad {\scriptstyle a\ \ i_1} \\ (2n\,|\,n) \\ \quad a \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n\,|\,n) \\ \quad a \\ (\overline{2n\,|n|\,k})(2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_1}\qquad\quad {\scriptstyle a\ \ i_1} \end{vmatrix}$$

$$= \sum_{i_1=1}^{n_k} \sum_{i_2=1}^{n_k} (-1)^{\nu} \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_1} \\ (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_2} \end{vmatrix} \left\{ \begin{vmatrix} (\overline{2n\,|n|\,k}) \\ {\scriptstyle a\ \ i_1} \\ (2n\,|n|\,k) \\ {\scriptstyle a\ \ i_2} \end{vmatrix} - \begin{vmatrix} (2n\,|n|\,k) \\ {\scriptstyle a\ \ i_1} \\ (\overline{2n\,|n|\,k}) \\ {\scriptstyle a\ \ i_2} \end{vmatrix} \right\}$$

where ν is the number of inversions in $\begin{matrix} (2n\,|\overline{n}|\,k)(2n\,|n|\,k) \\ {\scriptstyle a\ \ i_1}\qquad\quad {\scriptstyle a\ \ i_1} \end{matrix}$ *plus the number in* $\begin{matrix} (2n\,|\overline{n}|\,k)(2n\,|n|\,k) \\ {\scriptstyle a\ \ i_2}\qquad\quad {\scriptstyle a\ \ i_2} \end{matrix}$.

The proof of this theorem consists in the expansion by LAPLACE'S theorem of each minor contained in the first sum on the left in terms of minors formed from the $n-k$ rows having for indices the numbers in $\begin{matrix} (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_1} \end{matrix}$, with their complementaries; and of each minor in the second sum on the left in terms of minors from the $n-k$ columns having the same numbers for column indices, with their complementaries, and finally in the collection of the terms in pairs one from each sum. Thus the left-hand side of (3) is equal to

$$\sum_{i_1=n_k}^{i_1=n_k} \left\{ \sum_{i_2=n_k}^{i_2=n_k} (-1)^{\nu} \begin{vmatrix} (\overline{2n\,|n|\,k}) \\ {\scriptstyle a\ \ i_1} \\ (2n\,|n|\,k) \\ {\scriptstyle a\ \ i_2} \end{vmatrix} \cdot \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_1} \\ (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_2} \end{vmatrix} \right\}$$

$$- \sum_{i_1=1}^{i_1=n_k} \left\{ \sum_{i_2=1}^{i_2=n_k} (-1)^{\nu} \begin{vmatrix} (2n\,|n|\,k) \\ {\scriptstyle a\ \ i_2} \\ (\overline{2n\,|n|\,k}) \\ {\scriptstyle a\ \ i_1} \end{vmatrix} \cdot \begin{vmatrix} (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_2} \\ (2n\,|\overline{n}|\,k) \\ {\scriptstyle a\ \ i_1} \end{vmatrix} \right\},$$

which in turn is equal to the right-hand side.

When $k=1$ the theorem $B(n, k)$ reduces to MUIR'S theorem B.

9. Under the condition for centrosymmetry the right-hand side of (3) vanishes and we have the general linear relation connecting minors of order n of a centrosymmetric determinant of order $2n$. The type $k=1$ is that which was first pointed out by MUIR in 1888.

10. MUIR in his paper of 1900 states in connection with the identities of the type $k=1$ given by him; "that the number of such identities possible, when the centrosymmetry is complete and the parent determinant is of the $2n$th order,

is $C_{2n,n}$." It is to be observed, however, that we do not get a true vanishing aggregate for every selection of the $2n$ numbers n at a time. For if x and y are any two of the n numbers in any given selection, then whenever $x + y = 2n + 1$ some of the minors in the first sum have their conjugates in the second and cancel each other, while others will vanish identically. Those that remain, if any, form an aggregate which is the extensional of a vanishing aggregate of minors of lower order.

Thus for a determinant of order six if we take the selection 256 we get

$$\begin{vmatrix}251\\256\end{vmatrix} + \begin{vmatrix}226\\256\end{vmatrix} + \begin{vmatrix}556\\256\end{vmatrix} - \begin{vmatrix}256\\251\end{vmatrix} - \begin{vmatrix}256\\226\end{vmatrix} - \begin{vmatrix}256\\556\end{vmatrix},$$

in which the first and fourth are conjugate and therefore cancel, the others are indentically zero having identical rows or columns.

For a determinant of order eight, taking the selection 1678 we get

$$\begin{vmatrix}1671\\1678\end{vmatrix} + \begin{vmatrix}1628\\1678\end{vmatrix} + \begin{vmatrix}1378\\1678\end{vmatrix} + \begin{vmatrix}8678\\1678\end{vmatrix} - \begin{vmatrix}1678\\1671\end{vmatrix} - \begin{vmatrix}1678\\1628\end{vmatrix} - \begin{vmatrix}1678\\1378\end{vmatrix} - \begin{vmatrix}1678\\8678\end{vmatrix}$$

which reduces to the extensional of

$$\begin{vmatrix}62\\67\end{vmatrix} + \begin{vmatrix}37\\67\end{vmatrix} - \begin{vmatrix}67\\62\end{vmatrix} - \begin{vmatrix}67\\37\end{vmatrix}.$$

Neglecting these degenerate aggregates we have, if λ denotes the number of true vanishing aggregates of each type (*i. e.*, for each of the $n - 1$ values of k), for the determination of λ the following:

FORMULA I:

$$(\mathrm{I}) \qquad \lambda = \tfrac{1}{2}\left[C_{2n,n} - \sum_{r=1}^{r=[n/2]} \frac{n(n-1)\cdots(n-2r+1)}{(r!)^2}\, 2^{n-2r} \right].$$

This we see by counting the number of combinations containing one, two, three, etc., pairs of numbers satisfying the equation $x + y = 2n + 1$, and subtracting the result from $C_{2n,n}$ and observing that for every relation

$$\sum_{i_1=1}^{n_k} \begin{vmatrix} (\overline{2n}\,|\underset{a}{n}|\,\overline{k})\,(2n\,|\underset{a}{\overline{n}}|\,\underset{i_1}{k}) \\ (2n|\underset{a}{n}) \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n|\underset{a}{n}) \\ (\overline{2n}\,|\underset{a}{n}|\,\underset{i_1}{\overline{k}})\,(2n\,|\underset{a}{\overline{n}}|\,\underset{i_1}{k}) \end{vmatrix} = 0,$$

there is another

$$\sum_{i_1=1}^{n_k} \begin{vmatrix} (\overline{2n}\,|\underset{a}{n}|\,\underset{i_1}{\overline{k}})\,(\overline{2n}\,|\underset{a}{\overline{n}}|\,\underset{i_1}{k}) \\ (\overline{2n}\,|\underset{a}{n}) \end{vmatrix} - \sum_{i_1=1}^{n_k} \begin{vmatrix} (2n|\underset{a}{n}) \\ (\overline{\overline{2n}\,|\underset{a}{n}|\,\underset{i_1}{k}})\,(\overline{2n}\,|\underset{a}{\overline{n}}|\,\underset{i_1}{k}) \end{vmatrix} = 0,$$

which in virtue of the centrosymmetry of the determinant is not different from the first.

11. On examining a set of λ true vanishing aggregates of any type k it will be seen that:

(a) Each aggregate contains n_k terms together with the negative of their conjugates.

(b) Every minor $\left|\begin{smallmatrix}(a)\\(b)\end{smallmatrix}\right|$ enters two aggregates, in one with (b) as the selection of columns and in another with (a), $\left(\text{or } (\bar{a})\right)$, as the selection of columns.

(c) There are $n_k\lambda$ independent minors ($\frac{1}{2}n_k\lambda$ with their conjugates).

(d) For every set of $\frac{1}{2}\lambda$ aggregates there is a set of $\frac{1}{2}\lambda$ minors (one in each aggregate) whose conjugates are found in the remaining $\frac{1}{2}\lambda$ aggregates.

(e) The sum of the λ aggregates vanishes.

It follows from (a) and (b) that every minor enters one aggregate with the positive and one with the negative sign, and that any two aggregates have either two or no minors in common.

It follows from (d) that the sum of no fewer than λ aggegates can vanish.

If we denote the number of linearly independent aggregates of each type by μ then we have for the determination of μ the following:

FORMULA II:

$$\mu = \lambda - 1. \tag{II}$$

SYRACUSE UNIVERSITY, *March* 15, 1901.

UEBER DIE ANWENDUNG DER CAUCHY'SCHEN MULTIPLICATIONSREGEL AUF BEDINGT CONVERGENTE ODER DIVERGENTE REIHEN*

VON

ALFRED PRINGSHEIM

In einer vor nahezu zwanzig Jahren verfassten Arbeit *über die Multiplication bedingt convergirender Reihen* † habe ich unter anderem gezeigt, dass das nach der CAUCHY'schen Regel gebildete Product aus einer nur *bedingt convergenten* oder sogar *divergenten* und einer *unbedingt* convergenten Reihe unter Umständen *unbedingt* convergirt. ‡ Und ich habe ferner hervorgehoben, dass auch für das Product *zweier bedingt convergenter* Reihen die Möglichkeit eines solchen Verhaltens keineswegs ausgeschlossen erscheine, ohne freilich damals im Stande zu sein, das wirkliche Vorkommen dieses Falles durch bestimmte Beispiele erweisen zu können. § Seitdem habe ich in Folge vielfacher Beschäftigung mit der Reihenlehre längst erkannt, dass das fragliche Verhalten für ganz grosse Kategorien von Reihen geradezu *typisch* ist, habe aber bisher nicht Gelegenheit genommen, hierüber etwas zu veröffentlichen. Nachdem nun Herr CAJORI in Bd. 2 dieser Zeitschrift, p. 25 ff. jene Frage wieder aufgenommen und durch Construction gewisser Beispiele in bejahendem Sinne entschieden hat, möchte ich mir erlauben, auch meine eigenen, auf ganz anderer und allgemeinerer Grundlage beruhenden Betrachtungen hier mitzutheilen und daran noch einige weitere Bemerkungen über den vorliegenden Gegenstand anzuknüpfen.

§ 1.

Es sei A_ν $(\nu = 0, 1, 2, \cdots)$ eine unbegrenzte Folge reeller oder complexer Zahlen von der Beschaffenheit, dass $\sum|A_\nu|$ *divergent* und $\lim A_\nu = 0$ ist. Die Potenzreihe $\sum A_\nu x^\nu$ besitzt alsdann den Convergenzradius $|x| = 1$ und kann

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication May 24, 1901.

† Mathematische Annalen, Bd. 21 (1883), p. 327.

‡ a. a. o., pp 357–359.

§ a. a. o., p. 332.

auf dem Convergenzkreise *keinesfalls unbedingt*, wohl aber durchweg oder theilweise noch *bedingt* convergiren. Angenommen, es sei a, wo $|a|=1$, eine *Divergenz*-Stelle für $\sum A_\nu x^\nu$. Ordnet man alsdann $(x-a)\sum A_\nu x^\nu$ wiederum nach Potenzen von x, also:

$$(x-a)\sum_{\nu=0}^{\infty} A_\nu x^\nu = -A_0 a + \sum_{\nu=1}^{\infty}(A_{\nu-1} - A_\nu a)x^\nu,$$

so geht diese neue Potenzreihe für $x=a$ in die folgende über:

$$-A_0 a + \sum_{\nu=1}^{\infty}(A_{\nu-1}a^\nu - A_\nu a^{\nu+1}) = -A_0 a + \mathop{\mathrm{L}}_{n=\infty}(A_0 a - A_n a^{n+1}) = 0$$

d. h. sie *convergirt* für $x=a$ (gegen die Summe 0), während sie im übrigen durchaus *gleichzeitig* mit der ursprünglichen Reihe $\sum A_\nu x^\nu$ *convergirt* oder *divergirt*.

Es möge nun $\sum A_\nu x^\nu$ für $|x|=1$ noch *bedingt* convergiren mit *einziger* Ausnahme der auf dem Einheitskreise gelegenen Stellen $a_1, a_2, \cdots, a_n$. Bildet man alsdann die Potenzreihe:

$$(1^a) \qquad \mathfrak{P}_1(x, a) = \prod_{\kappa=1}^{m}(x-a_\kappa)\cdot\sum_{\nu=0}^{\infty} A_\nu x^\nu,$$

indem man wiederum das rechts stehende Product nach Potenzen von x ordnet, so wird nach dem eben gesagten diese Reihe auf dem Einheitskreise (der dann nicht einmal mehr der *Convergenz*kreis zu sein braucht) *ausnahmslos* convergiren. Daraus darf freilich noch nicht ohne weiteres geschlossen werden, dass die Convergenz dann allemal noch eine *unbedingte* (absolute) sein müsse.* Immerhin wird man sagen können, dass "*im allgemeinen*" oder, um diesen Ausdruck noch genauer zu praecisiren, in unendlich vielen, durch geeignete Auswahl der A_ν in beliebiger Anzahl leicht zu erzielenden Fällen $\mathfrak{P}_1(x, a)$ für $|x|=1$ *unbedingt* convergiren wird.

Sei ferner $\sum B_\nu x^\nu$ eine ganz ähnliche Potenzreihe, die für alle Stellen auf dem Einheitskreise mit einziger Ausnahme von $\beta_1, \beta_2, \cdots, \beta_n$ (wo $\beta_\lambda \neq a_\kappa$) noch *bedingt* convergirt, sodass also:

$$(2^a) \qquad \mathfrak{P}_2(x, \beta) = \prod_{\lambda=1}^{n}(x-\beta_\lambda)\cdot\sum_{\nu=0}^{\infty} B_\nu x^\nu$$

für $|x|=1$ widerum *ausnahmslos* und, bei geeigneter Auswahl der B_ν, auch *unbedingt* convergirt. Bildet man nun andererseits die beiden Potenzreihen:

$$(1^b) \qquad \mathfrak{P}_1(x, \beta) = \prod_{\lambda=1}^{n}(x-\beta_\lambda)\cdot\sum_{\nu=0}^{\infty} A_\nu x^\nu$$

* Mathematische Annalen, Bd. 25 (1885), p. 419. Münchener Berichte, Bd. 30 (1900), p. 71.

$$(2^b) \qquad \mathfrak{P}_2(x,\ a) = \prod_{\kappa=1}^{m}(x - a_\kappa)\cdot\sum_{\nu=0}^{\infty} B_\nu x^\nu,$$

so wird auf Grund der zu Anfang gemachten Bemerkung die *erstere* an den *Divergenzstellen* von $\sum A_\nu x^\nu$, also die Stellen a_κ und, entsprechend, die *zweite* an den Stellen β_λ noch *divergiren*, während im übrigen für $|x| = 1$ beide Reihen *bedingt convergiren.*

Nun ist aber offenbar:

$$(3) \qquad \mathfrak{P}_1(x,\ \beta)\cdot\mathfrak{P}_2(x,\ a) = \mathfrak{P}_1(x,\ a)\cdot\mathfrak{P}_2(x,\ \beta),$$

und diese *Gleichheit* wird zu einer vollkommenen *Identität*, wenn man ihre beiden Seiten wieder in Potenzreihen umformt, d. h. die CAUCHY'sche Multiplicationsregel darauf anwendet. Wegen der für $|x| = 1$ noch *unbedingten* Convergenz von $\mathfrak{P}_1(x,\ a)$, $\mathfrak{P}_2(x,\cdot\beta)$ besteht dann diese letztere auch für die *Productreihe*, zu deren Herstellung auch die für $|x| = 1$ nur *bedingt convergenten*, an den Stellen $x = a_\kappa$ bezw. $x = \beta_\lambda$ sogar *divergenten* Reihen $\mathfrak{P}_1(x,\ \beta)$, $\mathfrak{P}_2(x,\ a)$ dienen können.

Es hat keine Schwierigkeit, dieses Resultat auch auf den Fall zu übertragen, dass *eine* der beiden zu multiplicirenden Reihen *unbedingt* convergirt. Man hat dann nur $\sum B_\nu x^\nu$ als für $|x| = 1$ noch *unbedingt* convergent anzunehmen und etwa zu setzen:

$$(4) \qquad \mathfrak{P}_1(x) = \sum_{\nu=0}^{\infty} A_\nu x_\nu, \qquad \mathfrak{P}_2(x) = \sum_{\nu=0}^{\infty} B_\nu x^\nu.$$

Behalten dann $\mathfrak{P}_1(x,\ a)$, $\mathfrak{P}_2(x,\ a)$ die in Gl. (1^a), (2^b) festgesetzte Bedeutung, so hat man wiederum:

$$(5) \qquad \mathfrak{P}_1(x)\cdot\mathfrak{P}_2(x,\ a) = \mathfrak{P}_1(x,\ a)\cdot\mathfrak{P}_2(x)$$

sodass $\mathfrak{P}_1(x)$, $\mathfrak{P}_2(x,\ a)$, obschon die *erste* dieser beiden Reihen für $|x| = 1$ nur *bedingt convergirt* bezw. für $x = a_\kappa$ *divergirt*, wegen der *unbedingten* Convergenz von $\mathfrak{P}_1(x,\ a)\cdot\mathfrak{P}_2(x)$ wieder eine *unbedingt* convergirende Productreihe liefert.

Hernach ergiebt sich:

Die Anwendung der Cauchy'schen Multiplicationsregel auf zwei Reihen, von denen die erste nur bedingt convergirt oder sogar divergirt, während die zweite bedingt oder unbedingt convergirt, liefert in unendlich vielen Fällen eine unbedingt convergirende Reihe.

§ 2.

Um jetzt Reihen der fraglichen Art mit lauten *reellen* Gliedern zu erhalten, wird man vor allem die A_ν, B_ν und x *reell*, also $x = \pm 1$ anzunehmen haben. Die a_κ bezw. β_λ müssen sodann gleichfalls *reell* oder *paarweise conjugirt com-*

plex gewählt werden. Um auf dem Einheitskreise wenigstens *eine reelle Convergenz*-Stelle, etwa $x = 1$, zu behalten, bleibt dann als einzige mögliche *reelle Divergenz*-Stelle für $\sum A_\nu x^\nu$ die Stelle $x = -1$ (d. h. die a_κ reduciren sich auf den *einen* Werth $a_1 = -1$). Als einfachste Wahl für die *Divergenz*-Stellen von $\sum B_\nu x^\nu$ bleibt dann $\beta_1 = i$, $\beta_2 = -i$. Man genügt offenbar allen diesen Bedingungen, wenn man setzt:

$$A_\nu = (-1)^\nu a_\nu, \quad B_\nu = (-1)^{[\nu/2]} b_\nu,^*$$

wo die a_ν, b_ν positive, mit unbegrenzt wachsenden ν *monoton* gegen *Null* abnehmende Zahlen mit *divergenten* $\sum a_\nu$, $\sum b_\nu$ vorstellen. In der That sind dann die beiden Reihen:

$$\sum_{\nu=0}^{\infty} (-1)^\nu a_\nu x^\nu = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + - + - \cdots$$

$$\sum_{\nu=0}^{\infty} (-1)^{[\nu/2]} b_\nu x^\nu = b_0 + b_1 x - b_2 x^2 - b_3 x^3 + + - - \cdots$$

für $|x| = 1$ noch *bedingt convergent*, die *erste* mit einziger Ausnahme von $x = -1$, die *zweite* mit Ausnahme von $x = \pm i$. Bildet man ferner nach Analogie von Gl. (1^a), (2^a):

$$(5) \quad \begin{cases} (1+x) \sum_{\nu=0}^{\infty} (-1)^\nu a_\nu x^\nu = a_0 + \sum_{\nu=1}^{\infty} (a_{\nu-1} - a_\nu) x^\nu \\ (1+x^2) \sum_{\nu=0}^{\infty} (-1)^{[\nu/2]} b_\nu x^\nu = b_0 + b_1 x + \sum_{\nu=2}^{\infty} (-1)^{[\nu/2]-1} (b_{\nu-2} - b_\nu) x^\nu, \end{cases}$$

so erkennt man unmittelbar, dass diese Reihen für $|x| = 1$, speciell also für $x = \pm 1$ noch *unbedingt* convergiren. Andererseits bleibt von den nach Analogie des Gl. (1^b), (3^b) gebildeten Reihen:

$$(6) \quad \begin{cases} (1+x^2) \sum_{\nu=0}^{\infty} (-1)^\nu a_\nu x^\nu = a_0 - a_1 x + \sum_{\nu=2}^{\infty} (-1)^\nu (a_{\nu-1} + a_\nu) x^\nu \\ (1+x) \sum_{\nu=0}^{\infty} (-1)^{[\nu/2]} b_\nu x^\nu = b_0 + \sum_{\nu=1}^{\infty} (-1)^{[(\nu-1)/2]} (b_{\nu-1} + (-1)^{\nu-1} b_\nu) x^\nu \end{cases}$$

die *erste* für $x = 1$, die *zweite* für $x = \pm 1$ noch *bedingt convergent*, während die *erste* für $x = -1$ (die *zweite* nur für $x = \pm i$) *divergirt*.

Setzt man also einmal $x = +1$, sodann $x = -1$, so folgt, dass die Reihen:

$$(A) \quad \left\{ a_0 - a_1 + \sum_{\nu=2}^{\infty} (-1)^\nu (a_{\nu-2} + a_\nu) \right\} \cdot \left\{ b_0 + \sum_{\nu=1}^{\infty} (-1)^{[(\nu-1)/2]} (b_{\nu-1} + (-1)^{\nu-1} b_\nu) \right\}$$

---

* Dabei bedeutet $[\nu/2]$ die grösste in $\nu/2$ enthaltene ganze Zahl.

$$(B)\quad \left\{a_0 + a_1 + \sum_{\nu=2}^{\infty}(a_{\nu-2} + a_\nu)\right\}\cdot\left\{b_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu+[(\nu-1)/2]}(b_{\nu-1} + (-1)^{\nu-1}b_\nu)\right\}$$

unbedingt convergente Productreihen liefern, obschon die *beiden* Reihen (A) und die *zweite* (B) *nur bedingt convergiren*, während die *erste* (B) sogar *divergirt*. Die Ausführung der Multiplication nach der CAUCHY'schen Regel liefert nämlich genau dasselbe Resultat, wie für die aus (5) für $x = \pm 1$ resultirenden, *unbedingt* convergirenden Reihen:

$$(A')\quad \left\{a_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{b_0 + b_1 + \sum_{\nu=2}^{\infty}(-1)^{[\nu/2]-1}(b_{\nu-2} - b_\nu)\right\}$$

$$(B')\quad \left\{a_0 - \sum_{\nu=1}^{\infty}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{b_0 + b_1 - \sum_{\nu=2}^{\infty}(-1)^{\nu+[\nu/2]}(b_{\nu-2} - b_\nu)\right\}.$$

Will man in (A) und (B) an die Stelle des *zweiten* Factors eine *unbedingt* convergirende Reihe treten lassen, so hat man nur $\sum c_\nu$ als *unbedingt* convergent, im übrigen beliebig anzunehmen, und findet dann auf Grund der Beziehung:

$$\left\{\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu\right\}\cdot\left\{(1+x)\sum_{\nu=0}^{\infty}c_\nu x^\nu\right\} = \left\{(1+x)\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^\nu\right\}\cdot\left\{\sum_{\nu=0}^{\infty}c_\nu x^\nu\right\},$$

dass die Reihen:

$$(C)\quad \left\{\sum_{\nu=0}^{\infty}(-1)^\nu a_\nu\right\}\cdot\left\{c_0 + \sum_{\nu=1}^{\infty}(c_{\nu-1} + c_\nu)\right\}$$

$$(D)\quad \left\{\sum_{\nu=0}^{\infty}a_\nu\right\}\cdot\left\{c_0 + \sum_{\nu=1}^{\infty}(-1)^\nu(c_{\nu-1} + c_\nu)\right\}$$

wiederum *unbedingt* convergirende Productreihen liefern, nämlich solche, die identisch sind mit denjenigen der *unbedingt* convergirende Factoren:

$$(C')\quad \left\{a_0 + \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{\sum_{\nu=0}^{\infty}c_\nu\right\},$$

$$(D')\quad \left\{a_0 - \sum_{\nu=1}^{\infty}(a_{\nu-1} - a_\nu)\right\}\cdot\left\{\sum_{\nu=0}^{\infty}(-1)^\nu c_\nu\right\}.$$

Die auf diese Weise gewonnenen Beispiele können auch dazu dienen, um die Richtigkeit des am Schlusse von § 1 ausgesprochenen Resultates mit den *elementarsten* Mittel der Reihenlehre zu erweisen, d. h. *ohne* im übrigen von der Theorie der *Potenzreihen*, welche zur *Auffindung* dieser Beispiele gedient hatte, irgend welchen Gebrauch zu machen. Einerseits kann nämlich der Convergenz-

Charakter sämmtlicher in $(A)-(D)$, $(A')-(D')$ auftretenden Reihen ganz unmittelbar erkannt werden. Sodann liefert die Ausführung der Multiplication nach der CAUCHY'schen Regel für $(A)-(D)$ genau dieselben Productreihen, wie für $(A')-(D')$. Schliesslich resultirt dann deren *unbedingten* Convergenz aus derjenigen aller in $(A')-(D')$ vorkommenden Reihen.

Ferner bemerke man noch folgendes. Die *unbedingte* Convergenz der in (A'), (B') vorkommenden Reihen wird in keiner Weise alterirt, wenn man a_ν durch $a_\nu + a$ *oder* bezw. *und* b_ν durch $b_\nu + b$ ersetzt, wo a, b irgend zwei positive Zahlen bedeuten. Hierdurch werden aber andererseits die entsprechenden Reihen in (A), (B) durchweg *divergent*. Daraus folgt aber:

Auch die Multiplication zweier divergenter Reihen kann eine unbedingt convergirende Reihe liefern.

§ 3.

Die in den vorigen Paragraphen gewonnenen Resultate beruhten wesentlich darauf, dass man eine *bedingt convergente* oder *divergente* Reihe $\sum u_\nu$ als Specialfall von $\sum u_\nu x^\nu$ für $x=1$ auffasste. (N. B. Die Voraussetzung, dass dabei $\sum u_\nu x^\nu$ den Convergenzradius 1 besitzen sollte, ist allemal *eo ipso* erfüllt, wenn $\sum u_\nu$ *bedingt convergirt;* wenn dagegen $\sum u_\nu$ *divergirt*, so muss noch ausdrücklich angenommen werden, dass $\mathrm{L}\sqrt[\nu]{|u_\nu|}=1$). Es werde nun gesetzt:

$$\sum_{\nu=0}^{\infty} u_\nu x^\nu = f(x) \quad \text{für } |x|<1,$$

und sodann $f(X)$ für alle Stellen X auf den Convergenzkreise $|X|=1$ durch die Gleichung definirt:

$$f(X) = \mathrm{L}_{\rho=1-0} \sum_{\nu=0}^{\infty} u_\nu (\rho X)^\nu,$$

(sodass also nach dem bekannten ABEL'schen Grenzwerth-Satze:

$$f(X) = \sum_{\nu=0}^{\nu=\infty} u_\nu X^\nu,$$

sobald diese Reihe *convergirt*). Um unnöthige Complicationen zu vermeiden, mag angenommen werden, dass $f(x)$ auf dem Einheitskreise eine nur *endliche* Anzahl *gewöhnlichen* Unendlichkeits-Stellen (d. h. *ohne Oscillationen*) $x=a_\kappa$ besitze, im übrigen längs des Einheitskreises *stetig* und zum mindesten in der Nähe von $x=1$ *ohne Oscillationen* verlaufe. Die *Convergenz* oder *Divergenz* von $\sum u_\nu$ wird dann nach bekannten Sätzen aus der Theorie der FOURIER'schen Reihen einzig und allein von der *Art des Unendlichwerdens* von $f(x)$ für $x=a_\kappa$ abhängen. Befindet sich unter den Stellen a_κ mindestens *eine* solche, für welche $f(x)$ von der *ersten* oder noch *höheren* Ordnung unendlich wird, so ist $\sum u_\nu$ alle-

mal *divergent*, da die u_ν in diesem Falle für unbegrenzt wachsende ν *nicht* gegen Null convergiren. Wohingegen $\sum u_\nu$ *convergiren* wird, wenn $f(x)$ an den Stellen a_κ durchweg von *niederer* oder, etwas genauer gesagt, von "*hinlänglich*" *niederer** Ordnung, als der *ersten* unendlich wird.

Es bedeute ferner $\sum v_\nu x^\nu$ eine Potenzreihe mit ganz analogen Eigenschaften, und es werde gesetzt:

$$\sum_{\nu=0}^{\infty} v_\nu x^\nu = g(x) \quad \text{für } |x| < 1,$$

$$g(X) = \mathop{\mathbf{L}}_{\rho=1\,-0} \sum_{\nu=0}^{\infty} v_\nu (\rho X)^\nu,$$

während die Unendlichkeits-Stellen von $g(X)$ mit β_λ bezeichnet werden mögen.

Da die Multiplication von $\sum u_\nu \cdot \sum v_\nu$ nach der CAUCHY'schen Regel genau dasselbe Resultate ergiebt, als wenn man das Product $\sum u_\nu x^\nu \cdot \sum v_\nu x^\nu$ zunächst nach Potenzen von x ordnet und sodann $x = 1$ setzt, so wird die *Convergenz* oder *Divergenz* jener ersten Productreihe lediglich davon abhängen, ob überhaupt bezw. in welcher Weise $f(x)$, $g(x)$ auf dem Convergenzkreise *unendlich* wird.

Sind die β_λ durchweg von den a_κ *verschieden*, so wird hiernach die Productreihe $\sum u_\nu \cdot \sum v_\nu$ sicher dann *convergiren*, wenn $\sum u_\nu$, $\sum v_\nu$ einzeln *convergirten*. Daraus erklärt es sich ohne weiteres, warum zwei Reihen, wie die zu Anfang von § 2 eingeführten, nämlich:

$$a_0 - a_1 + a_2 - a_3 + - + - \cdots \qquad \left(a_{\nu+1} \leqq a_\nu,\ \mathbf{L}\, a_\nu = 0\right)$$
$$b_0 + b_1 - b_2 - b_3 + + - - \cdots \qquad \left(b_{\nu+1} \leqq b_\nu,\ \mathbf{L}\, b_\nu = 0\right)$$

allemal ein *convergentes* Product liefern,† mögen die a_ν, b_ν auch *noch so langsam* gegen Null abnehmen.

Es können sich aber, immer unter der voraussetzung $\beta_\lambda \neq a_\kappa$, die Convergenz-Chancen für die Productreihe auch erheblich *günstiger* gestalten, als für die Einzelreihen $\sum u_\nu$, $\sum v_\nu$, nämlich *dann*, wenn $\sum u_\nu x^\nu$ die β_λ, $\sum v_\nu x^\nu$ die a_κ in's gesammt oder doch theilweise zu *Nullstellen* hat. Eine Illustration hierzu liefern die Betrachtungen der beiden vorangehenden Paragraphen.

---

* d. h. *so*, dass $f(x)$ bei $x = a_\kappa$ *integrabel* bleibt, wie z. B.

$$\left(\frac{1}{x - a_\kappa}\right)^{1-\epsilon} \quad (0 < \epsilon < 1), \qquad \frac{1}{(x - a_\kappa)\,[lg(x - a_\kappa)]^{1+\rho}} \quad (\rho < 0).$$

Wird $f(x)$ zwar von *niederer*, als der *ersten*, jedoch *nicht integrabel-unendlich*, wie z. B.

$$\frac{1}{(x - a_\kappa)\, lg(x - a_\kappa)},$$

so muss noch eine gewisse Bedingung hinzukommen, wenn $\mathbf{L} u^\nu = 0$ werden soll. Vgl. RIEMANN, Ges. W., pp. 244–245. Münchener Berichte, Bd. 25 (1895), pp. 339–362.

† Mathematische Annalen, Bd. 21 (1883), p. 346.

Hingegen tritt nun allemal eine *Verschlechterung* der Convergenz-Chancen für die Productreihe ein, sobald irgend ein β_λ mit einem a_κ *zusammenfällt*, da hierdurch für $f(x)\cdot g(x)$ eine Unendlichkeits-Stelle von entsprechend zusammengesetzter *höherer* Ordnung erzeugt wird: sobald diese Ordnung die *erste* erreicht oder übersteigt, muss dann die Productreihe allemal *divergiren*.

Aus dieser Bemerkung erklärt sich nun aber in sehr einfacher Weise die von Herrn Cajori* hervorgehobene Thatsache, dass die Potenzen gewisser bedingt convergirender Reihen—wie $\sum_{\nu=1}^{\nu=\infty}(-1)^{\nu-1}\nu^{-r}$ $(0<r<1)$—*aufhören zu convergiren*, sobald der Exponent q eine bestimmte, von r abhängige Zahl erreicht oder übersteigt; während für andere Reihen—wie $\sum_{\nu=1}^{\nu=\infty}(-1)^{\nu-1}\nu^{-1}$ —jede noch so hohe Potenz eine *convergente* Reihe liefert. Von den beiden Reihen:

$$\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{x^\nu}{\nu^r},\quad \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{x^\nu}{\nu}$$

wird nämlich für $x=-1$ die *erste so* unendlich,† wie $-(1+x)^{-(1-r)}$, die *zweite*, wie $lg(1+x)$. Erhebt man also die *erste* Reihe in die q-te Potenz, so entsteht bei $x=-1$ eine Unendlichkeits-Stelle mit der Ordnungszahl $q(1-r)$, und es tritt daher *Divergenz* ein, sobald:

$$q(1-r)\geqq 1,\ \text{d. h. } q\geqq\frac{1}{1-r},\ \text{anders geschrieben: } r\leqq 1-\frac{1}{q}.$$

Dagegen werden die Potenzen der *zweiten* Reihe für beliebig grosse Exponenten q bei $x=-1$ immer nur eine Unendlichkeits-Stelle vom Typus $[lg(1+x)]^q$ besitzen, sodass also die *Convergenz* erhalten bleibt.

§ 4.

Ich möchte diese Gelegenheit benützen, um eine Lücke auszufüllen, die mir bei der Durchsicht meiner zu Anfang dieser Note citirten Abhandlung ("*Ueber die Multiplication bedingt convergenter Reihen*") aufgefallen ist. Daselbst (a. a. o., p. 363) wird der folgende Satz ausgesprochen:

Sind a_n, b_n positive, mit wachsendem n niemals zunehmende und für $n=\infty$ verschwindende Zahlen, so bildet die Beziehung:

$$\mathrm{L}_{n=\infty}\, b_n\sum_{\kappa=1}^{n}a_\kappa=\mathrm{L}_{n=\infty}\, a_n\sum_{\kappa=1}^{n}b_\kappa=0$$

die *nothwendige und hinreichende* Bedingung für Anwendbarkeit der Multiplicationsregel auf die bedingt convergenten Reihen:

---

* A. a. O., p. 25. American Journal of Mathematics, vol. 18, p. 201.

† Nach einem Satze von P. Appell: Comptes rendus, T. 87 (1878), p. 690. (Auch bei Picard, *Traité d'Analyse*, T. I, p. 210).

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}a_{\nu},\quad \sum_{\nu=0}^{\infty}(-1)^{\nu}b_{\nu}.$$

Der Beweis dafür, dass die obige Bedingung *hinreicht*, wird an der betreffenden Stelle richtig geführt. Dagegen ist der dort gegebene Beweis für deren *Nothwendigkeit* nicht nur von überflüssiger Umständlichkeit, sondern in Folge der Benützung gewisser aus den gemachten Voraussetzungen noch keineswegs folgenden Beziehungen, nämlich:

$$\mathop{\mathrm{L}}_{\nu=\infty}\frac{a_{\nu+1}}{a_{\nu}}=1,\quad \mathop{\mathrm{L}}_{\nu=\infty}\nu\left(1-\frac{a_{\nu+1}}{a_{\nu}}\right)>1,$$

sogar unzulänglich. In Wahrheit ergiebt sich aber die *Nothwendigkeit* der fraglichen Bedingung ganz unmittelbar in folgender Weise. Die Ausführung der Multiplication nach der CAUCHY'schen Regel liefert die Reihe:

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}c_{\nu},\quad \text{wo:}\quad c_n=\sum_{\kappa=0}^{n}a_{\kappa}b_{n-\kappa}\begin{cases}>b_n\sum_{\kappa=0}^{n}a_{\kappa}\\ >a_n\sum_{\kappa=0}^{n}b_{\kappa}.\end{cases}$$

Da aber die Bedingung $\mathop{\mathrm{L}} c_n=0$ eine für die *Convergenz* der Reihe $\sum(-1)^{\nu}c_{\nu}$ jedenfalls *nothwendige* ist, so gilt dies *a fortiori* von der Beziehung:

$$\mathop{\mathrm{L}}_{n=\infty}b_n\sum_{\kappa=0}^{n}a_{\kappa}=\mathop{\mathrm{L}}_{n=\infty}a_n\sum_{\kappa=0}^{n}b_{\kappa}=0.$$

MÜNCHEN, 31. März 1901.

UEBER DEN GOURSAT'SCHEN BEWEIS DES CAUCHY'SCHEN INTEGRALSATZES*

VON

ALFRED PRINGSHEIM

Der CAUCHY'sche Satz über das Verschwinden eines geschlossenen Integrals von der Form $\int f(z)dz$ hat durch die in Bd. 1, pp. 14–16 dieser Zeitschrift von Herrn GOURSAT publicirte Note eine ausserordentlich bemerkenswerthe Erweiterung erfahren, insofern alle bisherigen Beweise, um vollständig und exakt zu sein, die *Stetigkeit* von $f'(z)$ oder, was im wesentlichen auf dasselbe hinausläuft,† die *gleichmässige* Differenzirbarkeit von $f(z)$, entweder schlechthin oder zum mindesten in gewissem Umfange ‡ zur unentbehrlichen Voraussetzung hatten. In jener Note zeigt nun Herr GOURSAT, dass man denjenigen Hauptschluss, welcher bei dem früher (Acta math., T. IV (1884), pp. 197–200) von ihm mitgetheilten Beweise auf der Voraussetzung der *gleichmässigen* Differenzerbarkeit beruht, mit Hülfe einer weniger speciellen, *jeder* (im complexen Sinne) *differenzirbaren* Function *eo ipso* zukommenden Eigenschaft herleiten kann. Dieselbe ist enthalten in dem folgenden LEMMA:

Es sei $f(z)$ für alle Stellen eines von der geschlossenen Curve C begrenzten Bereiches A stetig und mit einer endlichen Derivirten $f'(z)$ begabt. Wird dann $\epsilon > 0$ beliebig vorgeschrieben, so lässt sich A (auf unendlich viele Arten) in hinlänglich kleine Theilstücke A_ν zerlegen, derart dass:

$$(\mathrm{I}) \qquad |f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)| \leqq \epsilon |z_\nu - \xi_\nu|.$$

Dabei bedeutet z_ν jeden beliebigen Punkt auf der Begrenzung von A_ν, ξ_ν einen bestimmten, im Innern oder auf der Begrenzung von A_ν allemal wirklich vorhandenen Punkt.

Nachdem Herr GOURSAT dieses Lemma bewiesen, begnügt er sich im übrigen mit der Bemerkung, man könne nun auf Grund desselben die für seinen oben citirten, ersten Beweis erforderliche Theilung des Gesammtbereiches immer so

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication May 24, 1901.

† Vgl. Münchener Berichte, Bd. 25 (1895), p. 295.

‡ Vgl. Münchener Berichte, Bd. 29 (1899), p. 61. W. F. OSGOOD, American M. S. Bulletin, Bd. 5 (1898), p. 85.

einrichten, *dass das übrige Raisonnement in nichts geändert zu werden brauche.** Dieser *cum grano salis* zwar zutreffende Ausspruch ist indessen seinem *Wortlaute* nach nicht ganz exakt, ja sogar dazu angethan, † bezüglich der *Tragweite* des obigen Lemmas und der *richtigen Art seiner Anwendung* gewisse Missverständnisse und Zweifel hervorzurufen, ‡ nämlich: Der ursprüngliche GOURSAT'sche Beweis des CAUCHY'schen Satzes beruht auf einer Theilung des Bereiches A in lauter *congruente*, einen gewissen Kleinheitsgrad § besitzende *Quadrate* bezw. (längs des Randes) *Bruchstücke solcher Quadrate*, wobei aber jedes in Betracht kommende Quadrat immer nur *ein* solches Bruchstück liefert, und nicht in *der* Weise zerfallen soll, dass *mehrere* Bruchstücke dem Bereiche A angehören. (Ich will diese Art der Theilung im folgenden stets durch den Ausdruck bezeichnen: es werde der Bereich A in Quadrate "*etc.*" zerlegt).

Fasst man also den obigen Ausspruch des Herrn GOURSAT nach seinem *Wortlaute* auf, so müsste man annehmen, es existire auf Grund jenes Lemmas allemal eine Eintheilung in lauter *congruente* Quadrate "*etc.*" A_ν, derart dass für jedes A_ν die Bedingung (I) erfüllt ist. Dies geht nun aber aus dem *Beweise* des fraglichen Lemmas *keineswegs* hervor, vielmehr lehrt derselbe nur so viel, dass *unter einer gewissen der Randcurve C aufzuerlegenden Beschränkung* zwar stets *quadratische* Eintheilungen der verlangten Art vorhanden sind, dass dieselben aber im allgemeinen aus Quadraten bezw. Bruchstücken von Quadraten *verschiedener* Grössen bestehen werden.

Vor allem hat man die Randcurve C als *abtheilungsweise monoton* d. h. *so* anzunehmen, dass jede ihrer beiden Coordinaten nur eine endliche Anzahl von Extremen aufzuweisen hat. ‖ Man kann alsdann eine Strecke λ_0 hinlänglich

* "sans modifier en rien le reste du raisonnement." (a. a. O., p. 16.)

† It is understood that the criticism relates merely to verbal form, and that neither the scope of GOURSAT'S lemma nor its manner of application in proof of GOURSAT'S form of CAUCHY'S theorem is subject to reasonable doubt. The lemma neither in statement nor in proof refers to an ultimate division into *congruent* squares, while the division into such squares utilized in GOURSAT'S original proof of CAUCHY'S theorem is evidently not an essential part of that proof. In view however of the importance of the subject the application of the lemma may well be given explicitly; cf. p. 416 of the text.—THE EDITORS.

‡ Herr E. V. HUNTINGTON (Strassburg) hatte die Güte, mich hierauf aufmerksam zu machen.

§ Ein von vornherein bestimmter (nämlich von der Natur der Randcurve abhängiger) Kleinheitsgrad des quadratischen Gitters ist erforderlich, wenn jedes am *Rande* auftretende Theilstück lediglich von *einem* zusammenhängenden Stücke der Randcurve, im übrigen von einem *Theile* eines Gitterquadrates begrenzt sein soll. Man kann im übrigen diese (von GOURSAT *stillschweigend*, von O. STOLZ, *Grundzüge der Diff.- und Integr.-R.*, II, pp. 213, 218, *ausdrücklich* gemachte) Voraussetzung in gewissem Umfange auch fallen lassen, ohne das betreffende Raisonnement im wesentlichen zu ändern. (Siehe weiter unten).

‖ Es würde nicht einmal genügen anzunehmen: Die Curve C soll von jeder zu den Axen parallelen Graden nur in einer *endlichen* Anzahl von Punkten *geschnitten* werden (wobei eine Gerade welche die Curve in unendlich vielen Punkten *berührt* oder durch unendlich viele *Spitzen* geht nicht als *Schnitt*linie anzusehen wäre): dabei braucht nämlich die Anzahl der Schnittpunkte

klein so fixiren, dass kein Quadrat, dessen Seiten $\leqq \lambda_0$ und den Coordinatenaxen parallel sind, von der Randcurve in mehr als zwei Stücke zerschnitten wird. Denkt man sich nun den Bereich A zunächst mit einem quadratischen Gitter von der Seitenlänge $\lambda \leqq \lambda_0$ überzogen, so folgt aus der beim Beweise des GOURSAT'schen Lemmas angewendeten Schlussweise für's erste nur so viel, dass bei hinlänglicher Verkleinerung von λ (etwa durch Untertheilung von λ_0) *ein* oder *mehrere* Theilbereiche A_ν der Bedingung (I) genügen müssen; sodann, dass jeder Theilbereich A'_ν, welcher der Bedingung (I) noch nicht genügt, durch weitere geeignete Untertheilung im Quadrate "*etc.*" mit der Eigenschaft (I) zerlegt werden könne. Es folgt aber *nicht*, dass die für irgend ein A'_ν geeignete Untertheilung dieses A'_ν in lauter *congruente* Quadrate "*etc.*" zerlegen müsse und, selbst wenn dies der Fall sein sollte, dass die Anwendung der *nämlichen* Untertheilung auf irgend einen der mit A_ν bezeichneten, *schon* der Bedingung (I) *genügenden* Theilbereiche *wieder* lauter Theilbereiche *mit der Eigenschaft* (I) liefern müsse.

Im übrigen wird man jene "geeignete" Untertheilung am einfachsten in *der* Weise erzielen, dass man jedes der Quadrate, welches einen Theilbereich A'_ν ausmacht oder in sich enthält, in 4 congruente Quadrate mit der Seitenlänge $\lambda/2$ zerlegt, jedes dieser letzteren, *sofern es nicht bereits einen der Bedingung* (I) *genügenden Theilbereich liefert* oder überhaupt keine Innenpunkte von A mehr enthält, wieder in 4 congruente Quadrate mit der Seitenlänge $\lambda/4$ u. s. f. Der GOURSAT'sche Schluss lehrt dann, dass der angedeutete Process, bei dem also jedesmal alle diejenigen Quadrate, welche der Bedingung (I) bereits genügende Theilbereiche liefern oder keine Innerpunkte von A mehr enthalten, *von der weiteren Untertheilung ausgeschlossen sind*, nach einer *begrenzten* Anzahl von Operationen zu dem gewünschten Endziele führen muss. Man gelangt also auf diese Weise zu der folgenden pseciellen Formulirung des fraglichen Lemmas.

Unter den oben über $f(z)$ und über die Randcurve C gemachten Voraussetzungen lässt sich der Bereich A in eine endliche Anzahl von Quadraten Q_μ und (oben näher qualificirten) Bruchstücken R_ν von Quadraten mit der Sei-

---

noch *nicht* (wie bei einer *abtheilungsweise monotonen* Curve) unter einer *festen* endlichen Zahl zu bleiben (*Beispiel:* $y = x^2 \sin^2 1/x$ in jedem endlichen die Stelle $x = 0$ enthaltenden Intervall). In Folge dessen könnten sich bei der Untertheilung die Schnittpunkte jedesmal so vermehren, dass am Rande von A immer noch Quadrate vorhanden bleiben, welche *mehrere* Theilstücke von A enthalten. Man könnte indessen in diesem Falle dem fraglichen Lemma die folgende, für den Beweis des Integralsatzes noch ausreichende (s. die folgende Fussnote) Form geben: „*Der Bereich A lässt sich mit einer endlichen Anzahl von Quadraten Q_ν überdecken, derart dass jedes Q_ν nur eine endliche Anzahl von Theilstücken A_{m_ν} des Bereiches A und überdies einen ebenfalls zu A (also zu einem der A_{m_ν}) gehörigen Punkt ξ_ν von folgender Eigenschaft enthält: man hat*

$$|f(z_{m_\nu}) - f(\xi_\nu) - (z_{m_\nu} - \xi_\nu) \cdot f'(\xi_\nu)| \leqq \varepsilon |z_{m_\nu} - \xi_\nu|,$$

wobei wiederum z_{m_ν} *jeden Punkt der Begrenzung aller A_{m_ν} bedeutet.*"

tenlänge $\lambda, \lambda/2, \cdots, \lambda/2^m$ *zerlegen, derart dass jedes* Q_μ *und* R_ν *der Bedingung* (I) *genügt.*

Legt man nun bei dem GOURSAT'schen Beweise des Integralsatzes eine in vorstehender Weise *modificirte* Gebietstheilung zu Grunde, so behalten nichtsdestoweniger alle *wesentlichen* Schlüsse ihre Geltung, da sie in Wahrheit von einer etwaigen *Congruenz* der betreffenden Quadrate völlig unabhängig sind. Man hat nämlich für jedes der oben mit Q_μ bezeichneten quadratischen Theilstücke, falls dessen Seitenlänge mit a_μ bezeichnet wird:

$$\int_{(Q_\mu)} |z_\mu - \xi_\mu| \, |dz_\mu| < (\sqrt{2}\, a_\mu)\, 4\, a_\mu = \sqrt{2}\, 4\, Q_\mu .$$

Und wenn R_ν ein Bruchstück eines Quadrates $Q_\nu = a_\nu^2$ bedeutet, welches zum Theil von dem Curvenstücke s_ν begrenzt wird:

$$\int_{(R_\nu)} |z_\nu - \xi_\nu| \, |dz_\nu| < (\sqrt{2}\, a_\nu)\, (4\, a_\nu + s_\nu) \leqq \sqrt{2}\, (4\, Q_\nu + \lambda s_\nu).^*$$

Bezeichnet man also mit $\mathfrak{A}$ die Summe der Quadrate des ursprünglichen λ-Gitters, soweit dieselben Innen-Punkte von A enthalten, mit S die Gesammtlänge der Randcurve von A, und fasst ausserdem wieder die Q_μ, R_ν unter der Beziehung A_ν zusammen, so folgt:

$$\sum \int_{(A_\nu)} |z_\nu - \xi_\nu| \, dz_\nu < \sqrt{2}\, (4\, \mathfrak{A} + \lambda S),$$

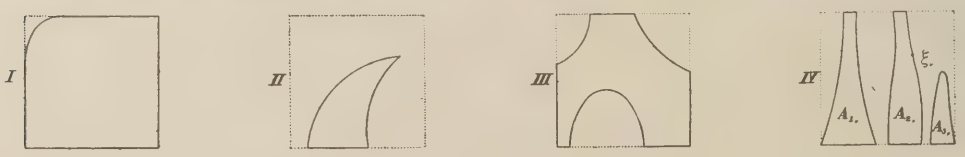
worauf dann alles übrige genau wie bei dem ursprünglichen GOURSAT'schen Beweise sich ergiebt.

Ich glaube nicht fehl zu gehen, wenn ich annehme, dass Herr GOURSAT den oben citirten Ausspruch ungefähr im Sinne der eben gemachten Auseinandersetzungen verstanden wissen will: immerhin dürften die vorstehenden Erläuterungen wohl nicht so ganz selbstverständlich und somit auch nicht überflüssig erscheinen.

Der Vollständigkeit halber möchte ich noch hinzufügen, dass die für den obigen Beweis, noch erforderliche Relation:

---

* Dabei könnte R_ν ausser der von Herrn GOURSAT (Acta Mathematica, a. a. O., p. 199) schematisch angedeuteten Gestalt (Fig. I) auch diejenige von Fig. II haben. Die obige Ungleichung bleibt aber auch richtig für ein Theilstück von der Form Fig. III und sogar, wenn der in der vorigen Fussnote charakterisirte Fall vertritt (Fig. IV): natürlich hat man dann unter s_ν die *Summe* der betreffenden Curvenbögen zu verstehen.



$$\int_{(c)} z\,dz = 0$$

(wo c eine geschlossene Curve bedeutet, die man im übrigen nur als stetig und rectificirbar anzunehmen braucht) entgegen einer früher von mir gemachten, die principielle Einfachheit des GOURSAT'schen Beweises mit Unrecht in Frage stellenden Bemerkung,* *ohne* jede umständlichere Grenzbetrachtung auf die allereinfachste Art gewonnen werden kann, sofern nur die Bedeutung und eindeutige Existenz von $\int f(z)\,dz$, erstreckt über eine Curve vom Charakter c in der üblichen Weise festgestellt ist. Darnach hat man nämlich, wenn $z_0, z_1, \cdots, z_{n-1}$, $z_n = z_0$ eine cyklische Folge von Punkten auf der Curve c bedeuten und ξ_ν einen willkürlichen Punkt des Bogenstückes $z_{\nu-1}z_\nu$ vorstellt:

$$\int_{(c)} z\,dz = \mathop{\mathrm{L}}_{n=\infty} \sum_{\nu=1}^{n} (z_\nu - z_{\nu-1})\,\xi_\nu \qquad (z_n = z_0).$$

Setzt man jetzt einmal: $\xi_\nu = z_\nu$, darauf: $\xi_\nu = z_{\nu-1}$, so folgt durch Addition der betreffenden Gleichungen:

$$2\int_{(c)} z\,dz = \mathop{\mathrm{L}}_{n=\infty} \sum_{\nu=1}^{n} (z_\nu^2 - z_{\nu-1}^2) = \mathop{\mathrm{L}}_{n=\infty} (z_n^2 - z_0^2) = 0, \quad \text{q.e.d.}$$

Durch die vorstehenden Betrachtungen wird der fragliche Integralsatz zunächst nur für den Fall *abtheilungsweise monotoner* Randcurven † gewonnen. Will man *rectificirbare* Curven im *allgemeinsten* (d. h. SCHEEFFER-JORDAN'schen) Sinne als Begrenzung zulassen ‡ so bleibt wohl kaum ein anderer Weg, als dass man dem obigen Resultate lediglich die Gültigkeit des Satzes für ein *Dreieck* entnimmt und sodann mit Hülfe einer Grenzbetrachtung auf *rectificir-*

---

* Münchener Berichte, Bd. 25 (1895), p. 42, Fussnote.

† Bezw. für Randcurven, welche von keiner Parallelen zu den Axen in unendlich vielen Punkten geschnitten werden.

‡ In Bd. 1 dieser Zeitschrift, p. 501, letzte Fussnote, bemerkt Herr E. H. MOORE, ich scheine leider mit dem heutzutage üblichen Begriffe der *allgemeinen rectificirbaren Curven* nicht in Fühlung zu sein ("*out of touch with* * * *"). Das dürfte indessen wohl nicht ganz zutreffen. Ich glaube sogar in Folge des intimen Umganges mit dem Schöpfer jenes Begriffes, meinem leider allzu früh verstorbenen Freunde LUDWIG SCHEEFFER, hierüber ebenso früh und ebenso genau orientirt gewesen zu sein, als irgend ein anderer. Immerhin hatte ich, als ich den von Herrn MOORE citirten Aufsatz (Münchener Berichte, 1895) über den CAUCHY'schen Integralsatz schrieb, nach dem Vorgange von DU BOIS-REYMOND gegen die definitive Einführung des verallgemeinerten Rectifications-Begriffes gewisse Bedenken (a. a. o., p. 55, Fussnote 2). Seitdem ist nun freilich mein Glaube an des letzteren Autorität, wie in manchen anderen Punkten, so auch in *diesem* merklich geschwunden. Und nachdem durch die *zweite* Auflage von C. JORDAN'S *Cours d'Analyse* T. I (welche zur Zeit der Abfassung jenes Aufsatzes zwar schon erschienen, mir jedoch unbekannt geblieben war) die *Fruchtbarkeit* des fraglichen Begriffes in das richtige Licht gesetzt worden ist, habe ich denselben auch bedingungslos acceptirt (s. z. B. Encyklop. der Math. W., Bd. II, p. 41).

bare Curven überträgt. * Hat man also dieses Ziel vor Augen, so erscheint es offenbar weit zweckmässiger, jenen Integralsatz zunächst *überhaupt nur* für den wesentlich einfacher gearteten Fall eines *Dreiecks* zu beweisen. Nach meinem Dafürhalten würde aber dieser Weg selbst dann jedem anderen vorzuziehen sein, wenn man sich—etwa aus didaktischen Rücksichten—schliesslich mit der Zulassung *abtheilungsweise monotoner* oder auch nur "*gewöhnlicher*" Randcurven begnügen will. Die Definition bezw. der Existenz-Beweis für das über eine Curve erstreckte Integral $\int f(z)\,dz$ erfordert lediglich die *Stetigkeit*, nicht aber die *Differenzirbarkeit* von $f(z)$. Auf *derselben* Grundlage ergiebt sich, dass eine infinitesimale Aenderung des Integrationsweges auch nur eine infinitesimale Aenderung des Integralwerthes nach sich zieht, dass also in's besondere jedes *Curven*-Integral mit beliebiger Annäherung durch ein *Polygon*-Integral ersetzt werden kann. Diese Erkenntniss ist *an sich* wichtig und für zahlreiche analytische Untersuchungen geradezu *unentbehrlich*. Hat man sie nun aber einmal gewonnen, so liegt in der That gar kein vernünftiger Grund vor, den Beweis des eigentlichen Integralsatzes mit denjenigen Complicationen zu belasten, die sich aus der mehr oder minder zusammengesetzten Natur der Randcurve ergeben. Der wahre *Kern* jenes Integralsatzes liegt in seiner Gültigkeit für irgend einen *Special*-Bereich *einfachster* Art z. B. ein *Dreieck* und unter Voraussetzung der *Differenzirbarkeit* von $f(z)$. Die Möglichkeit, ihn auf *krummlinig* begrenzte Bereiche zu übertragen, beruht dagegen lediglich auf *Stetigkeits*-Eigenschaften, welche den Integralen *jeder stetigen* Function zukommen. †

Dies vorausgeschickt, glaube ich vielleicht manchem einen kleinen Dienst zu leisten, wenn ich den fraglichen „*Dreiecks*"-Beweis, einschliesslich der hierfür zweckmässigen Formulirung und Herleitung des GOURSAT'schen Lemmas im folgenden noch vollständig durchführe.

LEMMA I. (*Specialform des Goursat'schen Lemmas*). *Ist $f(z)$ für jede einzelne Stelle z' im Innern und auf der Begrenzung des Dreiecks Δ eindeutig definirt, stetig und mit einem endlichen ‡ Differentialquotienten $f'(z)$ begabt, so*

* Vgl. C. JORDAN, a. a. O., § 196, 197. Auch der in Bd. 1 dieser Zeitschrift, pp. 499–506 mitgetheilte Beweis des Herrn MOORE liefert in der bezeichneten Richtung keinen anderen Weg (vgl. a. a. o., p. 505), da hierbei die Randcurven ebenfalls noch von gewissen Special-Bedingung, übrigens von ziemlich complicirtem Charakter genügen müssen.

† In der That braucht, bei der Beschränkung auf *abtheilungsweise monotone* Randcurven C, für die Gültigkeit des Integralsatzes nur die *Stetigkeit*, *nicht* die *Differenzirbarkeit* von $f(z)$ auf C selbst vorausgesetzt zu werden. Vgl. Münchener Berichte, a. a. O., p. 71. (Für den Fall, dass C „Schnabelspitzen" besitzt, wäre noch eine einfache Modification zu dem a. a. o. gesagten hinzuzufügen).

‡ Damit ist also nicht einmal vorausgesetzt, dass die Gesammtheit der Zahlen $|f'(z)|$ *unter einer endlichen Schranke* bleibt. Auch brauchen in Bezug auf die Stetigkeit von $f(z)$ und bei der Bildung von $f'(z)$ durchaus nur solche Punkte in Betracht gezogen zu werden, welche dem Innern oder der Begrenzung von Δ angehören (mit anderen Worten: $f(z)$ braucht „*nach aussen*" weder stetig, noch differenzirbar zu sein).

lässt sich Δ zu jedem $\epsilon > 0$ in eine endliche Anzahl unter sich und mit Δ ähnlicher (im allgemeinen aber nicht congruenter) Dreiecke Δ_ν zerlegen, derart dass:

$$(\mathrm{I}) \qquad |f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)| \leqq \epsilon |z_\nu - \xi_\nu|.$$

Dabei bedeutet z_ν jeden beliebigen Punkt auf der Begrenzung von Δ_ν, dagegen ξ_ν einen bestimmten, im Inneren oder auf der Begrenzung von Δ_ν allemal wirklich vorhandenen Punkt.

Beweis. Der auf die Existenz von $f'(z)$ bezügliche Theil der Voraussetzung besagt folgendes: Zu jeder Stelle z' und zu jedem $\epsilon > 0$ existirt eine bestimmte Zahl $f'(z')$ und eine positive Zahl ρ'_ϵ, sodass:

$$\left| \frac{f(z) - f(z')}{z - z'} - f'(z') \right| < \epsilon \quad \text{für:} \quad 0 < |z - z'| < \rho'_\epsilon,$$

und daher:

$$(1) \qquad |f(z) - f(z') - (z - z') f'(z')| \leqq \epsilon |z - z'| \quad \text{für:} \quad 0 \leqq |z - z'| < \rho'_\epsilon.$$

Man halbire nun zunächst die 3 Seiten $a^{(1)}$, $a^{(2)}$, $a^{(3)}$ von Δ und zerlege Δ durch geradlinige Verbindung der Halbirungs-Punkte in 4 congruente, dem ursprünglichen Δ ähnliche Dreiecke mit den Seiten $a^{(\kappa)}/2$ $(\kappa = 1, 2, 3)$. Jedes dieser 4 Dreiecke, sofern es nicht bereits der Bedingung (I) genügt, zerlege man auf die nämliche Art und setze dieses Verfahren analog d. h. in *der* Weise fort, dass jedes zum Vorschein kommende Theildreieck, welches der Bedingung (I) genügt, von der weiteren Untertheilung ausgeschlossen wird. Man muss dann allemal nach einer begrenzten Anzahl solcher Operationen zu einer Eintheilung gelangen, bei welcher *jedes* Theildreieck der Bedingung (I) genügt.

Denn *angenommen*, dies wäre *nicht* der Fall, so müsste wenigstens *eine unbegrenzte* Folge von Dreiecken

$$\Delta, \Delta'_1, \Delta'_2, \cdots, \Delta'_\nu, \cdots$$

mit den resp. Seitenlängen:

$$a^{(\kappa)}, \frac{a^{(\kappa)}}{2}, \frac{a^{(\kappa)}}{2^2}, \cdots, \frac{a^{(\kappa)}}{2^\nu}, \cdots \qquad (\kappa = 1, 2, 3)$$

resultiren, derart dass jedes dieser Dreiecke Δ'_ν einen Bestandtheil (nämlich ein Viertel) des vorangehenden bildet, also auch in *sämmtlichen* vorangehenden enthalten ist, während andererseits *keins* dieser Dreiecke der Bedingung (I) genügt. Es müsste dann *ein* bestimmter Punkt ξ existiren, welcher im Innern oder auf der Begrenzung *sämmtlicher* Δ'_ν (incl. Δ) liegt. Alsdann hätte man aber nach Ungl. (1):

$$(2) \qquad |f(z) - f(\xi) - (z - \xi) f'(\xi)| \leqq \epsilon |z - \xi|$$

für ein gewisses $\rho_\epsilon > 0$ und alle dem Bereiche Δ angehörigen, der Bedingung: $|z - \xi| < \rho_\epsilon$ genügenden Punkte z. Wird nun aber n so gross angenommen, dass: $a^{(\kappa)}/2^n < \rho_\epsilon$ $(\kappa = 1, 2, 3)$, so fallen sämmtliche Δ'_ν, für welche $\nu \geqq n$, *ganz innerhalb* des Geltungsbereiches der Ungleichung (2), würden also speciell auch in Bezug auf ihre *Begrenzungspunkte* der Bedingung (I) genügen, was der oben gemachten Annahme *widerspricht*. Dieselbe erscheint daher als unzulässig, womit dann das ausgesprochene Lemma bewiesen ist.

Lemma II. *Zerfällt bei der in dem vorigen Lemma beschriebenen Theilung das Dreieck Δ mit dem Umfange $a^{(1)} + a^{(2)} + a^{(3)} = s$ im ganzen in p ähnliche Dreiecke Δ_ν mit den resp. Umfängen s_ν, so ist:*

$$\text{(II)} \qquad \sum_{\nu=1}^{p} s_\nu^2 = s^2,$$

Beweis. Es besitze die *kleinste* Art von Dreiecken, die bei der fraglichen Eintheilung vorkommt, die Seitenlängen $a^{(\kappa)}/2^n$ $(\kappa = 1, 2, 3)$, also den Umfang $s/2^n$. Jedes etwa vorkommende *grössere* Dreiecke Δ_ν besitzt dann einen Umfang: $s_\nu = 2^{m_\nu} \cdot s/2^n$, wo: $1 \leqq m_\nu < n$, und lässt sich darnach in 2^{2m_ν} jener *Minimal*-Dreiecke zerlegen. Da nun: $s_\nu^2 = 2^{2m_\nu} \cdot (s/2^n)^2$, so darf man in dem Ausdrucke: $\sum_{\nu=1}^{\nu=p} s_\nu^2$ jedes s_ν^2, welches nicht schon von einem *Minimal*-Dreiecke Δ_ν herrührt, also den Werth: $1 \cdot (s/2^n)^2$ besitzt, gerade durch dasjenige *Multiplum* von $(s/2^n)^2$ ersetzen, welches die genaue *Anzahl* der in Δ_ν Platz findenden *Minimal*-Dreiecke angiebt. Da aber die Gesammtzahl der in Δ überhaupt unterzubringenden *Minimal*-Dreiecke 2^{2n} ist, so ergiebt sich:

$$\sum_{\nu=1}^{p} s_\nu^2 = 2^{2n} \cdot \left(\frac{s}{2^n}\right)^2 = s^2, \quad \text{q.e.d.*}$$

Hauptsatz. *Ist $f(z)$ für jede einzelne Stelle im Innern und auf der Begrenzung des Dreiecks Δ eindeutig definirt, stetig und mit einem endlichen $f'(z)$ begabt, so ist:*

$$\int_{(\Delta)} f(z)\, dz = 0.$$

Beweis. Man nehme $\epsilon > 0$ beliebig klein an und denke sich das Dreieck Δ auf Grund des Lemma I in p ähnliche Dreiecke Δ_ν zerlegt, welche der Bedingung (I) genügen. Man hat dann zunächst:

---

* Der Beweis gestaltet sich noch kürzer, wenn man den *geometrischen* Satz benützen will, dass die Flächen ähnlicher Dreiecke sich verhalten, wei die Quadrate homologer Seiten, also auch wie diejenigen der Perimeter. Mann findet alsdann ohne weiteres:

$$\sum_{\nu=1}^{p} s_\nu^2 : s^2 = \sum_{\nu=1}^{p} \Delta_\nu : \Delta, \quad \text{d.h.} \quad \sum_{\nu=1}^{p} s_\nu^2 = s^2.$$

$$\int_{(\Delta)} f(z)\,dz = \sum_{\nu=1}^{p} \int_{(\Delta_\nu)} f(z)\,dz = \sum_{\nu=1}^{p} \int f(z_\nu)\,dz_\nu ,$$

vorausgesetzt, dass alle Integrale in *demselben*, etwa dem *positiven* Sinne verstanden werden.

Nun ist identisch:

$$\int f(z_\nu)\,dz_\nu = \int \{f(z_\nu) - f(\xi_\nu) - (z_\nu - \xi_\nu) f'(\xi_\nu)\}\,dz_\nu$$

$$+ \{f(\xi_\nu) - \xi_\nu f'(\xi_\nu)\} \int dz_\nu + f'(\xi_\nu) \int z_\nu\,dz_\nu ,$$

und daher, wegen:

$$\int dz_\nu = 0 , \qquad \int z_\nu\,dz_\nu = 0 ,$$

und mit Berücksichtigung von Ungl. (I):

$$\left| \int f(z_\nu)\,dz_\nu \right| \leqq \int |f(z_\nu) - f(\xi_\nu) - (z - \xi_\nu) f'(\xi_\nu)|\,|dz_\nu| ,$$

$$\leqq \epsilon \int |z_\nu - \xi_\nu|\,|dz_\nu| .$$

Wird nun wiederum der Umfang von Δ_ν mit s_ν bezeichnet, so hat man unter allen Umständen:

$$|z_\nu - \xi_\nu| < \tfrac{1}{2} s_\nu ,$$

also:

$$\left| \int f(z_\nu)\,dz_\nu \right| < \tfrac{1}{2} \epsilon\, s_\nu \int |dz_\nu| = \frac{\epsilon}{2} s_\nu^2 ,$$

und daher mit Berücksichtigung von Lemma II:

$$\left| \int_{(\Delta)} f(z)\,dz \right| < \frac{\epsilon}{2} \sum_{\nu=1}^{p} s_\nu^2 = \frac{\epsilon}{2} s^2 ,$$

d. h. schliesslich:

$$\int_{(\Delta)} f(z)\,dz = 0 , \quad \text{q.e.d.}^*$$

MÜNCHEN, 5. Mai, 1901.

* In ganz analoger Weise kann der Beweis auch für den Fall eines *Rechtecks* geführt werden: der von mir in Bd. I (1900) der Bibliotheca mathematica, p. 478 skizzirte Beweis ist in diesem Sinne zu vervollständigen. In seiner jetzigen Fassung beruht er auf derjenigen unzutreffenden Anwendung des GOURSAT'schen Lemmas, welche auf den ersten Seiten der vorliegenden Note des näheren erörtert wurde.

NEW PROOF OF A THEOREM OF OSGOOD'S IN THE CALCULUS OF VARIATIONS*

BY

OSKAR BOLZA

In the last number of the Transactions,† Professor Osgood has proved an important characteristic property of a strong minimum of an integral of the form

$$I = \int_{\tau_0}^{\tau_1} F(x, y, x', y')\, d\tau. \tag{1}$$

His proof, however, is rather complicated, and the following note is intended to give a simpler proof of the theorem.

§ 1.

Introduction of curvilinear coördinates.

Suppose the integral (1) is taken along a continuous curve C with continuously turning tangents:

$$C: \qquad x = \phi(\tau), \quad y = \psi(\tau) \qquad (\tau_0 \leqq \tau \leqq \tau_1)$$

joining two fixed points $A(\tau_0)$ and $B(\tau_1)$; further $\phi'^2 + \psi'^2 \neq 0$ in (τ_0, τ_1).

Concerning the function $F(x, y, x', y')$ we make the same assumptions as Osgood on p. 277, l. c., except the assumption $F > 0$, which is not necessary for the present proof.

Now introduce instead of the rectangular coördinates x, y any curvilinear coordinates

$$u = U(x, y), \quad v = V(x, y), \tag{2}$$

where U, V are single-valued functions with continuous first and second derivatives in a region T of the x,y-plane containing the curve C; in the same region their Jacobian is supposed $\neq 0$. Interpret u, v as the rectangular co-

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication September 2, 1901.

† Transactions of the American Mathematical Society, vol. 2, p. 273.

ordinates of a point in a u,v-plane and denote by T', C', A', B' the images of T, C, A, B respectively. Suppose further that the correspondence between T and T' is a one-to-one correspondence and that accordingly the inverse functions

$$x = X(u, v), \quad y = Y(u, v), \tag{3}$$

are single-valued functions with continuous first and second derivatives in T' and

$$D = \frac{\partial(X, Y)}{\partial(u, v)} \neq 0$$

in T'.

Then the integral I is changed into

$$I' = \int_{\tau_0}^{\tau_1} G(u, v, u', v')\, d\tau, \tag{4}$$

the function G of the four arguments u, v, u', v' being defined by

$$G(u, v, u', v') \equiv F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'), \tag{5}$$

where $X_u = \partial X/\partial u$, etc.

The integral I' is taken, in the u,v-plane, along the image C' of C.

From $I = I'$ it follows that if the curve C minimize the integral I, its image C' will minimize I', and vice versa; and if C be an extremal for I, C' must be an extremal for I', and vice versa. Further WEIERSTRASS's function F_1 is an invariant for the above transformation, viz., if we denote the corresponding function derived from G by G_1, we obtain easily

$$G_1 = D^2 F_1. \tag{6}$$

Finally WEIERSTRASS's E-function is an absolute invariant, i. e., if we denote the new E-function by E' we have:

$$\left\{\begin{array}{l} E'(u, v; u', v'; \bar{u}', \bar{v}') = E(x, y; x', y'; \bar{x}', \bar{y}') \\ \qquad\qquad \text{where} \\ x' = X_u u' + X_v v', \quad \bar{x}' = X_u \bar{u}' + X_v \bar{v}', \\ y' = Y_u u' + Y_v v', \quad \bar{y}' = Y_u \bar{u}' + Y_v \bar{v}', \end{array}\right. \tag{7}$$

as follows immediately from (5).

§ 2.

Proof of Osgood's theorem.

Now let

$$x = \phi(t, a), \quad y = \psi(t, a) \tag{8}$$

be a set of extremals for the integral I, satisfying the following conditions:

1. The functions ϕ, ψ are single-valued functions of t, a with continuous first and second derivatives in the region:

$$T_0 - \epsilon \leqq t \leqq T_1 + \epsilon, \quad |a - a_0| \leqq \kappa \qquad (\epsilon > 0).$$

2. The extremal $C_0 : x = \phi(t, a_0)$, $y = \psi(t, a_0)$ has no multiple point for $T - \epsilon \leqq t \leqq T_1 + \epsilon$, and passes through the two given points $A(t_0)$ and $B(t_1)$ where $T_0 < t_0 < t_1 < T_1$.

3. If we denote by $\Delta(t, a)$ the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)},$$

then

$$\Delta(t, a_0) \neq 0 \qquad \text{in} \qquad (T_0 - \epsilon, \ T_1 + \epsilon). \tag{9}$$

4. The inequality

$$F_1\big(\phi(t, a_0), \ \psi(t, a_0), \ \cos\lambda, \ \sin\lambda\big) > 0 \tag{10}$$

holds for every t of the interval $T_0 - \epsilon \leqq t \leqq T_1 + \epsilon$ and for every λ.

Under these circumstances if we denote by R_κ the region:

$$R_\kappa : \qquad T_0 \leqq t \leqq T_1, \quad |a - a_0| \leqq \kappa,$$

and denote by S_κ the image of R_κ in the x,y-plane, then κ can be taken so small that

1. $\Delta(t, a) \neq 0$ in R_κ.
2. $F_1(x, y, \cos\lambda, \sin\lambda) > 0$ for every x, y in S_κ and for every λ.
3. The correspondence between R_κ and S_κ is a *one-to-one correspondence.**

The inverse functions

$$t = t(x, y), \qquad a = a(x, y)$$

will then be single-valued with continuous first and second derivatives in S_κ and their Jacobian will be $\neq 0$ in S_κ.

Then S_κ is a "field" surrounding the arc AB of the extremal C_0, and if

$$\bar{C} : \qquad x = \bar{\phi}(\tau), \quad y = \bar{\psi}(\tau), \qquad (\tau_0 \leqq \tau \leqq \tau_1),$$

be any other curve with continuously turning tangents drawn from A to B in S_κ, for which $\bar{\phi}'^2 + \bar{\psi}'^2 \neq 0$, then WEIERSTRASS'S theorem holds, according to which

$$\Delta I = I_{\bar{C}}(AB) - I_C(AB) = \int_{\tau_0}^{\tau_1} E(x, y \,; \, x', y' \,; \, \bar{x}', \bar{y}')\, d\tau, \tag{11}$$

* See KNESER, *Lehrbuch der Variationsrechnung*, § 14, and OSGOOD, l. c., p. 278. Both proofs have to be supplemented by the following preliminary lemma: If for every κ, however small, there existed points (x, y) in S_κ to which correspond in R_κ at least two distinct points (t', a') and (t'', a''), then there must exist, in R_κ, a point (t_2, a_0) such that in every vicinity of it pairs of distinct points can be found whose images in the x, y-plane coincide.

where x', y' refer to the extremal of the set (8) passing through x, y, and $\bar{x}'$, $\bar{y}'$ to the curve C. Hence WEIERSTRASS infers that $\Delta I > 0$ by making use of the following theorem † connecting the functions E and F_1:

$$(12)\quad E(x, y;\ \cos\vartheta, \sin\vartheta;\ \cos\bar{\vartheta}, \sin\bar{\vartheta}) = (1 - \cos\omega) F_1(x, y, \cos\vartheta^*, \sin\vartheta^*),$$

where $\omega = \bar{\vartheta} - \vartheta$; ϑ^* is an angle between ϑ and $\bar{\vartheta}$; and the angles are so measured that $|\omega| \leqq \pi$.

Now let $0 < h < \kappa$ and use S_h in the analogous signification as S_κ; then OSGOOD'S theorem may be stated as follows: *There exists a positive quantity depending upon* h, ϵ_h, *such that for every curve* C *joining* A *and* B *drawn in the interior of* S_κ *but not wholly contained in* S_h,

$$\Delta I > \epsilon_h.$$

To prove this theorem it is now only necessary to introduce instead of x, y the curvilinear coördinates

$$(13)\qquad u = t(x, y), \quad v = a(x, y)$$

which satisfy for the regions S_κ and R_κ the conditions of § 1, and to make use of the remarks made there.

Accordingly the extremals for the integral I' are the lines $v = \text{const.}$, and, therefore, in the u,v-plane WEIERSTRASS'S theorem takes the form:

$$\Delta I = \Delta I' = \int_{s_0}^{s_1} E'(u, v;\ \cos 0, \sin 0;\ \cos\omega, \sin\omega)\, ds$$
$$= \int_{s_0}^{s_1} (1 - \cos\omega) G_1(u, v, \cos\vartheta^*, \sin\vartheta^*)\, ds.$$

These integrals are taken along the image $\bar{C}'$ in the u,v-plane (i. e., t,a-plane) of the curve C; ω denotes the angle between the positive tangent to $\bar{C}'$ in the point u, v and the positive u-axis; and s is the arc of the curve $\bar{C}'$.

But from (9), (10), and (6) it follows that we can assign a positive quantity m such that

$$G_1(u, v, \cos\lambda, \sin\lambda) \geqq m > 0$$

for every u, v in R_κ and for every λ. Therefore

$$\Delta I \geqq m \int_{s_0}^{s_1} (1 - \cos\omega)\, ds,$$

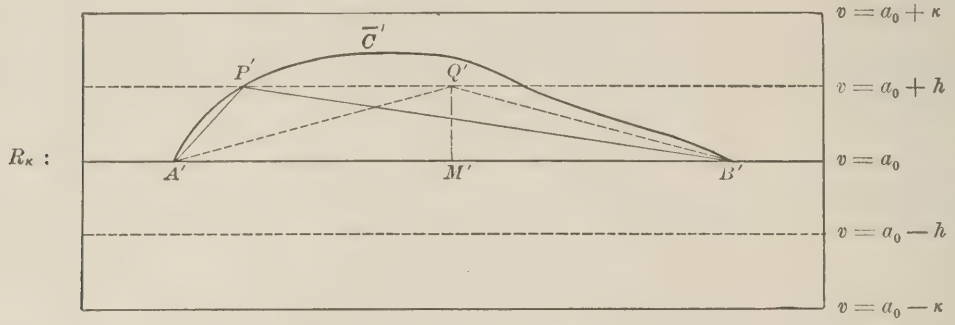
that is

$$\Delta I \geqq m\big(l - (t_1 - t_0)\big),$$

if l denotes the length of the curve $\bar{C}'$.

† *Lectures on the Calculus of Variations*, 1882.

Now suppose that $\bar{C}$ is not wholly contained in the interior of S_h and therefore passes through a point P of one of the two extremals $a = a_0 \pm h$, then $\bar{C}'$ passes through a point P' whose ordinate is $v = a_0 \pm h$. Hence l is greater



than or equal to the length of the broken line $A'P'B'$. But if we choose Q' on the same line $v = \text{const.}$ as P' so that $A'Q' = B'Q'$, then

$$A'P'B' \geqq A'Q'B' ,$$

and therefore

$$\Delta I \geqq 2m\left[\sqrt{h^2 + \left(\frac{t_1 - t_0}{2}\right)^2} - \frac{(t_1 - t_0)}{2}\right],$$

which proves OSGOOD'S theorem. *

OSGOOD'S theorem can easily be extended to the case where one endpoint, say B, is fixed, the other, A, movable on a given curve. For this purpose, it is only necessary to choose for the new coördinates KNESER'S curvilinear coördinates u, v (KNESER, § 16); a slight modification of the above reasoning leads then to the inequality

$$\Delta I \geqq m[\sqrt{h^2 + (u_1 - u_0)^2} - (u_1 - u_0)] .$$

---

* OSGOOD bases his proof upon the following lemma: "Let $f(x)$ be a single valued continuous function of x in the interval $a \leqq x \leqq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let $a < l \leqq b$ and $|f(l) - f(a)| = L > 0$. Then

$$\int_a^l f'(x)^2 dx \geqq \frac{L^3}{54(b-a)^2},$$

provided that $L < 3\sqrt{2}(b-a)$."

This lemma may be proved as follows: Since the value of the integral does not change if $f(x)$ is replaced by $M \pm f(x)$, M being a constant, we may confine ourselves to functions $f(x)$ for which $f(a) = 0$, $f(l) = L$. This remark reduces the question to the problem of minimizing the above integral with these given initial values. The solution is $y = L(x-a)/(l-a)$; it satisfies WEIERSTRASS' sufficient conditions for a minimum, and furnishes for the integral the minimum value: $(L^2)/(l-a) \geqq (L^2)/(b-a)$, which under the above inequality for L is greater than $\frac{1}{54}(L^3)/(b-a)^2$.

It is, however, to be remembered that the introduction of KNESER'S coördinates presupposes that

$$F\left(\phi(t, a),\ \psi(t, a),\ \phi'(t, a),\ \psi'(t, a)\right) > 0$$

in the region R_κ.

UNIVERSITY OF CHICAGO,
August 31, 1901.

ON CERTAIN PAIRS OF TRANSCENDENTAL FUNCTIONS WHOSE ROOTS SEPARATE EACH OTHER*

BY

MAXIME BÔCHER

Let us consider eight functions $p, q, \phi_1, \phi_2, \psi_1, \psi_2, \chi_1, \chi_2$ of the real variable x, which throughout the interval

$$(I) \qquad a \leqq x \leqq b$$

are real and continuous, and of which the last six have finite first derivatives at every point of this interval. Let y be a solution, which we assume to be not identically zero, of the differential equation

$$(1) \qquad y'' + py' + qy = 0.$$

The functions whose roots we wish to consider in this paper are the following:

$$\Phi = \phi_2 y' - \phi_1 y, \quad \Psi = \psi_2 y' - \psi_1 y, \quad \mathrm{X} = \chi_2 y' - \chi_1 y.$$

It should be noticed that, as a consequence of the assumptions made above, these functions are continuous and have finite first derivatives at every point of the interval (I).

The roots of functions of this form were considered by STURM† by a method which, however, yields results distinctly less general than those we shall obtain. The relation of our methods and results to those of STURM will be indicated at the proper places.

The third section of the present paper may appear at first sight to be of slight importance, the generalizations of the results of the first two sections which are contained in it being in themselves not very far-reaching. The method used in establishing these generalizations is, however, I believe, a valuable one apart from the special application here made of it. I have indicated one application to other questions in the footnote on p. 435.

* Presented to the Society August 19, 1901. Received for publication August 10, 1901.

† Liouville's Journal, vol. 1 (1836), pp. 149–164.

§ 1. *Two functions.*

We shall consider in the present section only the two functions Φ, Ψ. These functions satisfy a differential equation which may be obtained as follows:

We start from the identity:

$$\Phi(\psi_2 y' - \psi_1 y) - \Psi(\phi_2 y' - \phi_1 y) = 0.$$

From this we obtain by differentiation the equation:

$$(\Phi'\psi_2 - \Psi'\phi_2 + \Phi\psi_2' - \Psi\phi_2' - \Phi\psi_1 + \Psi\phi_1 - p\Phi\psi_2 + p\Psi\phi_2)y'$$
$$- (\Phi'\psi_1 - \Psi'\phi_1 + \Phi\psi_1' - \Psi\phi_1' + q\Phi\psi_2 - q\Psi\phi_2)y = 0.$$

Since y and y' cannot both vanish at any point of (I) we can eliminate them between the last two equations. This gives:

$$(2)\quad (\phi_1\psi_2 - \phi_2\psi_1)(\Phi'\Psi - \Phi\Psi') + \{\psi_1, \psi_2\}\Phi^2 - \{\psi_1, \phi_2, \phi_1, \psi_2\}\Phi\Psi + \{\phi_1, \phi_2\}\Psi^2 = 0,$$

where for the sake of abbreviation we have set

$$\{\psi_1, \psi_2\} = \psi_1'\psi_2 - \psi_1\psi_2' + \psi_1^2 + p\psi_1\psi_2 + q\psi_2^2,$$
$$\{\psi_1, \phi_2, \phi_1, \psi_2\} = \psi_1'\phi_2 - \psi_1\phi_2' + \phi_1'\psi_2 - \phi_1\psi_2' + 2\phi_1\psi_1 + p(\psi_1\phi_2 + \phi_1\psi_2) + 2q\phi_2\psi_2,$$
$$\{\phi_1, \phi_2\} = \phi_1'\phi_2 - \phi_1\phi_2' + \phi_1^2 + p\phi_1\phi_2 + q\phi_2^2.$$

Equation (2) is the differential equation we were seeking. It may be termed a homogeneous Riccati's equation for the pair of functions Φ, Ψ.*

If the two functions Φ and Ψ were to vanish at the same point we should have at this point

$$\phi_2 y' - \phi_1 y = 0,$$
$$\psi_2 y' - \psi_1 y = 0,$$

and, since y and y' cannot both vanish at this point, as otherwise y would be identically zero, we must have at this point

$$\phi_1\psi_2 - \phi_2\psi_1 = 0.$$

* By a Riccati's equation is ordinarily understood an equation of the form:

$$\omega' = A + B\omega + C\omega^2.$$

If for ω we write ω_1/ω_2 this equation takes the homogeneous form:

$$\omega_1'\omega_2 - \omega_1\omega_2' = A\omega_2^2 + B\omega_1\omega_2 + C\omega_1^2,$$

and this is the form of equation (2). Conversely if we divide (2) by Φ^2 and Ψ^2 respectively we obtain at once the ordinary Riccati's equation for Ψ/Φ and Φ/Ψ.

It should be noticed that the function $\{\phi_1, \phi_2\}$ (or $\{\psi_1, \psi_2\}$) when equated to zero gives the homogeneous form of the Riccati's equation which is satified by y'/y.

By assuming, as we shall now do, that $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish at any point of (I) we make it impossible for Φ and Ψ to vanish together.

In this section and the next we shall not use equation (2) itself, but only the following formulæ which follow from it at once:

At a point where $\Phi = 0$:

$$\Phi' = -\frac{\{\phi_1, \phi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}\Psi. \tag{3}$$

At a point where $\Psi = 0$:

$$\Psi' = \frac{\{\psi_1, \psi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}\Phi. \tag{4}$$

Since Φ and Ψ cannot vanish together we see from (3) that, if $\{\phi_1, \phi_2\}$ does not vanish in (I), Φ and Φ' cannot vanish together, and therefore Φ cannot vanish an infinite number of times in (I). Whatever the functions ϕ_1 and ϕ_2 may be, provided merely they do not both vanish at any point of (I), the functions ψ_1 and ψ_2 can be so chosen (e. g., $\psi_1 = -\phi_2$, $\psi_2 = \phi_1$) that $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish in (I). Moreover if ϕ_1 and ϕ_2 vanished at the same point, $\{\phi_1, \phi_2\}$ would also vanish there. We have therefore the theorem which it will be noticed relates to the function Φ alone:

I. *If $\{\phi_1, \phi_2\}$ does not vanish in (I), then Φ cannot vanish more than a finite number of times there and Φ and Φ' do not both vanish at any point of (I).**

Since at two successive roots of Φ the derivative Φ' has opposite signs, and since by (3) this can happen only when Ψ has opposite signs at these two points, we have the theorem:

II. *If neither of the two functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}$$

vanishes in (I), then in any portion of (I) in which Ψ does not vanish Φ cannot vanish more than once.†

* STURM (l. c., p. 150) proves this theorem for the special case $\phi_2 = e^{\int p dx}$ by using the special case of formula (2) obtained by letting $\psi_2 = 0$, $\psi_1 = -1$. This special case, written to be sure in a slightly modified form, is his equation (11). STURM also restricts himself to the case $\{\phi_1, \phi_2\} > 0$ (see his inequality (14)) and can thus add to our theorem the statement that Φ' and $y = \Psi$ have opposite signs when $\Phi = 0$. He does not point out the connection between this formula and the Riccati's equations which he uses on p. 159.

† This theorem is proved by STURM (l. c., p. 160) by a wholly different method, but only in the very special case

$$\psi_2 = 0, \ \psi_1 = -1, \ \phi_2 = e^{\int p dx}, \ q > 0, \ \phi_1' \gtrless 0.$$

This is the more remarkable as his formula (11) would have yielded him the result in the much more general case in which his last two conditions are replaced by the single condition $\{\phi_1, \phi_2\} > 0$. This would have brought his later results (pp. 158–164) into harmony with those which immediately precede them.

A similar result of course holds for the roots of the function Ψ, and by combining these facts we get the theorem:

III. *If none of the functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I) *then between two roots of* Φ *lies one and only one root of* Ψ, *and between two roots of* Ψ *lies one and only one root of* Φ.*

That is, the roots of Φ and Ψ separate each other.

Let us now consider two points at one of which Φ at the other Ψ vanishes while neither function vanishes between them. These points are two successive roots of the product $\Phi\Psi$, and therefore the derivative $\Phi'\Psi + \Phi\Psi'$ has opposite signs at them. From this it follows, as a glance at (3) and (4) shows, that if none of the functions

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I) then $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ must have the same sign. If, then, these two functions have opposite signs, our assumption that there exist two points, at one of which Φ at the other Ψ vanishes, must be false, and, therefore, by theorem III, neither of these functions can vanish twice. Hence the theorem:

IV. *If none of the functions:*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}$$

vanish in (I), *and if the last two of them have opposite signs, then neither* Φ *nor* Ψ *vanishes more than once in* (I), *and if one of these functions vanishes the other does not.*

A special case of this theorem deserves mention, namely that in which $\Psi = y$. In order that $\phi_1\psi_2 - \phi_2\psi_1$ shall not vanish, it is necessary and sufficient that ϕ_2 do not vanish in (I). If then we let $\phi = \phi_1/\phi_2$ we have the theorem:

V. *If* ϕ *is continuous, has a finite derivative at every point of* (I), *and satisfies the inequality:*

$$\{\phi, 1\} = \phi' + \phi^2 + p\phi + q < 0,$$

then neither y *nor* $y' - \phi y$ *vanishes more than once in* (I), *and if one of these functions vanishes the other does not.*†

* This theorem is also proved by STURM (l. c., pp. 161, 163) first in a case analogous to the one mentioned in the last footnote, then in the more general, but still very special, case

$$\phi_2 = \psi_2 = e^{\int p\,dx}, \; q > 0, \; \phi_1' \geqq 0, \; \psi_1' \geqq 0.$$

It should be noticed that the condition $\phi_1\psi_2 - \phi_2\psi_1 \neq 0$ now means that ϕ_1 is either everywhere greater or everywhere less than ψ_1.

† The part of this theorem which says that y cannot vanish more than once was proved by the writer by another method in the Bulletin of the American Mathematical Society, May, 1901, p. 335, footnote. Cf. also a less far reaching theorem of PICARD there quoted.

Thus for instance if $q < 0$, y and y' cannot both vanish in (I) nor can either of them vanish more than once.

§ 2. *Three functions.*

We will now establish the following theorem concerning the three functions Φ, Ψ, X.

VI. *If none of the six functions:*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \psi_1\chi_2 - \psi_2\chi_1, \quad \chi_1\phi_2 - \chi_2\phi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}, \quad \{\chi_1, \chi_2\}$$

vanish in (I), *if the last three of them have the same sign, and if the product of all six is negative, then between any root of* Φ *and a larger root of* X *lies a root of* Ψ, *between any root of* Ψ *and a larger root of* Φ *lies a root of* X, *and between any root of* X *and a larger root of* Ψ *lies a root of* Φ.*

Let us write:

$$i = \operatorname{sgn}\{\phi_1, \phi_2\} = \operatorname{sgn}\{\psi_1, \psi_2\} = \operatorname{sgn}\{\chi_1, \chi_2\}.\dagger$$

Then the assumption that the product of the six functions is negative is expressed by the equation:

$$i \cdot \operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn}(\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn}(\chi_1\phi_2 - \chi_2\phi_1) = -1.$$

Let x_1 and x_2 be two points of (I) such that

$$\Phi(x_1) = 0, \quad X(x_2) = 0, \quad x_1 < x_2.$$

If Φ vanished between x_1 and x_2, say at x', Ψ would vanish, by theorem II, between x_1 and x'. If Φ does not vanish between x_1 and x_2 we may proceed as follows:

Since $\operatorname{sgn}\Phi(x_1 + \epsilon) = \operatorname{sgn}\Phi'(x_1)$ we see by formula (3) that

$$\operatorname{sgn}\Phi(x_1 + \epsilon) = -i \cdot \operatorname{sgn}(\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn}\Psi(x_1). \tag{5}$$

Furthermore if in the identical relation

$$(\phi_1\psi_2 - \phi_2\psi_1)X + (\chi_1\phi_2 - \chi_2\phi_1)\Psi + (\psi_1\chi_2 - \psi_2\chi_1)\Phi = 0 \tag{6}$$

we let $x = x_2$ so that the first term drops out, and then note the signs of the two remaining terms, we get:

$$\operatorname{sgn}\Psi(x_2) = -\operatorname{sgn}(\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn}(\chi_1\phi_2 - \chi_2\phi_1) \cdot \operatorname{sgn}\Phi(x_2). \tag{7}$$

---

* A special case of this theorem and the next, analogous to the ones mentioned in previous footnotes, is proved by STURM (l. c., p. 165).

† We understand with KRONECKER by sgn Ω the value $+1$ or -1 according as Ω is positive or negative.

Combining equations (5) and (7), and remembering that

$$\operatorname{sgn} \Phi(x_2) = \operatorname{sgn} \Phi(x_1 + \epsilon),$$

we get:

$$\operatorname{sgn} \Psi(x_2) = i \cdot \operatorname{sgn} (\phi_1\psi_2 - \phi_2\psi_1) \cdot \operatorname{sgn} (\psi_1\chi_2 - \psi_2\chi_1) \cdot \operatorname{sgn} (\chi_1\phi_2 - \chi_2\phi_1) \cdot \operatorname{sgn} \Psi(x_1).$$

This equation shows that $\Psi(x_2)$ and $\Psi(x_1)$ have opposite signs. Therefore Ψ vanishes between x_1 and x_2—as was to be proved.

We have thus proved the first statement of our theorem and the proof of the other two statements follows at once from considerations of symmetry.

The case in which the product of the six functions considered in the last theorem is positive can be reduced to the case just considered by interchanging two of the three letters ϕ, ψ, χ. If, further, we remember that under the conditions of the last theorem the roots of any two of the functions Φ, Ψ, X separate each other (cf. theorem III) we obtain the following theorem:

VII. *If none of the six functions*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \psi_1\chi_2 - \psi_2\chi_1, \quad \chi_1\phi_2 - \chi_2\phi_1, \quad \{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}, \quad \{\chi_1, \chi_2\}$$

vanish in (I), *and if the last three of them have the same sign, then the roots of the functions* Φ, Ψ, X *follow each other cyclically in the order named or in the reverse order* X, Ψ, Φ *according as the product of the six functions above written is negative or positive.*

As an application of this theorem let us consider the three functions:

$$\Phi = y, \quad \Psi = y', \quad \mathrm{X} = py' + qy = -y''.$$

We must here assume that the coefficients p, q of the differential equation have finite first derivatives at every point of (I). We have in this case:

$$\phi_1\psi_2 - \phi_2\psi_1 = -1, \quad \psi_1\chi_2 - \psi_2\chi_1 = q, \quad \chi_1\phi_2 - \chi_2\phi_1 = p, \quad \{\phi_1, \phi_2\} = 1,$$

$$\{\psi_1, \psi_2\} = q, \quad \{\chi_1, \chi_2\} = qp' - q'p + q^2 = q^2\left[\frac{d}{dx}\left(\frac{p}{q}\right) + 1\right].$$

VIII. *If throughout the interval* (I):

$$q > 0, \quad \frac{d}{dx}\left(\frac{p}{q}\right) > -1,$$

and p *does not vanish in this interval, then the roots of* y, y', y'' *follow each other in the cyclic order named if* $p > 0$, *in the reverse order if* $p < 0$.

As an application of this theorem let us consider the positive roots of the three Bessel's functions $J_n(x), J_{n+1}(x), J_{n+2}(x)$ where n has any real value. Let us introduce the new independent variable $\xi = x^2/4$ and let

$$R_m(\xi) = \left(\frac{2}{x}\right)^m J_m(x).$$

Then we have:*

$$R_m'(\xi) = -R_{m+1}(\xi).$$

The relative position of the positive roots of J_n, J_{n+1}, J_{n+2} is then evidently the same as that of the positive roots of R_n, R_n', R_n''. Now R_n satisfies the differential equation:

$$y'' + \frac{n+1}{\xi} y' + \frac{1}{\xi} y = 0.$$

By applying theorem VIII to this equation we get at once the theorem:

IX. *The positive roots of J_n, J_{n+1}, J_{n+2} follow each other cyclically in the order named if $n > -1$, in the reverse order if $n < -1$.*†

§ 3. *An extension of the previous results.*

We shall begin by considering the non-homogeneous linear equation of the first order:

$$y' + py = r, \tag{8}$$

where p and r are throughout the interval (I) real and continuous functions of x.

It is well known that there exists one and only one solution of this equation which vanishes at an arbitrary point c of (I) and that this solution is given by the formula:

$$y = e^{-\int_c^x p dx} \int_c^x r e^{\int_c^x p dx} dx.$$

This formula shows us that if throughout (I) $r \geqq 0$ the following relations are satisfied:

$$y \geqq 0 \qquad (c < x \leqq b),$$

$$y \leqq 0 \qquad (c > x \geqq a).$$

* Cf. GRAY and MATHEWS: *Treatise on Bessel Functions*, p. 46.

† This theorem was established by the writer in the Bulletin of the American Mathematical Society, March, 1897, p. 207, although owing to an oversight the reversal of the order of the roots when $n < -1$ is not mentioned there. The part of this theorem which states that the roots of J_n and J_{n+1} separate each other is of earlier date, it having been stated without proof by GRAY and MATHEWS (l. c.) as being probably correct, and actually proved by VAN VLECK (American Journal of Mathematics, vol. 19 (1897), p. 75). Among a considerable number of other proofs which have since been given of this part of the theorem I should like to call special attention to an extremely brief and simple one by PORTER (American Journal of Mathematics, vol. 20 (1898), p. 199, and Bulletin of the American Mathematical Society, March, 1898, p. 274) which has recently been reproduced by H. WEBER in his edition of RIEMANN'S *Differentialgleichungen der mathematischen Physik*, vol. 1 (1900), p. 166.

If on the other hand $r \leqq 0$ these last two inequalities must be reversed. Moreover it is clear that in each of these cases y will vanish at a point x when and only when r vanishes at all points between c and x.

X. *If r does not change sign in (I) a solution of* (8) *vanishes either at no point of (I), at one and only one point, or at all the points of a connected portion of (I) throughout which $r = 0$, but at no other point of (I). In the last two cases:*

if $r \geqq 0$ $\qquad y(a) \leqq 0, \quad y(b) \geqq 0,$

if $r \leqq 0$ $\qquad y(a) \geqq 0, \quad y(b) \leqq 0.$*

Let us now consider the very much more general differential equation:

$$y' + p(x, y) \cdot y = r(x, y), \tag{9}$$

where p and r are continuous real functions of the two independent variables (x, y) so long as x remains in (I), y having any value.† Let us suppose that y_1 is a solution of (9) which is continuous throughout (I). Then y_1 is also a solution of the linear equation:

$$y' + p(x, y_1) \cdot y = r(x, y_1),$$

where p and r are now continuous functions of x only. Moreover, if $r(x, y)$ does not change sign the same will be true *a fortiori* of $r(x, y_1)$. Accordingly in this case *all that is said in theorem* X *about y is true of every solution of* (9) *which is continuous throughout (I).*

* I should like to indicate in passing how this theorem can be used to prove the theorems contained on pages 415, 416 in my paper in vol. 1 of these Transactions.

We had there two functions ω_1 and ω_2 continuous throughout (I) and satisfying respectively the RICCATI'S equations:

$$\omega' = A_1 + C_1\omega^2,$$
$$\omega' = A_2 + C_2\omega^2,$$

where the functions A_1, A_2, C_1, C_2 are continuous and satisfy throughout (I) the inequalities:

$$A_2 \geqq A_1, \quad C_2 \geqq C_1.$$

It is clear that the difference $\omega_2 - \omega_1$ satisfies the linear equation:

$$y' - C_2(\omega_2 + \omega_1)\, y = (A_2 - A_1) + (C_2 - C_1)\,\omega_1^2.$$

Since $A_2 - A_1 + (C_2 - C_1)\,\omega_1^2 \geqq 0$, theorem X shows that if $\omega_2(a) > \omega_1(a)$ then $\omega_2(x) > \omega_1(x)$, while if $\omega_2(a) = \omega_1(a)$ then $\omega_2(x) \geqq \omega_1(x)$, and in fact $\omega_2(x) > \omega_1(x)$ provided we exclude the special cases for which throughout the neighborhood of a, $A_2 = A_1$ and $(C_2 - C_1)\,\omega_1^2 = 0$.

While the proof here given is perhaps a little more artificial than the one I used before, it has the advantage, besides being slightly shorter, of dispensing with the use of theorems of a not altogether elementary character about differential equations involving parameters.

† The more general case in which p and r are continuous only when y is restricted to a finite interval might easily be considered.

In particular the RICCATI'S equation

$$\omega' = A + B\omega + C\omega^2, \tag{10}$$

where A, B, C are continuous real functions of x throughout (I), is of the form (9) if we let $r = A$.

XI. *If the function A does not change sign in (I) and does not vanish at all points of a continuous portion of (I), than a real solution ω of (10) which is continuous throughout (I) cannot vanish at more than one point of (I), and if it does vanish it passes with increasing x from negative to positive if $A \geqq 0$, from positive to negative if $A \leqq 0$.*

By applying this theorem to equation (2) we get the following generalization of theorem II:

XII. *If $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish in (I), and $\{\phi_1, \phi_2\}$ does not change sign and does not vanish at all points of a continuous portion of (I), and if ϕ_1', ϕ_2', ψ_1', ψ_2', are continuous throughout (I), then in any portion of (I) in which Ψ does not vanish Φ cannot vanish more than once.*

From this we deduce a generalization of theorem I:

XIII. *If ϕ_1 and ϕ_2 do not both vanish at any point of (I) and if $\{\phi_1, \phi_2\}$ does not change sign in (I) and does not vanish at all points of a continuous portion of (I), and if ϕ_1', ϕ_2' are continuous throughout (I), then Φ cannot vanish more than a finite number of times in (I), and when it vanishes it changes sign.*

It should be noticed that it is no longer true that Φ and Φ' cannot vanish together. In a similar manner the other theorems of §§ 1, 2 can be generalized by replacing the requirement that the functions $\{\phi_1, \phi_2\}$, $\{\psi_1, \psi_2\}$, $\{\chi_1, \chi_2\}$ do not vanish, by the requirement that they do not change sign and do not vanish at all points of a continuous portion of (I), and adding the requirement that ϕ_1', ϕ_2', ψ_1', ψ_2', χ_1', χ_2' are continuous. The proofs of some of the theorems must, however, be slightly modified. In particular, formulæ (3) and (4) must be replaced by the following:

At a point x_1 where $\Phi = 0$:

$$\operatorname{sgn}\frac{\Delta\Phi}{\Delta x} = \operatorname{sgn}\,\Phi(x_1+\epsilon) = -\operatorname{sgn}\,\Phi(x_1-\epsilon)$$
$$= -\operatorname{sgn}\,\{\phi_1, \phi_2\}\cdot\operatorname{sgn}\,(\phi_1\psi_2 - \phi_2\psi_1)\cdot\operatorname{sgn}\,\Psi(x_1).$$

At a point x_2 where $\Psi = 0$:

$$\operatorname{sgn}\frac{\Delta\Psi}{\Delta x} = \operatorname{sgn}\,\Psi(x_2+\epsilon) = -\operatorname{sgn}\,\Psi(x_2-\epsilon)$$
$$= \operatorname{sgn}\,\{\psi_1, \psi_2\}\cdot\operatorname{sgn}\,(\phi_1\psi_2 - \phi_2\psi_1)\cdot\operatorname{sgn}\,\Phi(x_2).$$

GRUND IM HARZ,
July 20, 1901.

ON THE SYSTEM OF A BINARY CUBIC AND QUADRATIC AND THE REDUCTION OF HYPERELLIPTIC INTEGRALS OF GENUS TWO TO ELLIPTIC INTEGRALS BY A TRANSFORMATION OF THE FOURTH ORDER*

BY

JOHN HECTOR McDONALD

Introduction.

1. If a hyperelliptic integral of genus 2 is reducible to an elliptic integral by a transformation of order k then there always exists a second integral † with the same irrationality which is reducible by a transformation of the same order. In the algebraic treatment of the problem the difficulty lies in the determination of the second integral after the first has been constructed.

For $k = 4$ the problem has been treated by Professor Bolza ‡ and in the sixth section of his dissertation the formulæ for the two reducible integrals are established algebraically. Professor Bolza used a particular system of variables x_1, x_2. These are chosen so that if

$$y_1 = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$$

and

$$y_2 = b_0x_1^4 + 4b_1x_1^3x_2 + 6b_2x_1^2x_2^2 + 4b_3x_1x_2^3 + b_4x_2^4$$

are the two forms of order 4 which are equal to the homogeneous variables in the elliptic integral, then $a_1 = 0$ and $b_1 = 0$, and the numerators of the two reducible integrals turn out to be x_1 and x_2. The derivation of the second integral depends on the use of this particular system of variables and the connection of these variables with the irrationality and with the reducing substitutions is unexplained.

* Presented to the Society (Chicago) April 6, 1901, under a slightly different title. Received for publication May 17, 1901.

† This is a consequence of the Weierstrass-Picard Theorem: Acta Mathematica, vol. 4 (1884), p. 400; Bulletin de la Société Mathématique de France, vol. 11 (1882–1883), p. 25.

‡ *Inaugural Dissertation*, Göttingen, 1886; Mathematische Annalen, vol. 28 (1887), p. 447.

The object of this paper is to supply the following desiderata: (*a*) a solution of the problem independently of a special system of variables, by using theorems on biquadratic involutions, (*b*) a methodical deduction of the second integral from the first, (*c*) the introduction of the system of variables and the normal form which are appropriate to the problem.

The integral

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}}$$

is a hyperelliptic integral of the first kind of genus 2 if $R_6(x)$ is a polynomial of the 6th degree, and it is reducible to an elliptic integral by a transformation of order 4 if

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}} = M\int \frac{dy}{\sqrt{R_4(y)}},$$

where $R_4(y)$ is a polynomial of order 3 or 4 and $y = U(x)/V(x)$, U and V being polynomials, the higher of whose orders is 4 and M a multiplier not depending on x or y.

In homogeneous variables $x = x_1/x_2$, $y = y_1/y_2$ and, with the usual notations, the above relation becomes

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}} = M\int \frac{(y\,dy)}{\sqrt{R_4(y_1y_2)}},$$

where

$$y_1 = \rho\, U(x_1x_2),$$
$$y_2 = \rho\, V(x_1x_2),$$
$$(xd) = x_1d_2 - x_2d_1.$$

Denoting more explicitly $R_6(x_1x_2)$ by $(xa)(x\beta)(x\gamma)(x\delta)(x\kappa)(x\lambda)$ and $R_4(y_1y_2)$ by $(y\mu)(y\nu)(y\pi)(y\omega)$ we see by reasoning similar to that used by JACOBI* for the transformation of elliptic integrals that it is possible to break up R_6 into quadratic factors $(xa)(x\beta)$, $(x\gamma)(x\delta)$, $(x\kappa)(x\lambda)$ and to determine linear forms (xd'), (xd''), (xd''') and a quadratic form $(x\xi)(x\eta)$ so that

$$\mu_2U - \mu_1V = (xa)(x\beta)(xd')^2,$$
$$\nu_2U - \nu_1V = (x\gamma)(x\delta)(xd'')^2,$$
$$\pi_2U - \pi_1V = (x\kappa)(x\lambda)(xd''')^2,$$
$$\omega_2U - \omega_1V = (x\xi)^2(x\eta)^2;$$

and if we denote by ϑ the Jacobian of U and V then

$$\vartheta = (xd)(xd')(xd'')(xd''')(x\xi)(x\eta).$$

*JACOBI, *Fundamenta Nova*, *Werke*, vol. 1, p. 57.

Conversely, if these relations are satisfied, the integral

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}}$$

is reducible by the substitution $y_1 : y_2 = U : V$; that is, these conditions are sufficient as well as necessary.

Accordingly with every reducible hyperelliptic integral is associated a biquadratic involution $I_4 = \lambda U + \nu V$ which is special since it contains a complete square, viz., $(x\xi)^2(x\eta)^2$. Every such involution contains, aside from the square $(x\xi)^2(x\eta)^2$, four forms having double roots. The quadratic factor which goes along with the double factor to make up any one of these four forms is called a branch quadratic and the forms themselves branch forms. It follows from the above equations that the product of three branch quadratics of an involution containing a square gives a sextic belonging to a reducible integral and that the numerator is furnished by the double element in the fourth branch form. To every reducible integral belongs an involution I_4 in which one of the four branch forms is distinguished from the other three, each of which plays the same rôle.

In section I it is shown that there exists one and only one I_4 which contains the square of an arbitrary quadratic f, and has the factors of an arbitrary cubic ϕ for three of its double elements. An I_4 determined in such a way has evidently one branch form distinguished from the rest and the three which contain the factors of ϕ are not distinguished from each other.

In sections II, III it is shown that if ϕ, f be a system (Σ) of a cubic and a quadratic there exists a covariant system (Σ) of a cubic and a quadratic ϕ, f, such that ϕ, f are the same covariants of ϕ, f as ϕ, f are of ϕ, f, or in other words, the relation between (Σ) and (Σ) is mutual. The forms ϕ and f are obtained as follows. Using transvectant notation put $\phi = l_x m_x n_x$, the product of linear factors, and let p and q be two linear forms and also let

$$D_{pq}(\phi) = 4^3(p^3q,\ l)_1(p^3q,\ m)_1(p^3q,\ n)_1 p^{-6}.$$

Then $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q. If we let p and q be the two linear covariants of ϕ and f so denoted in the notation of Clebsch (*Binäre Formen**) $D_{pq}(\phi)$ will be a covariant of ϕ and f. Also let $Q(\phi)$ be the cubic covariant of ϕ, $D_{pq}\big(Q(\phi)\big) = \phi' = c_x^3$ and $f' = d_x^2 =$ the Jacobian of pq and $(cq)c_x^2$. Further, let

$$\sigma = \frac{(cd)^2 c_x}{q}, \quad \rho = \frac{(cd)^2 (cd')\, d'_x}{p};$$

σ and ρ are invariants of ϕ, f. In these notations the following theorem holds.

* If $\varphi = a_x^3$, $f = b_x^2$, $p = (ab)^2 a_x$, $q = (ab)^2(ab')b'_x$.

THEOREM: *The forms*

$$\frac{\rho}{\sigma^2}\phi' = \bar{\phi}, \quad \frac{\sigma}{\rho}f' = \bar{f}$$

are covariants of ϕ, f *and if* $\bar{\bar{\phi}}, \bar{\bar{f}}$ *are the same covariants of* $\bar{\phi}, \bar{f}$ *as* ϕ, f *were of* ϕ, f *then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f.$$

Also, if the covariants $\bar{p}$, $\bar{q}$ *are formed from* $\bar{\phi}$ *and* $\bar{f}$;

$$\bar{p} = q, \quad \bar{q} = p.$$

It is on the existence of these mutual relations between the systems (Σ) and $(\bar{\Sigma})$ that the second reducible integral depends.

In section IV the involutions I_4 and $\bar{I}_4$ belonging to (Σ) and $(\bar{\Sigma})$ are determined and it is shown that the Jacobian of $I_4 = \phi f q$ and of $\bar{I}_4 = \bar{\phi}\bar{f}\bar{q} = \bar{\phi} f p$; moreover the three branch quadratics belonging to the three branch forms whose double elements are the factors of ϕ are identical with those in $\bar{I}_4$ whose double elements are the factors of $\bar{\phi}$, except as to sign. If these branch quadratics are denoted by R, S, T then

$$\int \frac{q(x\,dx)}{\surd RST}, \quad \int \frac{p(x\,dx)}{\surd RST}$$

are both reducible integrals with the same irrationality, the first by I_4, the second by $\bar{I}_4$. The statement of this theorem constitutes section V.

In section VI, the variables $x_1' = q$, $x_2' = p$ are introduced as leading to an appropriate canonical form, which is identical with that used by Professor BOLZA in his solution of the problem.

In section VII a number of miscellaneous results are collected relating to biquadratic involutions containing a complete square and their geometrical representations on a rational quartic and a twisted cubic.

I.

THEOREMS ON THE BIQUADRATIC INVOLUTION HAVING A COMPLETE SQUARE.

2. *A biquadratic involution* I_4 *containing a square is determined without ambiguity by the quadratic form whose square it contains and by any three of its other double elements, or in other words, by a cubic and a quadratic. These forms are independent and their introduction singles out one of the branch forms of the involution from the others.*

Before proving these statements another result must be proved.

LEMMA.—*Let* U *and* f^2 *be two biquadratic forms. Let the Jacobian of* U

and f^2, which must contain f, be $f\theta$. The form θ is apolar to the forms of the involution $\lambda U + \mu f^2$.*

Proof.—Let U and f be taken in a normal form found by introducing as variables the linear factors of f. Then

$$f = x_1 x_2$$

and

$$U = a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4.$$

The Jacobian of f^2, U is found on calculation to be:

$$\tfrac{1}{2}\, x_1 x_2 (a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^3 - a_4 x_2^4),$$

and so

$$\theta = \tfrac{1}{2}(a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^3 - a_4 x_2^4).$$

Then on calculation one finds the desired relations

$$(u, \theta)_4 = 0, \qquad (f^2, \theta)_4 = 0.$$

3. We now prove the following theorem:

THEOREM.—*Let ϕ denote any cubic and f any quadratic not apolar to ϕ. There exists one and but one I_4 containing the form f^2 and having three of its double elements given by $\phi = 0$.*

Proof.—We can always determine a linear form $(x\delta)$ so that $\phi(x\delta)$ is apolar to f^2, and in but one way. For $(\phi(x\delta)f^2)_4 \not\equiv 0$, since f^2 is not apolar to ϕ;† hence $\left(\phi(x\delta)f^2\right)_4 = 0$ is an equation of the first degree to determine δ_1/δ_2.

Let $f = x_1 x_2$, the factors of f being introduced as variables; then we may take

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 6c_2 x_1^2 x_2^2 + 4c_3 x_1 x_2^3 + c_4 x_2^4;$$

but $c_2 = 0$, since $\phi(x\delta)$ is apolar to $f^2 = x_1^2 x_2^2$, hence

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4,$$
$$f^2 = x_1^2 x_2^2.$$

First. It appears at once that there is one I_4 satisfying the required conditions, viz.:

$$I_4 = \lambda x_1^2 x_2^2 + \mu(c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4).$$

For calculating the form θ we find it is equal to

$$\tfrac{1}{2}(c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

hence I_4 contains f^2 and ϕ gives three of its double elements.

---

* Two forms $f = a_x^n$, $\phi = b_x^n$ are apolar if

$$(f, \phi)_n = (ab)^n = \sum_{i=0\cdots n} a_i b_{n-i} (-1)^i {}_nC_i = 0.$$

† If ϕ and f are apolar, that is, if $(\phi f)_2 \equiv 0$, then $\phi(x\delta)$ is apolar to f^2 whatever $(x\delta)$ may be.

Second. There can be only one such I_4. For suppose another to exist:

$$I_4' = \lambda x_1^2 x_2^2 + \mu(c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4),$$

we should have

$$x_1 x_2 \theta = \sigma x_1 x_2 \theta',$$

since the Jacobians of I_4 and I_4' must both be $\phi(x\delta)f$.

Hence

$$c_0' x_1^4 + 4c_1' x_1^3 x_2 + 4c_3' x_1 x_2^3 + c_4' x_2^4 = \sigma(c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

or

$$c_0' = \sigma c_0, \quad c_1' = \sigma c_1, \quad c_3' = \sigma c_3, \quad c_4' = \sigma c_4;$$

or

$$c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4 = \sigma(c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4);$$

that is

$$c_0' x_1^4 + 8c_1' x_1^3 x_2 - 8c_3' x_1 x_2^3 - c_4' x_2^4$$

belongs to I_4; hence I_4' is identical with I_4, which is contrary to hypothesis.

In virtue of this theorem we may consider a special I_4 containing a square as determined uniquely by a cubic and quadratic ϕ and f.

4. The following theorem determines $(x\delta)$ more exactly. In symbolic notation let $\phi = a_x^3, f = b_x^2$.

THEOREM: *$(x\delta)$ is proportional and may be taken equal to $(ab)^2(ab')b_x'$.*

To prove this we must show that

$$\left(a_x^3 \cdot (a'b)^2(a'b')b_x', \quad b_x''^2 b_x'''^2\right)_4 = 0.$$

Proof.—We have

$$(a_x^3, b_x''^2 b_x'''^2)_3 = \tfrac{4}{3}(ab'')^2(ab''')b_x''',$$

and therefore

$$\left((ab'')^2(ab''')b_x''', \quad (a'b)(a'b')b_x'\right)_1$$

must be zero, or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

If we interchange b', b'''; a, a'; b, b'', which has no effect on the value of the expression, we find

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = (ab'')^2(ab''')(a'b)(a'b')(b'''b')$$

$$= -(ab'')^2(ab''')(a'b)(a'b')(b'b'''),$$

or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

II.

The System of a Cubic and two Linear Forms and their Conjugate System.

5. We consider next a system of a cubic form ϕ resolved into its linear factors, $\phi = l_x m_x n_x$, and two linear forms p_x and q_x (denoting them for brevity by l, m, n, p, q), returning later to the system of a cubic and quadratic. The discussion will be much facilitated by three theorems.

Let

$$D_{pq}(\phi) = 4^3 (p^3 q, l)_1 (p^3 q, m)_1 (p^3 q, n)_1 p^{-6}.$$

Here $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q since $(p^3 q, l)_1$ is divisible by p^2.

Theorem I: *If*

$$D_{pq}(\phi) = 4^3 (p^3 q, l)_1 (p^3 q, m)_1 (p^3 q, n)_1 p^{-6},$$

where l, m, n are the linear factors of ϕ, then

$$D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6 \phi.$$

Proof.—We have

$$4\,(p^3 q, l)_1 = p^2 \big(p(ql) + 3q(pl)\big),$$

and find that

$$D_{pq}(\phi) = \prod_{l, m, n} \big(p(ql) + 3q(pl)\big) = l'_x m'_x n'_x = l'm'n', \quad 4\,(q^3 p, l')_1 = 3\,(pq)^2 q^2 l$$

by a short computation, making use of the identity

$$l_x(pq) + p_x(ql) + q_x(lp) = 0.$$

Hence

$$D_{qp}(l'm'n') = D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6 \phi.$$

Theorem II: *Let $Q(\phi)$ denote the cubic covariant of ϕ; then*

$$Q\big(D_{pq}(\phi)\big) = -\,27\,(pq)^6 D_{pq}\big(Q(\phi)\big).$$

Let

$$L = (ml) n_x + (nl) m_x = 2\,(l, nm)_1,$$

$$M = (nm) l_x + (lm) n_x = 2\,(m, ln)_1,$$

$$N = (ln) m_x + (mn) l_x = 2\,(n, lm)_1,$$

so that Ll is harmonic to mn. Then $LMN = \sigma Q$, since the cubic covariant of a cubic may be found by taking the conjugate of each factor with respect to the other two and forming the product. The factor σ is a numerical constant since LMN and Q are both of the 3d degree in the coefficients of ϕ; this appears at once from the expressions of L, M, N in l, m, n.

We have

$$D_{pq}\big(Q(\phi)\big) = D_{pq}(\sigma LMN) = \sigma \prod_{L,M,N} \big(p(qL) + 3q(pL)\big)$$

$$= \sigma \prod_{l,m,n} p\{(ml)(qn) + (nl)(qm)\} + 3q\{(ml)(pn) + (nl)(pm)\},$$

and

$$2(l', m'n') = \big(p(qm) + 3q(pm)\big)\big((pl')(qn) + 3(pn)(ql')\big)$$

$$+ \big(p(qn) + 3q(pn)\big)\big((pl')(qm) + 3(pm)(ql')\big);$$

but

$$(pl') = 3(pl)(pq), \quad (ql') = (qp)(qm).$$

Substituting in $(l', m'n')_1$, and making use of the relation

$$(pl)(qn) + (pq)(nl) + (pn)(lq) = 0,$$

derivable from $l_x(qn) + q_x(nl) + n_x(lq) = 0$ by putting $x_1 : x_2 = -p_2 : p_1$, we find

$$2(l', m'n')_1 = -3(pq)^2\big\{p\big((ml)(qn)+(nl)(qm)\big)+3q\big((ml)(pn)+(nl)(pm)\big)\big\}.$$

Put

$$\phi' = l'm'n';$$

then

$$Q(\phi') = \sigma 2^3 (l', m'n')_1 (m', l'n')_1 (n', m'l')_1$$

$$= \sigma \prod_{l,m,n} -3(pq)^2\big\{p\big((ml)(qn) + (nl)(qm)\big)$$

$$+ 3q\big((ml)(pn) + (nl)(pm)\big)\big\}$$

$$= -27(pq)^6 D_{pq}\big(Q(\phi)\big),$$

or

$$Q\big(D_{pq}(\phi)\big) = -27(pq)^6 D_{pq}\big(Q(\phi)\big).$$

6. THEOREM III: *Let R be the discriminant of ϕ; then*

$$D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4} 27^2 (pq)^{12}\phi.$$

Proof.—By definition

$$D_{qp}\big(Q(\phi')\big) = D_{qp}\big(Q\{D_{pq}[Q(\phi)]\}\big);$$

but we may interchange the two inside operators by theorem II; therefore

$$D_{qp}\big(Q(\phi')\big) = -27(pq)^6 D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big);$$

the factor $-27(pq)^6$ may be taken out because $D_{qp}(\phi)$ is linear in the coefficients of ϕ, viz., $D_{qp}(a\phi) = aD_{qp}(\phi)$. By theorem I,

$$D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big) = 27(pq)^6 Q\big(Q(\phi)\big);$$

hence

$$D_{qp}\big(Q(\phi')\big) = -27^2(pq)^{12}Q\big(Q(\phi)\big),$$

and

$$Q\big(Q(\phi)\big) = -4^{-1}R^2\phi$$

(CLEBSCH, *Binäre Formen*); therefore

$$D_{qp}\big(Q(\phi')\big) = -4^{-1}R^2 27^2(pq)^{12}\phi.$$

III.

THE SYSTEM OF A CUBIC AND QUADRATIC AND THEIR CONJUGATE SYSTEM.

7. It is necessary now to recall two properties of the form system of a cubic and quadratic c_x^3 and d_x^2: 1°. The first polar of c_x^3 with respect to the linear covariant $(cd)^2c_x$ of c_x^3 and d_x^2 is a quadratic harmonic to d_x^2; 2°. The linear covariant $(cd)^2(cd')\,d_x'$ is the polar of $(cd)^2c_x$ with respect to d_x^2, or the product of these two linear covariants is harmonic to d_x^2.

Proof.—1°. Let (xy) be any linear form; then $c_x^2c_y$ is harmonic to d_x^2 if $(c_x^2c_y, d_x^2)_1 = 0$ or $(cd)^2c_y = 0$; hence $(xy) = (cd)^2c_x$.

2°. If we form $\big(d_x^2, (cd)^2c_x\big)_1$ we find $(cd)^2(cd')\,d_x'$. Hence the truth of the above statements.

8. Consider the form $\phi' = D_{pq}\big(Q(\phi)\big) = c_x^3$ and also $(c_x^3, q)_1 = (cq)\,c_x^2$. Then there is a quadratic form $f' = d_x^2$ which is the Jacobian of $(cq)\,c_x^2$ and pq. Then from the first property of the previous paragraph $(cd)^2c_x = \sigma q$ and $(cd)^2(cd')\,d_x' = \rho p$, where σ and ϕ are proportionality factors,* for q and p have just the properties of the two linear covariants; which properties define them except as to multipliers.

Further, it follows that ρ and σ are invariants of the system ϕ, p, q. For ϕ' is a covariant of this system and so consequently is also $f' = d_x^2$, and therefore $(cd)^2c_x$, $(cd)^2(cd')\,d_x'$, and finally, $(cd)^2c_xq^{-1} = \sigma$, $(cd)^2(cd')\,d_x'p^{-1} = \rho$.

Take then the covariants $\rho\sigma^{-2}\phi' = C_x^3 = \bar{\phi}$, $\sigma\rho^{-1}f' = D_x^2 = \bar{f}$ of ϕ, p, q; then $(CD)^2C_x = \rho\sigma^{-2}\cdot\sigma\rho^{-2}(cd)^2c_x = q$ and

$$(CD)^2(CD')\,D_x' = \rho\sigma^{-2}\cdot\sigma^2\rho^2\cdot(cd)(cd')\,d_x' = p.$$

Hence q and p are these two linear covariants of $\bar{\phi}, \bar{f}$.

From the reasoning just used it is clear that if ϕ' is any cubic and q, p any two linear forms there is but one solution to the problem to find a quadratic f and a multiplier k so that the two linear covariants defined above of $k\phi$ and f should be q and p.

* We determine σ and ρ in § 11.

For let $\bar{\phi}$ and $\bar{f}$ be one solution, $\phi = C_x^3, f = D_x^2$; and let $u\bar{\phi} = \Gamma_x^3$, $v\bar{f} = \Delta_x^2$ be any solution. Then

$$(\Gamma\Delta)^2\Gamma_x = (CD)^2 C_x, \quad (\Gamma\Delta)^2(\Gamma\Delta')\Delta'_x = (CD)^2(CD')D'_x,$$

or

$$uv(CD)^2 C_x = (CD)^2 C_x, \quad uv^2(CD)^2(CD')D'_x = (CD)^2(CD')D'_x;$$

hence $uv = 1$, $uv^2 = 1$, the only solution of which is $u = 1$, $v = 1$, which was to be proved.

This remark will be useful in the sequel.

9. Suppose given a system (Σ) of a cubic ϕ and quadratic f, in symbolic notation a_x^3, b_x^2; let $(ab)^2 a_x = p$ and $(ab)^2(ab')b'_x = q$ where ϕ, p, q are to be used as in the preceding paragraphs. Let $\bar{\phi}, \bar{f}$ be formed according to the prescription given there, calling the system of $\bar{\phi}, \bar{f}$, $(\bar{\Sigma})$. Then $\bar{\phi}, \bar{f}$ are two covariants of (Σ).

THEOREM.—*If we form the same two covariants* $\bar{\bar{\phi}}, \bar{\bar{f}}$, *of* $(\bar{\Sigma})$ *as* $\bar{\phi}, \bar{f}$ *are of* (Σ) *then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f;$$

that is (Σ) *may be derived from* $(\bar{\Sigma})$ *as* $(\bar{\Sigma})$ *was from* (Σ).

Proof.—Remembering that the covariants of (Σ) which we call p, q are for (Σ) equal to q, p we operate with ϕ, q, p to produce $\bar{\phi}$ as we operated with ϕ, p, q to produce ϕ.

Since $Q(a\phi) = a^3 Q(\phi)$ and $D_{pq}\big(a^3 Q(\phi)\big) = a^3 D_{pq}\big(Q(\phi)\big)$ and $\bar{\phi} = \rho\sigma^{-2}\phi'$ it follows that

$$D_{qp}\big(Q(\phi)\big) = \frac{\rho^3}{\sigma^6} D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4} 27^2 (pq)^2 \frac{\rho^3}{\sigma^6}\phi = \phi''.$$

From ϕ'', p, q deduce $\bar{\bar{\phi}}, \bar{\bar{f}}$ as $\bar{\phi}, \bar{f}$ were deduced from ϕ', q, p. Then $\bar{\bar{\phi}}$ is proportional to ϕ'' and if $\bar{\bar{\phi}} = A_x^3$, $\bar{\bar{f}} = B_x^2$, we have

$$(AB)^2 A_x = p, \quad (AB)^2(AB')B'_x = q.$$

But ϕ is proportional to ϕ'' and $\phi = a_x^3$, $f = b_x^2$ have the property $(ab)^2 a_x = p$, $(ab)^2(ab')b'_x = q$; therefore, by the remark at the end of § 8,

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f.$$

We have then *complete reciprocity* between two systems of a cubic and quadratic (Σ) and $(\bar{\Sigma})$. If we denote forms derived analogously from the two systems by a stroke we have the scheme:

(Σ)	$(\bar{\Sigma})$
ϕ, f	$\bar{\phi}, \bar{f}$
p, q	$\bar{p}, \bar{q}$

and $p = q$, $\bar{q} = p$.

This duality has some resemblance to the relations existing between a binary cubic and its cubic covariant, the pencil of a binary biquadratic and its hessian and the hessians of the pencil,* and the pencil of a ternary cubic and its hessian and the hessians of the pencil.†

10. Let $\psi, \phi_1, \cdots \phi_n$ denote any covariants of (Σ) and $\psi, \phi_1, \cdots \phi_n$ the corresponding covariants of (Σ) and suppose we have a relation

$$\chi\left((\psi\bar{\psi})_\lambda, \phi_1 \cdots \phi_n\right) = 0;$$

then because of the duality of (Σ), $(\bar{\Sigma})$ we have also

$$\chi\left(\pm(\psi\bar{\psi})_\lambda, \phi_1 \cdots \bar{\phi}_n\right) = 0,$$

since

$$(\bar{\psi}\psi)_\lambda = \pm(\psi\bar{\psi})_\lambda,$$

according as λ is even or odd.

By means of such a relation we are able to show the duality which exists between the involutions I_4, $\bar{I}_4$ determined by (Σ), $(\bar{\Sigma})$.

11. It is easy to calculate ϕ and f in terms of the fundamental concomitants of ϕ and f since the two covariants p and q have been used to give a typical representation of ϕ and f. We start from the formulæ to be found in GORDAN's *Invariantentheorie*, vol. 2, p. 327 :

$$fF = \tfrac{1}{2} A_{ff} p^2 + q^2,$$

$$-F^3\phi = \left(F(pq) - \tfrac{1}{2}A_{ff}(sq)\right)p^3 - \tfrac{3}{2}MA_{ff}p^2q + \tfrac{3}{2}(A_{ff}L - A_{f\Delta}F)pq^2 + Mq^3,$$

where the meanings of M, A_{ff}, etc., are given in the preceding pages of GORDAN.

In calculating covariants, using the typical representation, we may proceed as if p and q were the variables x_1, x_2 provided we multiply the result by a power of (pq) equal to the weight of the covariant. For if I be a covariant of weight λ so that $I' = r^\lambda I$, r being the determinant of substitution, we may make the substitution $x_1' = p$, $x_2' = q$ of determinant $1/(pq)$; then $I' = [1/(pq)^\lambda]\ I$ or $I = (pq)^\lambda I'$ and I' is expressed in x_1', x_2' but $x_1' = p$, $x_2' = q$; hence the truth of the above statement.

Suppose then the cubic covariant $Q(\phi)$ of ϕ to be calculated from the typical representation; then the operation $D_{pq}\left(Q(\phi)\right)$ may be effected.

Let

$$Q(\phi) = C_0p^3 + C_1p^2q + C_2pq^2 + C_3q^3 = C_0(p - \alpha_1q)(p - \alpha_2q)(p - \alpha_3q),$$

* CLEBSCH, *Binäre Formen.*

† CLEBSCH-LINDEMANN, *Geometrie*, p. 559.

where C_0, C_1, C_2 and C_3 are expressed in terms of the fundamental invariants of ϕ, f.

Then

$$D_{pq}\big(Q(\phi)\big) = (pq)^3 C_0 (p + 3a_1 q)(p + 3a_2 q)(p + 3a_3 q),$$

as a short computation shows, or

$$D_{pq}\big(Q(\phi)\big) = (pq)^3 (C_0 p^3 - 3C_1 p^2 q + 9C_2 pq^2 - 27(C_3 q^3)$$
$$= \phi' = c_0 p^3 + 3c_1 p^2 q + 3c_2 pq^2 + c_3 q^3 = c_x^3,$$

where

$$c_0 = (pq)^3 C_0, \quad -c_1 = (pq)^3 C_1, \quad c_2 = 3(pq)^3 C_2, \quad -c_3 = 27(pq)^3 C_3.$$

Next let f' be the Jacobian of (c_x^3, q), and pq; we find

$$f' = \tfrac{1}{2}(pq)^2 (c_0 p^2 - c_2 q^2) = d_x^2.$$

If we calculate $(cd)^2 c_x$ and $(cd)^2 (cd') d_x'$ which are of weights 2 and 3 we find

$$(cd)^2 c_x = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2) q,$$

$$(cd)^2 (cd') d_x' = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3) p,$$

so that

$$\sigma = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2),$$

$$\rho = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3),$$

and consequently $\rho\sigma^{-2}\phi'$ and $\sigma\rho^{-2} f'$ are expressed in terms of the form system of (Σ); that is, $\bar{\phi}$ and $\bar{f}$ are so expressed, since

$$\rho\sigma^{-2}\phi' = \bar{\phi}, \quad \sigma\rho^{-1} f' = \bar{f}.$$

THEOREM.—*The forms $\bar{\phi}$, $\bar{f}$ are expressed in terms of the fundamental concomitants of ϕ, f by the following equations:*

$$\phi' = c_0 p^3 + 3c_1 p^2 q + 3c_2 pq^2 + c_3 q^3,$$

$$c_0 = (pq)^3 C_0, \quad -c_1 = (pq)^3 C_1, \quad c_2 = 3(pq)^3 C_2, \quad -c_3 = 27(pq)^3 C_3,$$

where C_0, C_1, C_2, C_3 are defined by

$$Q(\phi) = C_0 p^3 + C_1 p^2 q + C_2 pq^2 + C_3 q^3;$$

$$f' = \tfrac{1}{2}(pq)^2 (c_0 p^2 - c_2 q^2),$$

$$\sigma = \tfrac{1}{2}(pq)^4 (c_3 c_0 - c_1 c_2),$$

$$\rho = \tfrac{1}{4}(pq)^7 (c_0 c_1 c_2 - c_0^2 c_3),$$

and

$$\phi = \frac{\rho}{\sigma^2}\phi', \quad f = \frac{\sigma}{\rho} f'.$$

IV.

The Involutions Belonging to ϕ, f and $\bar{\phi}, \bar{f}$.

12. In this section the involutions I_4 and I_4 belonging to (Σ) and $(\bar{\Sigma})$ are determined.

Let

$$(pq, l)_1 = l_1, \qquad (pq, \bar{l})_1 = \bar{l}_1,$$
$$(pq, m)_1 = m_1, \qquad (pq, \bar{m})_1 = \bar{m}_1,$$
$$(pq, n)_1 = n_1; \qquad (pq, \bar{n})_1 = \bar{n}_1;$$

also

$$(p\bar{l}_1, ql_1)_1 = R_x^2 = R,$$
$$(p\bar{m}_1, qm_1)_1 = S_x^2 = S,$$
$$(p\bar{n}_1, qn_1)_1 = T_x^2 = T.$$

The three quadratics R, S, T are such that if we interchange the rôles of (Σ) and $(\bar{\Sigma})$ in forming them we find $-R$, $-S$, $-T$. Hence any relation between R, S, T and covariants of (Σ), $(\bar{\Sigma})$ is accompanied by a relation between $-R$, $-S$, $-T$ and the same covariants of $(\bar{\Sigma})$, (Σ). The three quadratics are branch quadratics of both I_4 and I_4.

Theorem: *The forms Rl^2, Sm^2, Tn^2, f^2 belong to I_4 and the Jacobian of I_4 is proportional to ϕfq.*

From this result it follows that $R\bar{l}^2$, $S\bar{m}^2$, $T\bar{n}^2$, $\bar{f}^2$ belong to $\bar{I}_4$ whose Jacobian is proportional to $\bar{\phi}\bar{f}p$.

13. For the proof of this theorem some results on apolar systems of biquadratic forms are necessary.*

1°. To every involution $(\Pi) = \lambda f_1 + \mu f_2$ there is an apolar linear system $(\bar{\Pi}) = \lambda\phi_1 + \mu\phi_2 + \nu\phi_3$ such that every form of $(\bar{\Pi})$ is apolar to every form of (Π) and (Π) determines $(\bar{\Pi})$ and conversely.

2°. If a biquadratic form is apolar to the three forms ϕ_1, ϕ_2, ϕ_3 it must belong to (Π).

If, therefore, we can show that Rl^2, Sm^2, Tn^2, f^2 are all apolar to three biquadratic forms they must belong to the involution apolar to the three forms.

We are able to find three such forms, viz., $l^3(fl)_1$, $m^3(fl)_1$, $n^3(fn)_1$.

1°. It must be shown that $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ are linearly independent, which is supposed of ϕ_1, ϕ_2, ϕ_3 above. Since $\phi = lmn$ we may regard l, m, n as arbitrary and also $(fl)_1$, $(fm)_1$ since f is so. For let a_x and b_x be any two linear forms, then if we take f to be the Jacobian of $a_x l_x$ and $b_x m_x$ we know that $(fl)_1$ and $(fm)_1$ are proportional to a_x and b_x, that is, we may choose f so that $(fl)_1$ and $(fm)_1$ are any two linear forms.

* Rosanes, Crelle's Journal, vol. 76.

Hence we may regard $l^3(fl)_1$ and $m^3(fm)_1$ as any two biquadratics having cubic factors.

If then a linear relation existed between $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ it would take the form

$$n^3(fn)_1 = Al^3(fl)_1 + Bm^3(fm)_1$$

or $n^3(fn)_1$ would belong to the involution I_4 of $l^3(fl)_1$ and $m^3(fm)_1$. But a cube factor in a form of I_4 enters as a square in the Jacobian of I_4, hence the Jacobian of I_4 must be proportional to $l^2m^2n^2$. To see that this is not the case make a linear transformation by putting $l = x_1'$, $m = x_2'$; then denote $(fl)_1$ by $\rho(x_1' + 4\lambda x_2')$, $(fm)_1$ by $\sigma(4\mu x_1' + x_2')$. Then the Jacobian ϑ of $\rho x_1'^3(x_1' + 4\lambda x_2')$ and $\sigma x_2'^3(4\mu x_1' + x_2')$ must be a square. The computation gives

$$\vartheta = x_1'^2 x_2'^2\{3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2\},$$

so that $3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2$ must be a square, which it is not, since $(8\lambda\mu + 1)^2 - 36\lambda\mu \neq 0$ if λ and μ are arbitrary.

2°. To prove Rl^2, Sm^2, Tn^2, f^2 apolar to three forms involves twelve relations but it suffices to prove three of them; thus if $l^3(fl)_1$ is apolar to Sm^2 it must be to Tn^2, etc. It is sufficient to prove $l^3(fl)_1$ apolar to (*a*) Rl^2, (*b*) f^2, (*c*) Sm^2. The proof is in each case facilitated by the use of a proper normal form.

(*a*) $l^3(fl)_1$ is apolar to Rl^2 independently of the particular values of $(fl)_1$ or R. Make the substitution

$$x_1' = l, \quad x_2' = x_2,$$

then

$$l^3(fl)_1 = x_1'^3(a_1x_1' + a_2x_2'), \quad Rl^2 = x_1'^2(b_0x_1'^2 + 2b_1x_1'x_2' + b_2x_2'^2),$$

and these two forms are at once seen to be apolar. This involves the proof of three relations.

(*b*) To prove $l^3(fl)_1$ apolar to f^2b let $l = x_1'$, $(fl)_1 = x_2'$; then since $(fl)_1 l$ is harmonic to f we must have $x_1'x_2'$ harmonic to f, that is $f = A_0x_1'^2 + A_2x_2'^2$ and $l^3(fl)_1 = x_1'^3x_2'$, but $x_1'^3x_2'$ is apolar to

$$(A_0x_1'^2 + A_2x_2'^2)^2 = A_0^2x_1'^4 + 2A_0A_2x_1'^2x_2'^2 + A_2^2x_2'^4.$$

This involves three relations.

(*c*) Since the relation $\left(l^3(fm)_1, Rm^2\right)_1 = 0$ is homogeneous in the coefficients of l, m, n we may in proving it use instead of l, m, n any forms proportional to them.

Let $x_1' = p$, $x_2' = q$, dropping the accents after transforming. Let ϕ be proportional to

$$x_1^3 + 3\lambda x_1^2x_2 + 3\mu x_1x_2^2 + \nu x_2^3 = a\phi = (x_1 + \xi_1x_2)(x_1 + \xi_2x_2)(x_1 + \xi_3x_2),$$

so that

$$\rho_1 l = (x_1 + \xi_1 x_2), \quad \rho_2 m = (x_1 + \xi_2 x_2), \quad \rho_3 n = (x_1 + \xi_3 x_2).$$

We know that f is harmonic to $x_1 x_2$, also to $(\phi p)_2$, that is, to

$$\lambda x_1^2 + \mu x_1 x_2 + \nu x_2^2;$$

so that f must be proportional to the Jacobian of $x_1 x_2$ and $(\phi p)_1$, or

$$\beta f = \lambda x_1^2 - \nu x_2^2,$$

where α and β are quantities which it is not necessary to determine.

If we put

$$\eta_i = \frac{3(\mu \xi_i - \nu)}{\lambda - \xi_i} \qquad (i = 1, 2, 3),$$

and use the method of forming $Q(\phi)$ described in section II we find $Q(\phi)$ proportional to

$$(3\xi_1 x_1 - \eta_1 x_2)(3\xi_2 x_1 - \eta_2 x_2)(3\xi_3 x_1 - \eta_3 x_2),$$

and performing the operation $D_{pq}\big(Q(\phi)\big)$ we find ϕ proportional to

$$(\xi_1 x_1 + \eta_1 x_2)(\xi_2 x_1 + \eta_2 x_2)(\xi_3 x_1 + \eta_3 x_2);$$

so that R is proportional to the Jacobian of $x_1(\xi_1 x_1 - \eta_1 x_2)$ and $x_2(x_1 - \xi_1 x_2)$, since $x_1 - \xi_1 x_2$, $\xi_1 x_1 - \eta_1 x_2$ are proportional to l_1, $\overline{l}_1$ or finally R, S, T are equal to

$$\sigma_1(x_1^2 - 2\xi_1 x_1 x_2 + \eta_1 x_2^2), \quad \sigma_2(x_1^2 - 2\xi_2 x_1 x_2 + \eta_2 x_2^2), \quad \sigma_3(x_1^2 - 2\xi_3 x_1 x_2 + \eta_3 x_2^2),$$

respectively.

Also $(fl)_1$ is proportional to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2;$$

hence

$$(x_1 + \xi_1 x_2)^3 \left(x_1 + \frac{\nu}{\xi_1 \lambda} x_2\right)$$

must be apolar to

$$(x_1^2 - 2\xi_1 x_1 x_2 + \eta_2 x_2^2)(x_1 + \xi_2 x_2)^2.$$

To show this we note that if $a_x^4 = (xa)(x\beta)(x\gamma)(x\delta)$ is apolar to b_x^4 then $b_\alpha b_\beta b_\gamma b_\delta = 0$, so that if we take the third polar of Sm^2 with respect to $x_1 + \xi_1 x_2$ and the first polar of the result with respect to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2,$$

the final result must be zero.

Doing this we obtain

$$\tfrac{1}{2}(\xi_2 - \xi_1)\{(\xi_1^2 + 2\xi_1\xi_2 + \eta_2)(\lambda\xi_1\xi_2 - \nu) + (\xi_2 - \xi_1)\big(\nu(\xi_1 + \xi_2) + (\xi_1\xi_2 + \eta_2)\lambda\xi_2\big)\}$$

which must be zero, or since $\xi_1 \neq \xi_2$,

$$(\xi_1^2 + 2\xi_1\xi_2 + \eta_2)(\lambda\xi_1\xi_2 - \nu) + (\xi_2 - \xi_1)\big(\nu(\xi_1 + \xi_2) + (\xi_1\xi_2 + \eta_2)\lambda\xi_2\big)$$

must be zero; or solving for η_2 and remembering that

$$3\lambda = \xi_1 + \xi_2 + \xi_3,$$

$$3\mu = \xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2,$$

$$\nu = \xi_1\xi_2\xi_3,$$

η_2 must equal

$$\frac{(\xi_1^2 + 2\xi_1\xi_2)(\xi_2\lambda - \xi_2\xi_3) - (\xi_1 - \xi_2)\big(\lambda\xi_1\xi_2 + \xi_1\xi_2(\xi_1 + \xi_2)\big)}{\xi_2\xi_3 - \lambda(2\xi_2 - \xi_1)}.$$

The denominator $\xi_2\xi_3 - \lambda(2\xi_2 - \xi_1)$ is equal to

$$(\lambda - \xi_2)(\xi_1 + \xi_2),$$

and the numerator is equal to

$$(\xi_1 + \xi_2)\{(\xi_2 - \xi_1)\xi_2\xi_3 + \xi_1(\xi_2\lambda - \xi_2\xi_3)\} + \xi_1\xi_2(\xi_2\lambda - \xi_2\xi_3) - (\xi_1 - \xi_2)\lambda\xi_1\xi_2,$$

and

$$\xi_1\xi_2(\xi_2\lambda - \xi_2\xi_3) - (\xi_1 - \xi_2)\lambda\xi_1\xi_2 = -(\lambda\xi_2\xi_1^2 + \xi_2\nu - 2\lambda\xi_2^2\xi_1)$$

$$= -\{\lambda\xi_1\xi_2(\xi_1 + \xi_2) - \xi_1^2\xi_2^2 - \xi_2^3\xi_1\} = (\xi_1 + \xi_2)(\xi_2^2\xi_1 - \lambda\xi_1\xi_2);$$

or the numerator of η_2 is equal to

$$(\xi_1 + \xi_2)(\xi_2^2\xi_3 - \xi_1\xi_2\xi_3 + \lambda\xi_1\xi_2 - \xi_1\xi_2\xi_3 - \lambda\xi_1\xi_2 + \xi_2^2\xi_1)$$

$$= (\xi_1 + \xi_2)\{\xi_2(\xi_2\xi_3 + \xi_3\xi_1 + \xi_1\xi_2) - 3\xi_1\xi_2\xi_3\} = 3(\xi_1 + \xi_2)(\mu\xi_2 - \nu)$$

or finally, η_2 must equal

$$\frac{3(\xi_1 + \xi_2)(\mu\xi_2 - \nu)}{(\lambda - \xi_1)(\lambda - \xi_2)} = \frac{3(\mu\xi_2 - \nu)}{\lambda - \xi_2},$$

which is the definition of η_2. Hence the relation from which we started must be true. This proves (c) involving six relations and the theorem is proved in all its parts if we show that the Jacobian of I_4 is proportional to $\phi f q$. This follows immediately from section I where it was proved that the Jacobian of I_4 determined from ϕ and f is proportional to $\phi f(x\delta)$, where $(x\delta) = (ab)^2(ab')b_x'$ which is the definition of q.

In virtue of the reasoning at the end of section III it follows that Rl^2, Sm^2, Tn^2, f^2 belong to I_4 whose Jacobian is proportional to $\bar{\phi}\bar{f}\bar{q}$, that is to $\bar{\phi}\bar{f}p$.

It follows that I_4 and I_4 determined by (Σ) and (Σ) have three branch quadratics in common and the two covariants p, q are the fourth double elements, q in I_4 and p in I_4.

I_4 derived from (Σ)	I_4 derived from (Σ)
Rl^2, Sm^2, Tn^2, f^2	$R\bar{l}^2$, $S\bar{m}^2$, Tn^2, $\bar{f}^2$
with double elements given by	with double elements given by
$\phi f q$.	$\bar{\phi}\bar{f}p$.

14. In section III ϕ and f were expressed in terms of the fundamental concomitants of ϕ and f by means of the typical representation by p and q. Since f^2 is a form of I_4 if we express one other form of I_4 in terms of the concomitants of ϕ, f we have every form of I_4 so expressed. The branch form whose double element is q will be rational in the concomitants of ϕ, f. Let it be equal to $F = q^2(ap^2 + 2\gamma pq + \beta q^2)$ and let f be $Ap^2 + Bq^2$ (there is no term in pq in the typical representation of f). Then the Jacobian of F and f^2 must be proportional to $\phi f q$, or the Jacobian of F, f to ϕq, or finally we may take α, β, γ so that

$$q \begin{vmatrix} Ap & , & Bq \\ q(ap + \gamma q), & & q(\gamma p + \beta q) + 2(ap^2 + \beta q^2 + \gamma pq) \end{vmatrix} = \phi q,$$

or

$$2A\alpha p^3 + 3A\gamma p^2 q + (3A\beta - \alpha B)pq^2 - B\gamma q^3 = \phi,$$

and since ϕ is expressed in terms of p and q, by equating coefficients α, β, γ may be found and consequently F expressed as concomitant to ϕ, f.

If we express the form of I_4, which contains p^2, in terms of p, q, we can express every form of I_4 in terms of p, q, for $\bar{f}^2$ is already so expressed.

V.

The reducible Hyperelliptic Integrals.

15. Since I_4 and I_4 have three branch quadratics in common and q and p are the fourth double elements, we have from the Introduction the

Theorem: *The integrals*

$$\int \frac{q(x\,dx)}{\sqrt{RST}}, \qquad \int \frac{p(x\,dx)}{\sqrt{R\bar{S}\bar{T}}}$$

are reducible to elliptic integrals by transformations of the fourth order I_4, I_4.

Having given one reducible integral and its reducing transformation we can solve the problem to find the second one belonging to the same irrationality and its reducing transformation. For the reducing transformation being known the form ϕ is known and also f; from ϕ and f we deduce the covariant $(ab)^2a_x$ which is the numerator of the second reducible integral; the second reducing transformation being given by the involution I_4 derived from ϕ and f, which can be found from ϕ and f.

We can enunciate the following theorem:

THEOREM: *If $\phi = a_x^3$ denotes the product of the three double elements complementary to the three quadratic factors of the sextic of a reducible integral, and $f = b_x^2$ denotes the form whose square occurs among the forms of the reducing involution, then the numerator of the second reducible integral belonging to the same irrationality is equal to $(ab)^2a_x$.*

16. It is possible to give another definition of the numerator of the second integral which resembles that given by Professor BOLZA* for the transformation of order 3 and also that for order 2. For we have shown that if l_x denotes a double element of an I_4 containing a complete square f^2, then $l^3(fl)_1$ is apolar to the forms of I_4; but q is a double element of I_4 and $(fq)_1$ is proportional to p, hence q^3p is apolar to I_4. We can enunciate the theorem:

THEOREM: *If $(x\delta)$ denotes the numerator of the first reducible integral and $(x\delta)$ of the second and I_4 the reducing involution, then $(x\delta)^3(x\delta)$ is apolar to the forms of I_4 and this is sufficient to determine $(x\delta)$.*

The following conjectural theorem suggests itself: *If $(x\delta)$ and $(x\delta)$ denote as before the numerators of the integrals—reducible by a transformation of order k—and I_k the reducing involution, then $(x\delta)^{k-1}(x\delta)$ is apolar to the forms of I_k.* This surely holds for $k = 2$, 3 and, as has been shown in this paper, for $k = 4$.

VI.

NORMAL FORM OF REDUCIBLE INTEGRALS.

17. The normal form used by Professor BOLZA† is found by introducing p and q as variables, that is, by introducing the numerators of the two integrals. This has been done for a special purpose in section IV, and the formulæ there developed have only to be supplemented; we found

$$R = \sigma_1(x_1^2 - 2\xi_1x_1x_2 + \eta_1x_2^2),$$
$$S = \sigma_2(x_1^2 - 2\xi_2x_1x_2 + \eta_2x_2^2),$$
$$T = \sigma_3(x_1^2 - 2\xi_3x_1x_2 + \eta_3x_2^2),$$

*Mathematische Annalen, vol. 50, p. 314.

†Dissertation, Mathematische Annalen, vol. 28.

$$RST = \sigma_1\sigma_2\sigma_3\{\nu' x_1^6 - 6\lambda\nu' x_1^5 x_2 + 3(4\mu\nu' + \lambda\mu')x_1^4 x_2^2 + 2(\lambda\lambda' + 5\nu\nu')x_1^3 x_2^3 + 3(4\mu'\nu' + \lambda'\mu)x_1^2 x_2^4 - 6\lambda'\nu x_1 x_2^5 + \nu x_2^6\},$$

where

$$\lambda' = -\tfrac{1}{3}\frac{2\lambda^2\nu - \lambda\mu^2 - \mu\nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\mu' = \tfrac{1}{9}\frac{\lambda^2\mu + \lambda\nu - 2\mu^2}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\nu' = -\tfrac{1}{27}\frac{2\lambda^3 - 3\lambda\mu + \nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3}.$$

Professor Bolza used implicitly a form corresponding to ϕ, viz.:

$$x_2^3 + 3\lambda' x_2^2 x_1 + 3\mu' x_2 x_1^2 + \nu' x_1^3,$$

and made the remark that if the cubic covariant of

$$x_1^3 + 3\lambda x_1^2 x_2 + 3\mu x_1 x_2^2 + \nu x_2^3$$

is

$$C_0 x_1^3 + 3C_1 x_1^2 x_2 + 3C_2 x_1 x_2^2 + C_3 x_2^3,$$

then

$$\lambda' = -\tfrac{1}{3}\frac{C_2}{C_3}, \quad \mu' = \tfrac{1}{9}\frac{C_1}{C_3}, \quad \nu' = -\tfrac{1}{27}\frac{C_0}{C_3},$$

which was the starting point for discovering the relations in sections III and IV.

The reducible integrals are

$$\int \frac{x_1(x\,dx)}{\sqrt{R}}, \quad \int \frac{x_2(x\,dx)}{\sqrt{R}},$$

where $R = RST$ is expressed in x_1, x_2 above.

The reducing substitution for the first integral is given by any two of the forms

$$\sigma_i(x_1^2 - 2\xi_i x_1 x_2 + \eta_i x_2^2)(x_1 + \xi_i x_2)^2 \qquad (i = 1, 2, 3).$$

The form f is equal to $(\lambda x_1^2 - \nu x_2^2)$.

VII.

Miscellaneous Results on Biquadratic Involutions Containing a Complete Square.

18. In section VII are collected some miscellaneous results relating to special biquadratic involutions containing a square and their geometrical representations.

(*A*) It is a known theorem that the Jacobians of two apolar systems are the same.* Since the roots of the Jacobian are multiple roots in forms of the system from which it is derived this theorem states that if (xa) is a double element of a biquadratic involution (II) it is a triple element of the apolar system $(\overline{\text{II}})$; hence $(xa)^3(x\bar{a})$ is a form of $(\overline{\text{II}})$. When (II) is a special involution we are able to determine $(x\bar{a})$ from section IV for it was there shown that $(xa) = \left(f, (xa)\right)_1$, f being the form whose square belongs to (II).

THEOREM.—*If* (xa) *is a double element of a special biquadratic involution* I_4 *containing a square* f^2, *then* $(xa)^3\left(f, (xa)\right)_1$ *belongs to the system apolar to* I_4.

(*B*) The theorem of section I which says that the form there denoted by θ is apolar to the involution $\lambda u + \mu f^2$ admits of a geometrical interpretation. Let a_x^4, β_x^4, γ_x^4 be three forms apolar to u_x^4 and v_x^4. The rational quartic C_4† given by the parameter representation

$$\rho x_1 = a_x^4,$$

$$\rho x_2 = \beta_x^4,$$

$$\rho x_3 = \gamma_x^4,$$

is intersected by a straight line $\lambda x_1 + \mu x_2 + \nu x_3$ in 4 points whose parameters satisfy the equation:

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0,$$

that is, they are given by a form apolar to u_x^4 and v_x^4.

If $\lambda x_1 + \mu x_2 + \nu x_3$ is an inflexional tangent of C_4 the equation

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0$$

has a triple root which is the parameter value for the point of inflexion. By the theorem already quoted in this section this must be a root of the Jacobian of u_x^4 and v_x^4. When this is applied to the special involution, $\theta = 0$ must give 4 points of inflexion; but θ is a form of the system apolar to the involution, that is $\theta = 0$ gives 4 collinear points, hence the

THEOREM: *Four of the points of inflexion of a special* C_4 *are collinear.*

(*C*) Another definition can be given of the linear factors of ϕ, which is adapted to their determination when a twisted cubic C_3 is used to carry the binary variables.‡ Using the notation of section II, we have the

THEOREM: Lq^2 *is apolar to* $(pq, \bar{l})_1 \underline{p}^2$.

---

*STEPHANOS: *Sur les faisceaux ayant une même jacobienne*, Mémoires par divers savants, vol. 27.

† MEYER: *Apolarität und Rationale Curven.*

‡ STURM, Crelle's Journal, vol. 86.

Proof.—Referring to the definition, l is proportional to $p(qL) + 3q(pL)$, hence $(pq, l)_1$ is proportional to $p(pq)(qL) - 3q(pq)(pL)$.

To express L in a convenient form we notice that $\big(pq, (pq, L)_1\big)_1$ is proportional to L, for we have taken the harmonic $(pq, L)_1$ of L with regard to pq and then of the linear form so found with regard to pq again, which must give the original L up to a factor.

Hence,

$$(pq, L)_1 = \tfrac{1}{2}\big(p(qL) + q(pL)\big),$$

and therefore

$$\big(pq, (pq, L)_1\big)_1 = \tfrac{1}{4}\{-p(pq)(qL) + q(pq)(pL)\},$$

therefore

$$\{-p(pq)(qL) + q(pq)(pL)\}q^2$$

must be apolar to

$$\{(pq)(qL) - 3q(pq)(pL)\}p^2;$$

that is, if we put $x_1' = p$, $x_2' = q$,

$$-x_1'x_2'^2(pq)(qL) + x_2'^3(pq)(pL)$$

must be apolar to

$$x_1'^3(pq)(qL) - 3x_1'^2x_2'(pq)(pL),$$

that is,

$$(pq)^2(qL)(pL) - (pq)^2(qL)(pL)$$

must be zero. This is true and therefore the theorem is proved.

The apolarity of cubic forms admits of geometrical representation (STURM, loc. cit.). Let a_x^3 be a cubic, to it will correspond a plane having a null point; if b_x^3 is another cubic apolar to a_x^3 then the plane corresponding to b_x^3 will pass through the null point of the plane corresponding to a_x^3, and conversely; if the plane passes through that null point b_x^3 will be apolar to a_x^3.

There is also a geometrical representation of the cubic covariant Q of $\phi = a_x^3$ on C_3. Call A the plane corresponding to ϕ, then A contains one line which is the intersection of two osculating planes of C_3; call these B and C. Then the plane D, which in the pencil containing A, B, C divides A harmonically from B and C, meets C_3 in 3 points given by $Q(\phi) = 0$.

Suppose we start with ϕ and f, then we can construct p and q (STURM, loc. cit.); also $Q(\phi) = LMN$. Then by the theorem last proved we can find the point $(pq, l)_1 = 0$, viz: the plane containing the tangent to C_3 at $p = 0$ and the null point of the plane through the point $L = 0$ and the tangent at $q = 0$ must meet C_3 in three points among which is the point $p = 0$ counted twice and the third of which is the point $(pq, l)_1 = 0$.

If we take the harmonic of the point $(pq, \bar{l})_1 = 0$ with respect to $pq = 0$ we reach the point $\bar{l} = 0$ and thus the points $\phi = 0$ may be found.

THEOREM: *Given the plane corresponding to* $\phi = 0$ *and the line corresponding to* $f = 0$, *it is possible to find on the twisted cubic* C_3 *the plane corresponding to* $\phi = 0$.

The harmonic of a point with respect to two others may be found (MEYER, loc. cit.) by considering the hyperboloids through C_3. But if we wish the harmonic of a with respect to b and c it will be found among the points given by the cubic covariant of a, b, c, viz: it is that one which is separated from a by b and c in virtue of the definition of cubic covariant used in section II.

This is a simpler way of reaching it because of the simple geometrical construction of the cubic covariant.

ON THE THEORY OF IMPROPER DEFINITE INTEGRALS*

BY

ELIAKIM HASTINGS MOORE

§ 1.

INTRODUCTION. THE FOUR CURRENT TYPES OF IMPROPER INTEGRALS.

1°. In this paper I wish to define a system of types of improper simple definite integrals, a system embracing in particular the four current types; of the theory of the general type I give at present merely the elements, the methods employed, however, being characteristic.

The four current types are compared in § 1 2°–13°. By way of generalization of their diversities the new types arise (11°–17°). As the desirable basis for the new types I propose (16°) an extension of the notion of the proper simple definite integral; this involves likewise an extension of the notions of the four current types.

In § 2 I state in convenient notations a body of elementary properties of the general type of integrals. These properties with two definitional processes of induction developed in §§ 3, 4 serve as the basis for the definition in § 5 of the system of types of improper integrals related to the (extended) type of proper integrals defined in 16°.

2°. If the definite integral

$$\int_a^b F(x)\,dx \tag{1}$$

is a proper integral, the interval ab of integration is finite and the integrand function $F(x)$ exists (everywhere defined but not necessarily in a single valued way) on ab as a limited function.

3°. IMPROPER INTEGRALS OVER FINITE INTERVALS.—We call to mind the four current types of improper integrals (1) over finite intervals ab. The following presuppositions are common to the four types. On ab the function $F(x)$ has a singular point-set Z. In every neighborhood of a point ζ of Z the function $F(x)$ is unlimited. On every interval of ab containing no point ζ the function $F(x)$ is properly integrable.

* Presented to the Society at the Ithaca meeting August 19, 1901. Received for publication October 2, 1901.

4°. TYPE I.—In the first type the set Z is reducible, that is, for some (lowest) ordinal * a of CANTOR'S first or second class of ordinals, the derived set $Z^{(a)} \equiv 0$; hence the set Z has the content zero, and it is finite or numerably infinite; and conversely, any set finite or numerably infinite is reducible. CAUCHY and RIEMANN considered the case of finite sets $Z(a = 1)$, and DU BOIS-REYMOND (1875: Crelle's Journal, vol. 79, pp. 36, 45), and DINI (1878: DINI-LÜROTH, pp. 404–445) the case of sets Z of the first kind † $(a = \nu = 1, 2, 3, \cdots)$, and quite recently SCHOENFLIES (1900: *Bericht*, pp. 185–186, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 8, no. 2) the general case $(a = a)$.

5°. TYPE II.—In the second type the set Z is any set of content zero. For a systematic exposition of the theory of these the (general) HARNACK integrals of 1883–1884 I refer to my paper in the last number of this journal (pp. 296–330); in the introduction are indicated the relations with the equivalent JORDAN theory of 1894.

The theories of these two types have this fundamental difference: the integral of the second type is defined by a definition covering all cases at once, while all cases of the first type are covered in that the integral for the case $a = 1$ is defined directly and then two definitional processes are given which serve to define, in the first place, the integral for the case $a = a_0 + 1$ in terms of integrals for the case $a = a_0$ and, in the second place, the integral for the case $a = a_\omega$ in terms of a numerably infinite set of integrals for the cases $a = a_\nu (\nu = 1, 2, 3, \cdots)$, where the ordinal a_ω is next higher than the set $\{a_\nu\}$. Thus the definition of a particular integral of the second type involves a single limiting process, and that of one of the first type involves a finite or a numerably infinite system of limiting processes.

6°. In the comparison of two types of integrals, presupposing that in each theory every integral (1) when existent has a definite finite value, we say for brevity that the two types are *accordant* or *in accord* in case every integral existent in each theory has in the two theories equal values; and of two accordant types we say that the first *contains* or *includes* the second if every integral existent in the second theory exists also in the first. Two types are *equivalent* if each includes the other.

7°. Now to compare the first two theories, we see that the second theory has wider application than the first theory, since the second theory covers the cases of irreducible sets Z. As to integrals with reducible sets Z, the two types are

* That is ordinal number (*Ordinalzahl*).

† *Gattung*.

well known to be equivalent for the case $a = 1$. For the cases* $a > 1$, however, while it is readily seen that the first type contains the second, the two types are not equivalent; for example ($a = 2$), the integral (31) p. 330 of my paper cited above exists or not according as it is of the first or second type.

8°. TYPE III.—The third type is that of HÖLDER (Mathematische Annalen, vol. 24 (1884), p. 90 ff.); the set Z is reducible. HÖLDER defines primarily an indefinite integral

$$\int F(x)\,dx \tag{2}$$

on the x-interval ab, viz., as any function $\phi(x)$ continuous on ab and such that for any two points x' x'' of ab, whose interval $x'x''$ contains no point ζ one has the relation

$$\phi(x'') - \phi(x') = \int_{x'}^{x''} F(x)\,dx. \tag{3}$$

This definition would apply to any set Z. The restriction to reducible sets Z is made in order that two such indefinite integrals on ab may differ on ab by a constant, and thus that the definite integral:

$$\int_{a'}^{b'} F(x)\,dx = \phi(b') - \phi(a'), \tag{4}$$

where $a'b'$ are any two points of ab, may be independent of the indefinite integral $\phi(x)$ used in its definition.

The first and the third types of definite integrals are easily seen to be equivalent. Indeed it is possible that HÖLDER had in mind the final extension, later made by SCHOENFLIES, of the integrals of the first type to the case of any ordinal α.

9°. TYPE IV.—The fourth type is that of DE LA VALLÉE-POUSSIN (1892: Liouville's Journal, ser. 4, vol. 8, pp. 421–467). For the general set Z of content† zero a primary definition is given by a single limiting process which is of such a nature that the integral (1) if existent is absolutely existent. Then, on

*I have found only the two following notes of comparison of the two types for these cases:

STOLZ (Wiener Berichte, vol. 107_{2a} (1898), pp. 207–224: cf. §2) states that for the case $\alpha = \nu > 1$ the second type is not evidently ("nicht ohne Weiteres") to be identified with the first.

SCHOENFLIES (loc. cit., p. 188, ll. 21–24) discriminates in general the cases of sets Z reducible and irreducible by introducing the terms: integrals of the first and second kinds (*Arten*), and thus distinctly implies that for reducible sets Z the two types are equivalent.—The proper discrimination is by the type.

†SCHOENFLIES (loc. cit., pp. 178, 187, §5) points out and emphasizes the fact that this condition if not inserted in the definition may be deduced as a conclusion, in any case in which the (modified) definition yields an integral.

the basis of the proper and of these absolutely existent improper integrals, for the general reducible set Z_0 of all points ζ of Z in whose neighborhood the function $F(x)$ is not absolutely integrable the non-absolutely existent integrals are defined * as in the first theory.

It is to be noted that SCHOENFLIES (loc. cit., pp. 188, 189) gives vague indications that the case of irreducible sets Z_0 may be included by suitable phrasing of the secondary definition, and indeed that the primary definition may be so modified as to include the non-absolutely existent integrals.

The second and the fourth types are equivalent in so far as absolutely existent integrals are concerned,† and, likewise, as to non-absolutely existent integrals, in the case $a = 1$ for the reducible set Z_0, while in the cases $a > 1$ the fourth type contains but is not equivalent to the second type; the example (for $a = 2$) cited above (7°) serves here also.

10°. To summarize: The HARNACK (second) type of integral (1) has the widest range of application and it involves a single limiting process. The other types involve in general a system of limiting processes. Every type contains the corresponding subtype of the HARNACK type. Every subtype involving only one limiting process is equivalent to the corresponding HARNACK subtype.

In every theory the integral symbol (1) has a unique definition.

11°. For a given singular set Z the system of types of improper integrals (1) to be defined in this paper arises from the four current types by way of generalization of their diversities. The reader will doubtless have foreseen the character of the general type to be proposed. It bears to the HARNACK type a relation which generalizes (not uniquely) the relation borne by the first type ($a = a$) to the CAUCHY subtype ($a = 1$). The integral symbol (1) for any type of the system receives a unique definition depending on the type, or, otherwise expressed, on the way in which the set Z is utilized in the limiting process or processes involved in the definition.

At present in connection with the definition of the system of types I consider briefly the theory of the general type of the system but not the relations amongst the various types. I conjecture however that this relation is (as amongst the current types) simply that every type is in accord with every other type. If this relation is indeed valid, we may obviously unite or fuse all the types of the system into a new single type in which the symbol (1) has again a unique definition.†

* The definition given by DE LA VALLÉE-POUSSIN (loc. cit., p. 453, § 55) is restricted to the case of reducible sets Z_0 of the first kind.

† Cf. STOLZ (loc. cit., introduction).

‡ If in any particular type of the system a symbol (1) has a value, it shall have that value in the new type. If in every type of the system the symbol (1) has no value, it shall have no value in the new type.

12°. Since in the theory of a particular type of integrals the definite integral function

$$J(x) = \int_a^x F(x)\,dx \qquad (x \text{ on } ab) \tag{5}$$

of a function $F(x)$, properly or improperly integrable from a to b, is a continuous function of x on ab, and quâ $\phi(x)$ satisfies the relation (3), we have in that theory, even for irreducible sets Z, means of selecting from amongst all HÖLDER'S functions $\phi(x)$ a system every two of which differ on ab by a constant, viz., the system

$$\phi(x) = J(x) + C, \tag{6}$$

where C is an arbitrary constant. For this theory these functions $\phi(x)$ may then properly receive the indefinite integral notation:

$$\phi(x) = \int F(x)\,dx. \tag{7}$$

In case we may fuse all the types of the system into a new type, as suggested in 11°, we obtain in the new theory a unique system of indefinite integrals (7).

13°. IMPROPER INTEGRALS OVER INFINITE INTERVALS.—In each of the four current types the improper integral (1) over an infinite interval ab is defined either explicitly or presumptively as the limit of the proper or improper integral over the finite subinterval $a'b'$ of ab as $a'b'$ converges to ab.* Thus for each type the theory of the improper integrals over infinite intervals depends upon that of the improper integrals over finite intervals.

It is, however, possible to define the former integrals as transforms of the latter integrals. The transformation

$$x = \psi(\bar{x}) = -(\bar{x} \pm 1)^{-1} \pm 1 \qquad (x \text{ on the interval } {}^{-\infty \cdots 0}_{0 \cdots \infty}) \tag{8}$$

throws the complete† x-line $(-\infty \leqq x \leqq \infty)$ in a one-to-one way on the $\bar{x}$-interval $-1 \leqq \bar{x} \leqq +1$; as x traverses the x-line in the positive sense x traverses the x-interval in the same sense; $\psi(\bar{x})$ is a continuous function of $\bar{x}$ with a continuous positive derivative $\psi'(\bar{x}) = (\bar{x} \pm 1)^{-2}$ on the $\bar{x}$-interval excluding its extremities ∓ 1. Then, setting

* We define in like manner the content $\gamma(\Xi)$ of a point-set Ξ lying on an infinite interval ab as the limit of the content $\gamma(\Xi')$ of the subset Ξ' of Ξ lying on the finite subinterval $a'b'$ of ab as $a'b'$ converges to ab.

† It is here permissible and convenient to admit the definite infinities ∞, $-\infty$ as arguments and functional values. For a function $f(x)$ the two numbers

$$f(\infty), \quad f(\infty - 0) \equiv \mathop{\mathrm{L}}_{x=\infty} f(x)$$

are definitionally distinct, and so are $f(-\infty), f(-\infty + 0)$. The function $f(x)$ is continuous at ∞ if $f(\infty), f(\infty - 0)$ exist and are equal.

$$(9)\qquad F(\bar{x}) \equiv F\big(\psi(\bar{x})\big)\,\psi'(\bar{x}) = F\big(\psi(\bar{x})\big)\,(\bar{x}\pm 1)^{-2} \quad (x \text{ on the interval } {}^{-\infty\cdots 0}_{0\cdots\infty}),$$

and denoting by a, $\bar{a}$ and b, $\bar{b}$ two pairs of corresponding values of x, $\bar{x}$ respectively, we have the relation of transformation:

$$(10)\qquad \int_a^b F(x)\,dx = \int_{\bar{a}}^{\bar{b}} F(\bar{x})\,d\bar{x}.$$

This relation is valid for finite intervals ab.—We may use the relation to define the integrals over infinite intervals, viz., as transforms of those over finite intervals.

Similarly, for any type of integrals, we may define the integrals over infinite intervals either as limits of integrals of the same type or as transforms of integrals of a transformed type, these integrals being in each case over finite intervals.

It seems to be desirable to adopt the former and usual definition as the definitive definition for use in the applications of the theory of any type of integrals, but in the development of the theory to adopt the latter new definition. For it is evident that an integral under the new definition is one under the old definition, while an integral under the old definition, if not one under the new definition, is one under the new definition for a closely related type of integrals; and so the theory is richer by the adoption of the old definition while its development (if made for the general type of integrals) is somewhat simpler by the adoption of the new definition.

For the first, third, and fourth types the two definitions are equivalent. In the second type, however, it is possible to have, for example, a HARNACK integral $\int_1^\infty F(x)\,dx$ (according to the old definition) whose transformed HARNACK integral $\int_{\frac{1}{2}}^1 \overline{F}(\bar{x})\,d\bar{x}$ does not exist: the example of §3 9° of my paper already cited is to the point,—on the x-interval $\frac{1}{2}1 = ab$ the present function $\overline{F}(x)$ being the function $F(x) + G(x)$ of the reference.

14°. For the sake of clearness of thought and also of greater generality we replace the point-set Z of singularities ζ of the integrand function $F(x)$ by any point-set Ξ of content zero.* The general type of integral is then in a certain sense a function or an aspect of Ξ. The integral (1) depends upon the type, the interval ab, and the integrand function $F(x)$. If the symbol (1) has value, the function $F(x)$ has certain relations to Ξ, e. g., on every interval of ab containing no point ξ of Ξ the function $F(x)$ is properly integrable; and further, the limiting relations implied in the type hold.

15°. For the purpose of the later theory the content $\gamma(\Xi)$ of Ξ needs to be zero. But the definitions and theory of the present paper are effective also for

* But cf. 15°.

sets Ξ of arbitrary content $\gamma(\Xi)$; since this fact has interest, the more general form of presentation is adopted.

16°. The more general proper definite integrals.—I extend the notion of the proper definite integral (1) as follows:

The interval ab is a finite interval. The integrand function $F(x)$ exists on ab, or, at least, on ab apart from a point-set Ω of content zero, as a function everywhere defined* and limited;† and a function $G(x)$, arising from $F(x)$ by modification of its determination on Ω so that $G(x)$ is on ab everywhere defined and limited, is properly integrable from a to b in the usual sense. Then we set in general

$$\int_a^b F(x)\,dx = \int_a^b G(x)\,dx. \tag{11}$$

In case on the set Ω $F(x)$ itself is everywhere defined and limited, the integral (1) exists as a proper integral in the usual sense, and the equality (11) is well known.

In other cases the equality (11) serves to define the integral (1) as a definite finite number, a proper definite integral in the new sense.

By virtue of this definition of the type of proper integrals every existing definite integral (1) of this type or of any derivative type defined in this paper (§5) retains its type and value under any modification of the determination of the integrand function on a point-set of content zero. This statement is embodied in theorem II of §2; and it is to be compared with the remark of §5 7° (p. 329) of my paper cited above (5°).

§2.

Notations and elementary properties of the general type of integrals.

1°. With respect to any linear point-set Ξ (§1 15°) we are to consider a system of types of improper definite integrals (§1 (1)), and especially the general (any particular) type of the system. For this type we use the notation $\mathbf{I}$ ("integral"), instead of the integral sign $\int$ with an affix indicating the type in question. Thus, the symbol:

$$\mathbf{I}_a^b F(x)\,dx, \tag{1}$$

where a and b are definite points (finite or infinite) and where $F(x)$ is a definite function of x on the interval ab, denotes a definite integral of the type $\mathbf{I}$,

* Not necessarily in a single valued way.

† On the set Ω $F(x)$ need not be defined, and, so far as it is defined, it need not be limited.

which is either existent and of definite finite value or non-existent. We speak of the $\mathbf{I}$-integral (1) and of functions $F(x)$ $\mathbf{I}$-integrable from a to b.

The closed set $\Xi + \Xi'$ is supposed to contain the two infinite points $(\pm \infty)$. If $\Xi = \{+\infty, -\infty\}$ the (only) type $\mathbf{I}$ of the set Ξ is the type of the proper definite integrals of §1 16° over finite intervals and

The integrals of type $\mathbf{I}$ over subintervals of a fixed interval ab constitute a type $\mathbf{I}_{ab}$. With respect to the type $\mathbf{I}_{ab}$ all sets Ξ having the same subset Ξ_{ab} lying on ab are equivalent.

2°. The following theorems (I–VIII) of the theory of the proper definite integrals defined in §1 16° hold also in the theory of the integrals of the type $\mathbf{I}$.

I*. *The function* 0 *is* $\mathbf{I}$*-integrable from* a *to* b, *and* $\mathbf{I}_a^b 0\,dx = 0$.

II. *A change in the determination of the integrand function* $F(x)$ *in the points of a point-set of content* † *zero makes no change in the determination of the* $\mathbf{I}$*-integral* $\mathbf{I}_a^b F(x)\,dx$.

FIXED HYPOTHESIS.—*The function* $F(x)$ *in* III–VIII (IV′ *apart) is supposed to be* $\mathbf{I}$*-integrable from* a *to* b.

III. *The function* $F(x)$ *is likewise* $\mathbf{I}$*-integrable from* x_1 *to* x_2, *where* $x_1 x_2$ *are any two points of* ab.

IV. *For any three points* $x_1 x_2 x_3$ *of the interval* ab

$$\mathbf{I}_{x_1}^{x_2} F(x)\,dx + \mathbf{I}_{x_2}^{x_3} F(x)\,dx = \mathbf{I}_{x_1}^{x_3} F(x)\,dx;$$

and accordingly

$$\mathbf{I}_{x_1}^{x_1} F(x)\,dx = 0, \quad \mathbf{I}_{x_2}^{x_1} F(x)\,dx = -\mathbf{I}_{x_1}^{x_2} F(x)\,dx.$$

IV′. *If* $F(x)$ *is* $\mathbf{I}$*-integrable from* x_1 *to* x_2 *and from* x_2 *to* x_3, *then it is* $\mathbf{I}$*-integrable from* x_1 *to* x_3, *and the equality of* IV *holds.*

V. *For every positive number* ϵ *there exists a positive number* δ_ϵ *such that* $|\mathbf{I}_{x_1}^{x_2} F(x)\,dx| < \epsilon$ *for every two points* $x_1 x_2$ *of* ab *for which* $|x_1 - x_2| < \delta_\epsilon$.

In the light of theorem III, theorem V is the affirmation of the uniform continuity on ab of the definite integral function

$$J(X) = \mathbf{I}_a^X F(x)\,dx \qquad (X \text{ on } ab), \tag{2}$$

an affirmation known to include and to follow‡ from the statement:

V′. *The definite integral function* $J(X)$ (2) *is a continuous function of the variable upper limit* X *on the* X*-interval* ab.

VI. *The function* $c\,F(x)$, *where* c *is any constant, is* $\mathbf{I}$*-integrable from* a *to* b, *and*

$$\mathbf{I}_a^b c\,F(x)\,dx = c\,\mathbf{I}_a^b F(x)\,dx.$$

* Theorem I follows as a corollary of theorem VI for *any* function $F(x)$ known to be I-integrable from a to b.

† The content of a point-set lying on an infinite interval is defined in a footnote of § 1 13°.

‡ Even if ab is an infinite interval.

VII. *If the functions $F(x)$, $G(x)$ are $\mathbf{I}$-integrable from a to b, so is the function $F(x) + G(x)$, and*

$$\mathbf{I}_a^b \big(F(x) + G(x)\big)\,dx = \mathbf{I}_a^b F(x)\,dx + \mathbf{I}_a^b G(x)\,dx.$$

VIII. *The transformation of definite integrals.*

From the hypotheses:

1) *The x-interval ab and an $\bar{x}$-interval $\bar{a}\bar{b}$ (each interval being finite or infinite) are set in one-to-one point correspondence with preservation of sense by the transformation*

$$x = \phi(\bar{x})\,^* \qquad (\bar{x} \text{ on } \bar{a}\bar{b}), \tag{3}$$

where

$$a = \phi(\bar{a}), \quad b = \phi(\bar{b}); \tag{4}$$

2) *The x-set Ξ_{ab} and an $\bar{x}$-set $\bar{\Xi}_{\bar{a}\bar{b}}$ correspond under the transformation* (3);

3) *On the $\bar{x}$-interval $\bar{a}\bar{b}$ apart from points $\bar{\xi}$ and limit-points $\bar{\xi}'$ of $\bar{\Xi}_{\bar{a}\bar{b}}$ the function $\phi(\bar{x})$ has a continuous finite and non-zero derivative† $\phi'(\bar{x})$,*

follow the conclusions:

1) *Under the transformation* (3) *the type $\mathbf{I}$ on the interval ab with respect to the set Ξ_{ab} transforms into a type $\mathbf{I}$ on the interval $\bar{a}\bar{b}$ with respect to the set $\Xi_{\bar{a}\bar{b}}$;*

2) *The function*

$$\bar{F}(\bar{x}) \equiv F\big(\phi(\bar{x})\big)\,\phi'(\bar{x}) \tag{5}$$

is $\mathbf{I}$-integrable from $\bar{a}$ to $\bar{b}$;

3) *The formula*

$$\mathbf{I}_a^b F(x)\,dx = \mathbf{I}_{\bar{a}}^{\bar{b}} F(\bar{x})\,d\bar{x} \tag{6}$$

of transformation of definite integrals has validity.

3°. These theorems, with the exception of the second, are valid for proper definite integrals in the usual sense; they belong to the elements of the theory.

---

* Where then $\phi(\bar{x})$ and its inverse $\bar{\phi}(x)$ are continuous functions of their respective arguments $\bar{x}$, x on the respective intervals $\bar{a}\bar{b}$, ab. (Cf. the second footnote of § 1 13°.) It is to be noted that a point-set Ω of content zero transforms under a transformation (3) subject to the condition 1) into a point-set $\bar{\Omega}$ likewise of content zero.

† Which has then the sign of $(b-a)/(\bar{b}-\bar{a})$ as permanent sign.—No statement is made concerning the existence or properties of the derivative $\phi'(\bar{x})$ at points $\bar{x} = \bar{\xi}$, $\bar{\xi}'$.—If the function $\phi(\bar{x})$ satisfies the conditions 1)–3), the inverse function $\bar{\phi}(x)$ satisfies the corresponding conditions on the x-interval ab. Similarly there is a composition of transformations of this character. Here one needs to *prove* (indirectly) that $\bar{a}$ or $\bar{b}$ belongs to $\bar{\Xi} + \bar{\Xi}'$ if it is infinite, so that, for example, no proper definite integral (over a finite interval) is (by theorem VIII) transformable into an integral over an infinite interval; this state of affairs accords with the general postulate of the close of § 1.

As to theorem VIII, cf. JORDAN'S *Cours d'Analyse*, ed. 2, vol. 1, nos. 141–144. The validity of all the theorems for proper integrals in the extended sense is easily perceived.

4°. *The* $\mathbf{I}$*-integrals over infinite intervals* ab *are defined* (cf. § 1 13°) *as transforms of* $\bar{\mathbf{I}}$*-integrals over finite intervals* $\bar{a}\,\bar{b}$ *by the definite transformation* $x = \psi(\bar{x})$ *of* § 1 13°.

That transformation $(\text{quâ } x = \phi(\bar{x}))$ satisfies the conditions of theorem VIII on the $\bar{x}$-interval* $-1 \leqq \bar{x} \leqq +1$.

By the notion of composition of transformations (3) one readily proves that the general validity of the theorems of § 2 follows from their validity for all finite intervals ab, $\bar{a}\,\bar{b}$ (the latter entering only in theorem VIII), and again that that validity follows from their validity for a single finite interval ab with all finite intervals $\bar{a}\,\bar{b}$.

§ 3.

FIRST DEFINITIONAL PROCESS OF INDUCTION.

1°. *By the first process of induction we are to define* (in 2°–4°) *from any set* Ξ_0 *and any type* $\mathbf{I}$ *of the set* Ξ *a* (*related*) *type* $\mathbf{I}_{\Xi_0}$ *of the aggregate set* $\Xi + \Xi_0$, that is, we are to characterize the functions $F(x)$ $\mathbf{I}_{\Xi_0}$-integrable from a to b. We agree at once that if $a = b$ every function $F(x)$ is so integrable, the integral having the value zero.

We assume the validity of theorems I–VIII *of* § 2 *for the type* $\mathbf{I}$, *and* (in 5°–7°) *we establish their validity for the type* $\mathbf{I}_{\Xi_0}$.

In accordance with § 2 4°, in these definitions and proofs we may and do restrict attention to the case of finite intervals ab.

In 8° I indicate a necessary and sufficient condition for the existence of the $\mathbf{I}_{\Xi_0}$-integral, which is very useful in the construction of integrals of type $\mathbf{I}_{\Xi_0}$ from those of type $\mathbf{I}$.

2°. The type $\mathbf{I}_{\Xi_0}$ depends upon the type $\mathbf{I}$ just as HARNACK'S general or narrow type† $\int_{(\Xi_0)}$ depends upon the type $\int$ of proper definite integrals, except that now the set Ξ_0 is of arbitrary content $\gamma(\Xi_0)$ and does not necessarily lie on a finite interval. As far as convenient I adopt the phrasings of my paper in the preceding number of this journal: On HARNACK'S Theory of Improper Definite Integrals,—and for brevity refer to it by the notation H. T.

3°. The following notations are needed.

(*a*) With respect to any interval ab and point-set Ξ or interval-set I the notation Ξ_{ab} or I_{ab} denotes the point-set of Ξ or interval-set lying on I which lies on ab.

* The extremities ± 1 are the transforms of $\pm\infty$ and so belong to the closed set $\bar{\Xi} + \bar{\Xi}'$.

† H. T. p. 304.

(β) When an interval-set * I encloses a point-set Ξ narrowly,* in which case it is denoted by the notation $I(\Xi)$, then its length D_I is greater than the content $\gamma(\Xi)$ of Ξ; this positive difference $D_I - \gamma(\Xi)$ we call the Ξ-length of I, and denote it by $D(I, \Xi)$; on the set of all such interval-sets $I(\Xi)$ the lower limit of $D(I, \Xi)$ is zero.

(γ) With respect to an interval-set I and a function $F(x)$ the function $F_I(x)$ is defined as having at every point x of I the value zero, and at every point x not of I the same determination as $F(x)$ at that point.

(δ) We denote the perfect set $\Xi_0 + \Xi_0'$ by Ξ^0.

4°. Our function $F(x)$ $\mathbf{I}_{\Xi_0}$-integrable from a to b (ab unequal and finite) is, in the first place, $\mathbf{I}$-integrable over every subinterval of ab containing no point ξ^0 of Ξ^0. This implies, according to theorems I, II, III, IV′ for the type $\mathbf{I}$, that $F_I(x)$ is $\mathbf{I}$-integrable from a to b, where I is any interval-set enclosing the point-set Ξ^0_{ab} narrowly.

Then in the second place, the (finite) limit:

$$\mathop{\mathbf{L}}_{D\left(I,\,\Xi^0_{ab}\right)=0}^{I\,|\,I(\Xi^0)} \mathbf{I}_a^b F_I(x)\,dx,$$

in notation †

$$\mathbf{I}_{a\,\Xi_0}^{\,b} F(x)\,dx,$$

exists; that is, for every positive ϵ a positive δ^1_ϵ ‡ exists such that

$$\left|\mathbf{I}_a^b F_I(x)\,dx - \mathbf{I}_{a\,\Xi_0}^{\,b} F(x)\,dx\right| < \epsilon$$

for every interval-set I enclosing Ξ^0_{ab} narrowly and of Ξ^0_{ab}-length less than δ^1_ϵ.—This limit is provably definite.—Thus in terms of the type $\mathbf{I}$ the type $\mathbf{I}_{\Xi_0}$ has been completely defined.

It is to be noticed that

$$\mathbf{I}_a^b F_I(x)\,dx = \mathbf{I}_{J_a^b} F(x)\,dx,$$

where J_a^b denotes the interval-set of ab complementary to I_{ab}, its intervals taken in the sense ab, and where the notation on the right denotes the sum of the $\mathbf{I}$-integrals of $F(x)$ over these directed intervals of J_a^b, or zero if J_a^b is non-existent. If in our definition we adopt $\mathbf{I}_{J_a^b} F(x)\,dx$ as the limitand expression, we have a form of definition closely analogous to that of JORDAN for the HARNACK-JORDAN integrals.

* H. T. § 1 5°, 7°.

† The notation $\mathbf{I}_{\Xi_0\,a}^{\,b} F(x)\,dx$ would (except for its appearance) be preferable.

‡ In this paper superscripts to δ_ϵ notations are not exponents but mere affixes of discrimination.

5°. *For the type* $\mathbf{I}_{\Xi_0}$ *(quâ* $\mathbf{I}$*) the theorems* I–VIII *of* § 2 *hold.*—We admit the validity of these theorems for the type $\mathbf{I}$, and, as indicated in 1°, we suppose that a and b are finite, and furthermore distinct, since the theorems obviously hold if $a = b$. The truth of I and II for $\mathbf{I}_{\Xi_0}$ follows at once from their truth for $\mathbf{I}$. Theorems III, IV, IV′, V, V′, VI, VII for $\mathbf{I}_{\Xi_0}$ depend upon the lemmas given in 6°, essentially as the theorems H. T. § 3 V, VIII, § 2 III, § 3 IX, II, XVII depend upon the theorems H. T. § 3 IV, V, VI. Theorem VIII is considered in 7°.

6°. Lemma I. *For every positive* ϵ *there is a positive* δ^2_ϵ *such that*

$$\left| \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx - \mathbf{I}_{x_1}^{x_2} F_{I_2}(x)\,dx \right| < \epsilon \tag{1}$$

for every two points $x_1 x_2$ *of the finite interval* ab *and every pair of interval-sets* $I_1 I_2$ *each enclosing* Ξ^0_{ab} *narrowly and of* Ξ^0_{ab}*-length less than* δ^2_ϵ.—Here the uniformity with respect to $x_1 x_2$ is of central importance.

This lemma corresponds to H. T. § 3 IV, the proof of which was involved with that of a more general theorem H. T. § 3 X, the general analogue of which is not now in question. I give as imple proof of the lemma.

The determination $\delta^2_\epsilon = \frac{1}{2}\delta^1_{\epsilon/2}$ is effective.

If the lemma is not true, then for a certain determination of ϵ, x_1, x_2, I_1, I_2 satisfying the specified conditions the inequality (1), call it (1′), is invalid. We may confine attention to the case $a < b$, $x_1 < x_2$.

Let the interval-set I enclose Ξ^0_{ab} narrowly and have Ξ^0_{ab}-length less than δ^2_ϵ. Let the interval $a\beta\,(a < \beta)$ enclose ab $(a < b)$ so that

$$a < a \leqq x_1 < x_2 \leqq b < \beta.$$

Then each interval-set:

$$I_3 = I_{ax_1} + I_{1x_1x_2} + I_{x_2\beta}\,,\quad I_4 = I_{ax_1} + I_{2x_1x_2} + I_{x_2\beta}\,, \tag{2}$$

encloses Ξ^0_{ab} narrowly and has* Ξ^0_{ab}-length less than $2\delta^2_\epsilon = \delta^1_{\epsilon/2}$.

Hence, obviously, recalling the definition, we have

$$\left| \mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx \right| < \epsilon. \tag{3}$$

On the other hand, by (2) and § 2 IV, for the type $\mathbf{I}$, we have

$$\mathbf{I}_a^b F_{I_3}(x)\,dx = \mathbf{I}_a^{x_1} F_I(x)\,dx + \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx + \mathbf{I}_{x_2}^b F_I(x)\,dx,$$

* We observe that I_3, I_4 are parts of $I_5 = I + I_1$, $I_6 = I + I_2$ respectively and use the following theorem of the theory of contents:

If the interval-sets I^1 *and* I^2 *enclose the point-sets* Ξ^1 *and* Ξ^2 *respectively, then the interval-set* $I = I^1 + I^2$ *encloses the point-set* $\Xi = \Xi^1 + \Xi^2$, *and*

$$D(I, \Xi) \leqq D(I^1, \Xi^1) + D(I^2, \Xi^2).$$

with a similar equation involving I_4 and I_2, so that

$$\mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx = \mathbf{I}_{x_1}^{x_2} F_{I_1}(x)\,dx - \mathbf{I}_{x_1}^{x_2} F_{I_2}(x)\,dx,$$

whence by (1′) we have, in contradiction with (3),

$$\left|\mathbf{I}_a^b F_{I_3}(x)\,dx - \mathbf{I}_a^b F_{I_4}(x)\,dx\right| \geqq \epsilon. \tag{4}$$

Thus lemma I is proved.

Obviously lemma I holds also if the interval-sets $I_1 I_2$ enclose $\Xi^0_{x_1 x_2}$ (instead of Ξ^0_{ab}) narrowly, for the integrals in (1) are from x_1 to x_2. Hence by the usual limit-considerations theorem II is proved for the type $\mathbf{I}_{\Xi_0}$, and likewise

LEMMA II. *For every ϵ and every two points $x_1 x_2$ of the finite interval ab*

$$\left|\mathbf{I}_{x_1}^{x_2} F_I(x)\,dx - \mathbf{I}_{x_1\,\Xi_0}^{x_2} F(x)\,dx\right| \leqq \epsilon$$

for every interval-set $\mathbf{I}$ enclosing Ξ^0_{ab} narrowly and of Ξ^0_{ab}-length less than δ^2_ϵ.

Lemma II is of fundamental importance in the deduction of theorems of linear form for the type $\mathbf{I}_{\Xi_0}$ from similar theorems for the type $\mathbf{I}$.

7°. The truth of theorem VIII (with ab, $\bar{a}\bar{b}$ finite intervals) for the type $\mathbf{I}_{\Xi_0}$(quâ $\mathbf{I}$) of the set $\Xi + \Xi_0$(quâ Ξ) follows easily from its truth for the type $\mathbf{I}$ of the set Ξ.

We consider a transformation

$$x = \phi(\bar{x}) \qquad (\bar{x} \text{ on } \bar{a}\bar{b}) \tag{5}$$

which satisfies the conditions of theorem VIII for the type $\mathbf{I}_{\Xi_0}$ of the set $\Xi + \Xi_0$.

Using notations of evident meaning we observe that the sets $\Xi_0\,\Xi^0$ correspond to similarly related sets $\bar{\Xi}_0\bar{\Xi}^0$; that every $\bar{I}$ enclosing $\bar{\Xi}^0$ narrowly corresponds to an I enclosing Ξ^0 narrowly; that in the notation of 4° every $\bar{J}^{\bar{b}}_{\bar{a}}$ corresponds to a J^b_a; and finally we observe, by considerations of the theory of contents of linear point-sets, in view of the continuity of the transformation (5) between $ab\ \bar{a}\bar{b}$, that for a positive ϵ a positive δ^1_ϵ exists such that the Ξ^0_{ab}-length of I is less than δ^1_ϵ in case the $\bar{\Xi}^0_{\bar{a}\bar{b}}$-length of $\bar{I}$ is less than $\bar{\delta}^1_\epsilon$.

Then

$$\left|\mathbf{I}_{J_b^a} F(x)\,dx - \mathbf{I}_{a\,\Xi_0}^{b} F(x)\,dx\right| < \epsilon \tag{6}$$

for every interval-set $\bar{I}$ enclosing $\bar{\Xi}^0_{\bar{a}\bar{b}}$ narrowly and of $\bar{\Xi}^0_{ab}$-length less than $\bar{\delta}^1_\epsilon$; and simultaneously, for every (directed) interval j of J^b_a,

$$\mathbf{I}_j F(x)\,dx = \mathbf{I}_{\bar{j}} F(\bar{x})\,d\bar{x}, \tag{7}$$

since the transformation (5) on $j\bar{j}$ satisfies the conditions of theorem VIII for the type $\mathbf{I}$ of the set Ξ; and thus

$$\mathbf{I}_{J_a^b} F(x)\,dx = \overline{\mathbf{I}}_{\bar{J}_a^b} \bar{F}(\bar{x})\,d\bar{x}. \tag{8}$$

From (6) and (8) by elimination we have the inequality needed to prove the validity of theorem VIII for the type $\mathbf{I}_{\Xi_0}$.

8°. The theorem of H. T. § 5 2° may readily be generalized to the case of $\mathbf{I}_{\Xi_0}$-integrals. In the proof of the generalized theorem one uses instead of the theorem H. T. § 3 XIII′ considerations of continuity and definitional considerations concerning the analogues of the integrals, loc. cit. (3). This theorem is of fundamental use in the construction (as in H. T. § 5) from the functions $\psi_{01}(x)$, $\chi_{01}(x)$ of H. T. § 5 4° of functions having integrals of type $\mathbf{I}$ of a set Ξ partitioned into a finite sequence $\Xi_1, \Xi_2, \cdots, \Xi_m$, viz.

$$\mathbf{I} \equiv \int_{\Xi_1 \Xi_2 \cdots \Xi_m}.$$

§ 4.

The second definitional process of induction.

1°. In the second process of induction we have to do with a simply ordered set* $M = \{m\}$ of symbols or marks m and with a related simply ordered set $\{\mathbf{I}^m\}$ of types $\mathbf{I}^m$ of the point-set Ξ^m, where Ξ^{m_1} is a proper subset of Ξ^{m_2} if $m_1 \prec m_2$.

We are to define in 4° *a type* $\mathbf{I}^M$ *of the aggregate set* Ξ^M, *where* Ξ^M *denotes the (least) point-set containing every set* Ξ^m *as a subset.*

From this definition, on the assumption of the validity of theorems I–VIII *of* § 2 *for the types* $\mathbf{I}^m$, *follows their validity for the type* $\mathbf{I}^M$, as one readily assures himself.

2°. The simply ordered set $M = \{m\}$ may have a highest mark m, $m = m'$; in this case we say that it terminates with the mark m'. Every finite set M terminates. A transfinite set M may or may not terminate.

3°. Even if the set $\{\Xi^m\}$ does not terminate the set $\{\Xi^m\}$ may on an interval i essentially terminate, i. e., there may be a mark m_0 such that $\Xi_i^m \equiv \Xi_i^{m_0}$ for every $m \succ m_0$, where Ξ_i^m denotes the subset of Ξ^m lying on i.

* With Cantor (Mathematische Annalen, vol. 46, p. 496) a set M of marks m is simply ordered if of every two distinct marks $m_1 m_2$ one, say m_1, is distinguished, and if this distinction is transitive, i. e., if of the marks $m_1 m_2$ the mark m_1 is distinguished, and of $m_2 m_3$ m_2 is distinguished, then of the marks $m_1 m_3$ the mark m_1 is distinguished. One says that according to the ordering in question m_1 has the lower and m_2 the higher rank, in notation:

$$m_1 \prec m_2, \quad m_2 \succ m_1.$$

A point $x = x_0$ is with respect to the non-terminating transfinite set $\{\Xi^m\}$ critical, in case on no interval i enclosing x_0 the set $\{\Xi^m\}$ is essentially a terminating set. We denote these critical points x by the notation κ and the set of all such points by the notation K. The set K is closed.

If an interval i contains no critical point, the set $\{\Xi^m\}$ on i essentially terminates, as one proves by the usual interval-halving process.

Of course, the set K exists only if $\{\Xi^m\}$ is non-terminating; and even then it may contain no finite point.

4°. We are to define an (existent) integral

$$\mathbf{I}_a^{b\,M} F(x)\,dx \tag{1}$$

of the type $\mathbf{I}^M$, that is, to characterize the function $F(x)$ in order that the symbol (1) may have a (definite finite) value.

If $b = a$, we give to the symbol (1) the value 0.

If on the interval ab the function $F(x)$ vanishes except perhaps at a set of points of content zero, we give to the symbol (1) the value 0.

Further, if the interval ab contains no critical point κ and so on ab the set $\{\Xi^m\}$ is essentially terminating, and if there is a mark m_0 such that for all marks m, $m \geqq m_0$, the $\mathbf{I}^m$-integrals of $F(x)$ from a to b exist, and for every subinterval $a'b'$ of ab and mark $m > m_0$

$$\mathbf{I}_{a'}^{b'\,m} F(x)\,dx = \mathbf{I}_{a'}^{b'\,m_0} F(x)\,dx, \tag{2}$$

then we denote by $\mathbf{I}_a^{b\,M} F(x)\,dx$ the common value of the existent integrals

$$\mathbf{I}_a^{b\,m} F(x)\,dx \qquad (m \geqq m_0).$$

Finally, if the finite or infinite interval ab splits into a finite number n of intervals $a_k b_k (k = 1, \cdots, n)$ taken in the sense of ab in such a way that by the specifications already made the integrals $\mathbf{I}_{a_k}^{b_k\,M} F(x)\,dx$ are defined, we set

$$\mathbf{I}_a^{b\,M} F(x)\,dx \equiv \sum_{k=1}^{n} \mathbf{I}_{a_k}^{b_k\,M} F(x)\,dx. \tag{3}$$

Thus the symbol (1) is given, if any, in fact a definite finite value.

It is clear that if the set M terminates, with m say, then the types $\mathbf{I}^M$ $\mathbf{I}^m$ are identical.

§ 5.

The definition of the general type $\mathbf{I}$ of the set Ξ.

1°. With Cantor* a well ordered set $W = \{w\}$ of marks w is a simply ordered set with the property that its every subset (including the set itself) has a mark

* Mathematische Annalen, vol. 49 (1897), pp. 207, 208.

of lowest rank. Every subset of W is itself a well ordered set. I denote by $W_1 = \{w_1\}$ the set of all marks $w = w_1$ having immediately preceding marks $w = w_1^0$; the mark of lowest rank by w_0; and the residual set by $W_2 = \{w_2\}$. It is to be noticed that the marks $w = w_1^0$ include all the marks of W except the terminal mark, if there is one.

2°. For the definition of the general type $\mathbf{I}$ of improper integrals with respect to the point-set Ξ I consider the general well ordered set W such that Ξ admits an exhibition, in the notation of § 4 1°, of the form $\Xi \equiv \Xi^W$, and I consider the general exhibition of Ξ in this form. The understanding is, however, that $\Xi^{w_0} \equiv \{\pm\infty\}$ and that for every w_2 the set Ξ^{w_2} is the least set containing as subsets the sets Ξ^w, $w \prec w_2$.

We set $\Xi_{w_1} \equiv \Xi^{w_1} - \Xi^{w_1^0}$; thus Ξ_{w_1} denotes an existing point-set.

3°. From the type $\mathbf{I}^{w_1^0}$ of the set $\Xi^{w_1^0}$ with the properties I–VIII of § 2 and the set Ξ_{w_1} we can, by the first process of induction, define the type $\mathbf{I}^{w_1}$ of the set Ξ^{w_1} with the same properties.

Likewise from the types $\mathbf{I}^w$, $w \prec w_2$, of the respective sets Ξ^w we can define the type $\mathbf{I}^{w_2}$ of the set Ξ^{w_2} by the second process of induction.

Likewise from all the types $\mathbf{I}^w$, w of W, of the respective sets Ξ^w we can define the type $\mathbf{I}^W$ of the set $\Xi^W \equiv \Xi$ by the second process.

4°. Now we take as type $\mathbf{I}^{w_0}$ of the set $\Xi^{w_0} \equiv \{\pm\infty\}$ on finite intervals the (extended) type of proper definite integrals defined in § 1 16°. From this type follow the definitions of all the types $\mathbf{I}^w$ and of the type $\mathbf{I}^W$. For, if not, there is an undefined type $\mathbf{I}^{w'}$ of lowest rank w'; and this is impossible, for, by the statements just made, w' is neither w_0 nor a w_1 nor a w_2.—*Thus the general type* $\mathbf{I}$ *of the set* Ξ *has definition uniquely related to the general exhibition of* Ξ *in the specified partitional form* $\Xi \equiv \Xi^W$, *where* W *denotes a well ordered set and where similar well ordered sets* W *are for the purposes of the definition equivalent.*

5°. In conclusion, we point out how the system of types of improper integrals thus defined embraces the four current types as extended (§ 1 1°, 16°).

TYPE I.—In the first type the set Ξ is reducible, that is, for some (lowest) ordinal $a \geqq 1$ of CANTOR's first or second class of ordinals the derived set $\Xi^{(a)} \equiv 0$. We consider the set $\{\beta\}$ of ordinals β, $1 \leqq \beta \prec a$, arranged in their natural ascending order. This set $\{\beta\}$ is a well ordered set. We consider the similar well ordered set $\{w_{1\beta}\}$ and make it the set W_1 of a well ordered set W, $W = w_0 + W_1 + W_2$, by introducing the initial mark w_0 and a set W_2 of marks $w_2 = w_{2\beta_2}$; here β_2 denotes an ordinal > 1 having no immediately preceding ordinal, while in well ordered set W the mark $w_{2\beta_2}$ immediately precedes the mark $w_{1\beta_2}$. Then the set Ξ^{w_0} is $\{\pm\infty\}$, the set $\Xi^{w_{1\beta}}$ contains all

points ξ not belonging to the derived set $\Xi^{(\beta)}$, and the set $\Xi^{w_2\beta_2}$ is the least set containing as subsets the sets $\Xi^{w_1\beta}$, $\beta < \beta_2$, and so the least set containing as subsets the sets Ξ^{w}, $w < w_{2\beta_2}$. The set $\Xi^{w'}$ is a proper subset of $\Xi^{w''}$ if $w' < w''$.

TYPE II.—In the second type the set Ξ is any set containing $\pm\infty$; the set W consists of two ordered elements w_0, w_1; and $\Xi^{w_0} = \{\pm\infty\}$, $\Xi^{w_1} = \Xi$.

TYPE III.—The third type is essentially the same as the first type.

TYPE IV.—The fourth type is in a sense (sufficiently indicated in § 1 9°) a combination of the second and the first types.

THE UNIVERSITY OF CHICAGO,
October 2, 1901.

ON THE CONVERGENCE AND CHARACTER OF THE CONTINUED FRACTION

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots.$$*

BY

EDWARD B. VAN VLECK

The principal object of the present paper is to give two theorems (nos. I and IV) concerning the character of the function represented by the continued fraction

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots.$$

Incidentally a criterion for the convergence of

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} \cdots$$

is obtained, which appears to be more general than any hitherto discovered except for the special case in which the elements b_n are positive. This is stated in theorem III.

§ 1.

The first part of the paper relates to the following theorem:

THEOREM I. *If k denotes the greatest modulus of a point of condensation of the coefficients a_n of the continued fraction*

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots, \tag{1}$$

the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the z-plane as center, and the only singularities of the function contained within the circle will be poles.

* Presented to the Society August 20, 1901, at its summer meeting as a part of the paper entitled *On the Convergence of the Continued Fraction of Gauss.* Received for publication September 23, 1901. The remainder of the paper is published in the Annals of Mathematics, October, 1901.

If, furthermore, any circle of smaller radius is drawn about the origin as center and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region which remains the continued fraction will converge uniformly to the analytic function as its limit.

A special case of this theorem is that in which $|a_n|$ diminishes indefinitely with increasing n. The continued fraction then represents a function which is meromorphic over the entire finite plane. This particular result, though in a yet more restricted form, has been previously discovered by STIELTJES.* Confining his attention to a continued fraction in which a_n is real and positive, he proved that it represents a meromorphic function when $\lim. a_n = 0$, and only then.

§ 2.

The proof of the above theorem is as follows: Let C be a circle whose center is the origin and whose radius is $e/4k$, in which e is taken to denote some fixed number less than unity. If e_1 is any other assigned number greater than e but less than unity, the value of $|a_n z|$ throughout the circle—inclusive of the perimeter—may be made less than $e_1/4$ by sufficiently increasing n. Let this be true when $n > \mu$, and consider the continued fraction

$$\frac{1}{1} + \frac{a_{\mu+1}z}{1} + \frac{a_{\mu+2}z}{1} + \cdots. \tag{2}$$

We shall first demonstrate the convergence of (2). Let N_n and D_n be the numerator and denominator of its nth convergent. Obviously $2|D_2| > |D_1|$. Suppose also that

$$2|D_n| > |D_{n-1}| \tag{3}$$

for any particular value of n. Since

$$D_{n+1} = D_n + a_{n+\mu} z D_{n-1}, \tag{4}$$

it follows that

$$|D_{n+1}| > |D_n| \left(1 - \frac{e_1}{2}\right),$$

and hence $2|D_{n+1}| > |D_n|$. The inequality (3) therefore holds for all values of n and accordingly

$$\frac{|D_{n+1}|}{|D_{n-1}|} > \tfrac{1}{4}.$$

A similar result holds for the numerators of the convergents.

---

*Annales de la Faculté des Sciences de Toulouse, vol. 9, *A* pp. 42-47.

Consider next the equation

$$\frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n} = (-1)^n a_{\mu+1} a_{\mu+2} \cdots a_{\mu+n} z^n \left(\frac{1}{D_n D_{n+1}} - \frac{a_{\mu+n+1} z}{D_{n+1} D_{n+2}} \right.$$

$$(5) \qquad \left. + \frac{a_{\mu+n+1} a_{\mu+n+2} z^2}{D_{n+2} D_{n+3}} - \cdots + (-1)^r \frac{a_{\mu+n+1} a_{\mu+n+2} \cdots a_{\mu+n+r} z^r}{D_{n+r} D_{n+r+1}} \right).$$

Since $|a_{\mu+i} z| < e_1/4$ and $4|D_{i+1} D_i| > |D_i D_{i-1}|$, the modulus of the right hand member is less than

$$\frac{e_1^n}{4^n |D_n D_{n+1}|} (1 + e_1 + e_1^2 + \cdots + e_1^r),$$

and hence less than

$$\frac{e_1^n}{4(1 - e_1)|D_1 D_2|}.$$

But by increasing n the last quantity may be made as small as desired. If, therefore, an arbitrarily small quantity ϵ is prescribed, a value m can be found such that

$$\left| \frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n} \right| < \epsilon \qquad (n > m).$$

It follows that N_{n+r+1}/D_{n+r+1} converges uniformly to a limit in the circle C. The convergence of the continued fraction (2) has accordingly been established.

The character of the limiting function can next be easily determined. The inequality (3) and the similar inequality connecting N_n and N_{n-1} show, in the first place, that neither the numerator nor the denominator of the nth convergent ever vanishes. The convergents are, therefore, holomorphic throughout C. Since also they converge uniformly to a limit, their limit must be holomorphic in the interior of C. Furthermore, whenever a series of holomorphic functions converge uniformly to a limit, the zeros of the limiting function are the condensation points of the zeros of the rational functions.* We conclude, therefore, that the limiting function cannot vanish within C.

A case of special interest is that in which $|a_{\mu+n}| \leqq k$ for all values of n in the continued fraction (2). The radius of C may then be taken as nearly equal to $1/4k$ as desired, and we have in consequence the following result:

THEOREM II. *If in any continued fraction*

$$\frac{1}{1} + \frac{b_2 z}{1} + \frac{b_3 z}{1} + \cdots$$

$|b_n|$ is equal to or less than a fixed quantity k for all values of n, the continued fraction converges within a circle of radius $1/4k$ described about the origin of the

* HURWITZ, Mathematische Annalen, vol. 33.

z-plane as center, and it represents a function which is holomorphic within the circle and vanishes nowhere in its interior.

§ 3.

We return now to our continued fraction (1). If the mth convergent is denoted by N'_m/D'_m, we have

$$\frac{N'_{\mu+n-1}}{D'_{\mu+n-1}} = \frac{N'_{\mu-1} + a_\mu z \dfrac{N_n}{D_n} N'_{\mu-2}}{D'_{\mu-1} + a_\mu z \dfrac{N_n}{D_n} D'_{\mu-2}}. \tag{6}$$

But in § 2 it was shown that N_n/D_n converges uniformly to a function—call it $\phi(z)$—which is holomorphic within C. Hence the numerator and denominator of the right hand member of (6) converge uniformly, and their quotient approaches the limit

$$\frac{N'_{\mu-1} + a_\mu z N'_{\mu-2}\phi(z)}{D'_{\mu-1} + a_\mu z D'_{\mu-2}\phi(z)}.$$

It is impossible that the denominator of this expression should vanish identically, for we have

$$\frac{D'_{\mu-2}}{D'_{\mu-1}} = \frac{1}{1} + \frac{a_{\mu-1}z}{1} + \frac{a_{\mu-2}z}{1} + \cdots + \frac{a_2 z}{1},$$

which would give a second continued fraction for $\phi(z)$ inconsistent with (2).* The limit is therefore an analytic function.

Furthermore, since the numerator and denominator of the above expression are holomorphic in C, its only singularities will be poles. Let each of these poles be excluded from C by drawing a circle of small but finite radius around it. We then have left a region within which not only the numerator and denominator of the right hand member of (6) converge, but also their quotient, $N'_{\mu+n-1}/D'_{\mu+n-1}$. The convergence of the continued fraction is thus established. Since also by increasing μ the radius of C may be made as nearly equal to $1/4k$ as we please, the theorem enunciated at the beginning of the paper follows.

It will be observed that the proof involved the fact that $N'_{\mu-1}$, $N'_{\mu-2}$, $D'_{\mu-1}$, $D'_{\mu-2}$ are polynomials, and did not depend upon the monomial form of the numerators and denominators of the first $\mu - 1$ partial quotients. The theorem will therefore hold if a finite number of partial quotients are replaced by arbitrarily selected rational functions.

§ 4.

No information is afforded in theorem I concerning the character of the continued fraction without the circle considered. This question is discussed for

* See PADE'S thesis, Annales de l'Ecole Normale, sér. III, vol. 9 (1892), supplément.

an important case in the Annals of Mathematics, October, 1901, namely, for the case in which a_n approaches a finite limit k in such a manner that a_{2n-1}/a_{2n} and a_{2n}/a_{2n+1} for sufficiently great values of n can be expanded into series in ascending powers of $1/n$. It is shown that *the domain within which the function represented by the continued fraction is meromorphic can then be extended over the entire plane with the exception of a rectilinear cut from* $x = -1/4k$ *to* $x = \infty$, *drawn in a direction which is the continuation of the line from the origin to* $x = -1/4k$. Application of this result is made to the continued fraction of GAUSS.

§ 5.

Theorem II gave a sufficient condition that the limiting function shall be holomorphic within the circle in question. This condition is, however, by no means a necessary one. To obtain a more general condition that there shall be no poles of the function within the circle, consider the continued fraction

$$\frac{1}{1+}\ \frac{r_2 e^{i\theta_2}}{1+}\ \frac{r_3(1-r_2)e^{i\theta_3}}{1+}\ \frac{r_4(1-r_3)e^{i\theta_4}}{1+}\ \cdots, \tag{7}$$

in which r_n is a positive number less than 1. Obviously

$$|D_2| \geqq (1-r_2)\,|D_1| = 1 - r_2.$$

If, now, for any value of n

$$|D_n| \geqq (1-r_n)\,|D_{n-1}| \geqq (1-r_2)(1-r_3)\cdots(1-r_n), \tag{8}$$

we shall have also

$$|D_{n+1}| \equiv |D_n + r_{n+1}(1-r_n)e^{i\theta_n}D_{n-1}| \geqq |D_n| - r_{n+1}(1-r_n)\,|D_{n-1}|,$$

and hence

$$|D_{n+1}| \geqq |D_n|(1-r_{n+1}) \geqq (1-r_2)(1-r_3)\cdots(1-r_{n+1}).$$

The inequality (8) therefore holds for all values of n.

By a well known formula, the nth convergent is the sum of n terms of the series

$$1 - \frac{r_2 e^{i\theta_2}}{D_2} + \frac{r_2 r_3(1-r_2)e^{i(\theta_2+\theta_3)}}{D_2 D_3} - \frac{r_2 r_3 r_4(1-r_2)(1-r_3)e^{i(\theta_2+\theta_3+\theta_4)}}{D_3 D_4} + \cdots. \tag{9}$$

In the particular case for which $\theta_n = \pi$ we have

$$D_n = (1-r_2)(1-r_3)\cdots(1-r_n),$$

and the ratio of the nth term of the series to the preceding is then $r_n/(1-r_n)$. When $r_n < 1$, this is clearly the case which is least favorable to the convergence of the continued fraction (7). The following result is now evident:

THEOREM III. *If r_n is a positive number, a necessary condition that the continued fraction*

$$\cfrac{1}{1+\cfrac{r_2 e^{i\theta_2}}{1+\cfrac{r_3(1-r_2)e^{i\theta_3}}{1+\cfrac{r_4(1-r_3)e^{i\theta_4}}{1+\cdots}}}}$$

shall converge for all values of the arguments θ_n is that the series

$$1+\frac{r_2}{1-r_2}+\frac{r_2 r_3}{(1-r_2)(1-r_3)}+\frac{r_2 r_3 r_4}{(1-r_2)(1-r_3)(1-r_4)}+\cdots \tag{10}$$

shall likewise converge. This is also a sufficient condition if $r_n < 1$ for all values of n.

§ 6.

The continued fraction

$$\cfrac{1}{1+\cfrac{b_2}{1+\cfrac{b_3}{1+\cdots}}} \tag{11}$$

is usually capable of being expressed in the form (7). Often we shall also have $r_n < 1$. For example, this is true when

$$|b_2| < \tfrac{1}{2}, \quad |b_{2n+1}| + |b_{2n+2}| \leqq \tfrac{1}{2} \qquad (n>0). \tag{12}$$

For if we put

$$|b_2| = r_2, \quad |b_n| = |1 - r_{n-1}|\, r_n \qquad (n>2),$$

and assume that $r_{2n} < \frac{1}{2}$, it follows from the second of the inequalities (12) that

$$r_{2n+1} < 1, \quad r_{2n+2} < \tfrac{1}{2}. \tag{13}$$

But $r_2 < \frac{1}{2}$, and hence, by induction, the inequalities (13) hold for all values of n.

PRINGSHEIM * has shown that the conditions (12) suffice to ensure the convergence of a continued fraction of the type (11). This appears to be the most general criterion for convergence hitherto discovered except for the special case in which all the elements b_n are positive. It is clear, however, from (13) that PRINGSHEIM'S criterion is included as a special case under theorem III.

§ 7.

Assuming now r_n to be less than 1, we shall next consider the effect of multiplying a single partial numerator, $r_m(1-r_{m-1})e^{i\theta_m}$, by a quantity δ, the absolute value of which is less than 1. If we put $r'_m = |\delta_m|\, r_m$ and seek to preserve the form of the continued fraction unaltered, we must make

$$(1-r_m)\, r_{m+1} = (1-r'_m)\, r'_{m+1}.$$

*Sitzungsberichte der mathematisch-physikalischen Classe der Münchener Akademie, vol. 28, p. 322.

Consequently $r'_{m+1} < r_{m+1}$. In like manner it follows that $r'_{m+2} < r_{m+2}$, and so on. All the terms in (10) after the $(m-1)$th will therefore be diminished, the new values being less than the old values multiplied into $|\delta|$.

Let now δ be introduced into each numerator of (7). In accordance with what has just been said, the nth convergent must be less than the sum of n terms of the series

$$1 + |\delta| \frac{r_2}{1 - r_2} + |\delta^2| \frac{r_2 r_3}{(1 - r_2)(1 - r_3)} + |\delta^3| \frac{r_2 r_3 r_4}{(1 - r_2)(1 - r_3)(1 - r_4)} + \cdots.$$

Suppose that this converges for $|\delta| = \delta'$. Then if δ in the continued fraction is replaced by a variable z, the fraction will converge uniformly in a circle having its center in the origin of the z-plane and a radius equal to δ'. Its limit is accordingly holomorphic within the circle. Furthermore,

$$|D_n| > (1 - \delta' r_2)(1 - \delta' r_3) \cdots (1 - \delta' r_n).$$

Hence it is impossible for the denominators of the convergents to vanish anywhere in the circle, and this is true also of the numerators, inasmuch as the law for their formation is similar to (4). Finally, by the theorem of HURWITZ previously cited, we conclude that the limiting function does not vanish. The result which we have thus reached may be recapitulated as

THEOREM IV. *If in the continued fraction*

$$\cfrac{1}{1 + \cfrac{r_2 e^{i\theta_2} z}{1 + \cfrac{r_3(1 - r_2) e^{i\theta_3} z}{1 + \cfrac{r_4(1 - r_3) e^{i\theta_4} z}{1 + \cdots}}}}$$

r_n is a positive number less than 1, the continued fraction will converge uniformly in a circle described about the origin of the z-plane with a radius equal to unity, provided the series (10) is convergent. If this series is divergent, the radius is at least equal to the largest positive value of δ for which

$$(14)\quad 1 + \delta \frac{r_2}{(1 - r_2)} + \delta^2 \frac{r_2 r_3}{(1 - r_2)(1 - r_3)} + \delta^3 \frac{r_2 r_3 r_4}{(1 - r_2)(1 - r_3)(1 - r_4)} + \cdots.$$

converges. The limit of the fraction is an analytic function which is holomorphic and nowhere vanishes within the circle, and the roots of the numerators and denominators of the convergents all lie without it.

An interesting application may be made to the well-known expansion

$$\tfrac{1}{2} x \log \frac{x + 1}{x - 1} = \cfrac{1}{1 - \cfrac{a_2 x^{-2}}{1 - \cfrac{a_3 x^{-3}}{1 - \cdots}}},$$

in which

$$a_{n+1} = \frac{n \cdot n}{(2n - 1)(2n + 1)}.$$

In this case

$$\theta_n = \pi, \quad r_n = \frac{n-1}{2n-1},$$

and (14) becomes

$$1 + \tfrac{1}{2}\delta + \tfrac{1}{3}\delta^2 + \tfrac{1}{4}\delta^3 + \cdots.$$

Now this series converges when $\delta < 1$. The continued fraction therefore possesses the properties indicated in the theorem over the portion of the plane which lies without a circle of unit radius with its center in the origin. On the other hand, when $\delta = 1$, the series diverges. As the sum of n terms for $\delta = 1$ and the value of the nth convergent for $x^{-2} = 1$ are identical, the continued fraction must diverge at the points $x = \pm 1$ upon the boundary of the circle.

The properties which this continued fraction possesses exterior to the circle hold also throughout the entire imaginary domain. This can be shown by throwing it into the form

$$\frac{x}{x} - \frac{1}{b_2 x} - \frac{1}{b_3 x} - \cdots,$$

in which

$$b_{2n} = \frac{1 \cdot a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \cdots a_{2n}}{a_3 a_6 \cdots a_{2n+1}},$$

Since b_n is positive and $\sum b_n$ is divergent, a theorem given on page 231 of the present volume of the Transactions can be applied. We conclude that, (1) the continued fraction converges everywhere except in the segment of the real axis between the points $x = \pm 1$, (2) the limit is holomorphic without this segment, and (3) the roots of the numerators and denominators of the convergents are contained within it. It will be noticed that the proof of this result here given, unlike previous demonstrations, is based *solely* upon the character of the continued fraction.

Wesleyan University, Middletown, Conn.

NOTES AND ERRATA: VOLUMES 1 AND 2

VOLUME 1.

O. BOLZA: *The elliptic σ-functions* · · ·.

P. 54, l. 8 up. *For* J' *read* T'.

W. F. OSGOOD: *On the existence of the Green's function* · · ·.

Pp. 310–314. I desire to point out the relation of my paper "On the existence of the Green's function for the most general simply connected plane region" to the analysis contained in HARNACK's *Logarithm. Potential* (1887), § 39. HARNACK there proposes the problem of showing the existence of a Green's function corresponding to an arbitrary simply or multiply connected continuum, i. e., precisely the problem that I have solved for a simply connected continuum, the extension of my results to multiply connected continua being obvious. (The extension is, namely, this: A Green's function for a multiply connected continuum will always exist when the boundary of the region does not contain isolated points, but is such that with each point of the boundary may be associated two other points so chosen that the three points lie on a Jordan curve.) In the solution which follows he restricts himself to a simply connected continuum F bounded by a Jordan curve C (cf. footnote, p. 310 of my article) and by an arbitrary set of curves (Einschnitte), finite in number, which lie within C, meet C each in a single point, and do not cut themselves or each other. In order to solve the problem, he constructs a set of nested polygons lying within F and having the boundary points of F as their points of condensation. The Green's functions belonging to these polygons are shown to converge toward a limit g, corresponding to the function u of my paper, which is a function similar in character to the Green's functions just considered. Up to this point both HARNACK's methods and mine are substantially the same as those of POINCARÉ, Bulletin de la Société mathématique de France, vol. 11 (1883), p. 112; cf. also HARNACK's

reference to SCHWARZ, loc. cit., p. 121. It remains to show that the function g (or u) assumes the required boundary values. To do this HARNACK employs as a majorante the Green's function belonging to a polygon Q lying wholly without F and having a point of its boundary in common with a point A of the boundary of F. His analysis suffices to show that the function g (or u) will take on the required boundary value in the point A, but not that this will be the case for a point of the boundary of F that cannot be reached by a polygon Q. Thus an ordinary beak-shaped cusp (Schnabelspitze) could not be treated by HARNACK's method. It appears, then, that HARNACK did not solve the problem he proposed even for regions F bounded by a finite number of pieces of analytic curves, to say nothing of regions, some of the points of whose boundaries cannot be approached along a continuous curve lying wholly within F. In my solution, I have employed the same method of the majorante (the function U) adopted by HARNACK, but have so chosen U that my proof covers *all* cases; and I have pointed out that there are here included cases which, I believe, had never been thought of before.—W. F. O.

P. 312, l. 1 up. *For* 167 *read* 67.

P. 314, l. 10. *After* whether *insert* if.

E. KASNER: *The invariant theory of the inversion group* ···.

P. 431, l. 6 up. The complete reference is: MAURER, *Ueber die Endlichkeit der Invarianten-Systeme,* Münchener Sitzungsberichte, vol. 29 (1899), pp. 147–175.

P. 440, l. 18. *For* $F(\lambda f + MQ)$ *read* $F_{\lambda f + MQ}$.

P. 443, l. 9. " $(ABCD)$ " $(ABCu)$.

P. 445, l. 12. The lower right hand element of the determinant g_{123} should be $\lambda_1\mu_1$.

P. 448, l. 17. *For* circles *read* cyclics.

P. 449, l. 3. " I_{ik} " I_{ik}^2.

P. 467, l. 13. " $\sum'$ " $\sum$.

P. 469, l. 18. " x " ψ.

P. 469, l. 5 up. " ϕ " Φ.

P. 475, l. 15. " $a_4 - a_1$ " $a_4 - a_2$.

P. 477, l. 8 up. The expression in braces should be squared.

P. 480, l. 20. *For* l_1 *read* l.

P. 489, l. 5 up. " WEITER " WEILER.

E. H. MOORE: *A simple proof of the fundamental Cauchy-Goursat theorem.*

P. 501, l. 5. *For* $<\epsilon$ *read* $\leqq\epsilon$.
P. 504, ll. 3, 7, 11 up. " $<$ " $\leqq$.

VOLUME 2.

E. J. WILCZYNSKI: *Invariants of systems of linear differential equations.*

P. 9, l. 12. For semivariants *read* seminvariants.
P. 11, l. 17. " $y_k =$ " $\bar{y}_k =$.
P. 22, l. 3. Make the expression into an equation by the addition of $= 0$.

J. C. FIELDS: *On the reduction of the general Abelian integral.*

P. 79, l. 19. *For* $r + s + 2 = \sigma$ *read* $r + s - 2 = \sigma$.
P. 80, l. 2 up. " $n + 3$ " $n - 3$.
P. 85, l. 2. " $+\sum_{\lambda=1}^{d+p}$ " $-\sum_{\lambda=1}^{d+p}$.

H. F. STECKER: *On the determination of surfaces ....*

P. 155, l. 11 up. Replace $d\mu$ in the expression for F_2 by $d\nu$.
" l. 3 up. The exponent of F_2 should be $-\frac{2}{3}$.
P. 159, l. 7. *For* $m + \beta - 1$ *read* $m - \beta - 1$.
P. 163, l. 17 up. " V_1 " V_2.
" l. 1 up. " $+\phi_2(\nu)$ " $-\phi_2(\nu)$.

E. B. VAN VLECK: *On the convergence of continued fractions ....*

Pp. 223, 224. The last line of p. 224 is to be set at the top of p. 223.
P. 226, l. 9 up. *For* $-a_n M_{n-1}^2$ *read* $a_n M_{n-1}^2$.
P. 233, l. 16. " $|a_n|/|\beta_n|$ " $a_n/|\beta_n|$.

W. F. OSGOOD: *On a fundamental property of a minimum ....*

P. 293, l. 7. *For* its longest side *read* the greatest of the differences $\tau_{i+1} - \tau_i$.

E. H. MOORE: *Concerning Harnack's theory* ···.

P. 296, l. 4 up. *For* DE LA VALLÉE POUSSIN *read* DE LA VALLÉE-POUSSIN.

P. 302, l. 19 up. " otherwise " , otherwise, .

P. 304, l. 2. *After* there *insert* exists.

P. 305, l. 10. In formula (8) insert an upper limit b.

P. 309, l. 2. }
P. 311, l. 8. } Clearly in this connection Z_{ab} should contain a or b only in case a or b is singular towards the inside of the interval ab.

P. 309, l. 14. *For* $\delta^1_{\epsilon/2}$ *read* $\delta_{\epsilon/2}$.

P. 319, l. 13 up. " J " $J,$.

P. 324, l. 10. " every " every ν.

P. 329, l. 15. " ν " ν'.

UNIVERSITY OF ILLINOIS-URBANA
510.6AMS C001
TRANSACTIONS OF THE AMERICAN MATHEMATICA
2 1901
3 0112 016864776